The Valuation of Stochastic Cash Flows

Leigh J. Halliwell

Abstract

A stochastic cash flow depends on a random outcome. Since insurance and reinsurance contracts in exchange for financial considerations provide financial compensations against random outcomes, their compensations are perfect examples of stochastic cash flows. This paper develops a theory for the valuation of such cash flows from the four principles of present value, utility, optimum, and equilibrium. The most important implications of the theory are that the optimal amount for an economic agent to purchase depends on price, that (price, amount) loci are preliminary to market value, and that market value is the unique price at which all interested economic agents purchase optimal amounts. The theory belies the prevalent practices of risk-adjusted discounting and capital allocation (Appendix A), faulting them for naïve and erroneous conceptions of time. Accordingly, although the information that the theory presumes of economic agents is formidable, each agent is realistically burdened with ascertaining its own present opinions and preferences, rather than impossibly burdened with omniscience. And it sets the agents to the virtuous task of extracting value from projects, rather than from one another. The theory lays claim to fundamental principles of financial economics; it derives from the work of European risk theorists Karl Borch, Hans Bühlmann, and Hans Gerber, and gains support from a small but growing number of American actuaries. Though the paper remains theoretical throughout (especially in Appendices B-D), it furnishes several examples of sufficient detail for actuaries to apply it to pricing traditional insurance and reinsurance contracts.

Keywords: present value, utility, optimum, equilibrium

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1. INTRODUCTION

The insurance and reinsurance industry is in an impasse. The muddled and contradictory thinking on how to value its contracts is the stuff of a Dilbert cartoon. Industry leaders act inconsistently, belittling theory as stifling action while grasping at every idea of the month. Actuaries too are swept along with the tide. What is needed, and what this paper hopes to accomplish, is to define basic economico-financial concepts and to develop a theory for the valuation of insurance and reinsurance contracts.

In the next section we define the basic concepts of stochastic cash flow and present value within the framework of modern probability theory. Then, in Section 3, we argue that present value is a random variable with a probability distribution, something that strikes at the heart of current financial theory. From there we turn in Section 4 to utility theory, for it alone seems to provide a criterion for preferring one probability distribution to another. But Section 5 gives a new twist to utility theory, according to which one purchases at a given price the amount that maximizes one's expected utility, as opposed to determining the price for a unit amount that maintains that utility. So the issue of price remains open until in Section 6 we see how it is collectively determined. Sections 5 and 6 contain what might aptly be called a market-tempered utility theory. Section 7 illustrates this theory with a reinsurance example, and suggests how to dampen the swings of the reinsurance market. Before the conclusion, Section 8 draws out two implications of the theory, especially how it could change actuaries *per se* from followers to leaders.

The theory to be developed is complicated. However, it is no more complicated than the various risk modeling and capital allocation projects in which companies have for years invested large sums of money and time, so far with little success. Moreover, very little herein is new; as radical as it may seem, the theory is just culled from the best actuarial, financial, and economic sources, from sources of unquestionable orthodoxy.

2. DEFINITIONS AND COMMENTS

We begin with the unexciting but important task of defining four essential terms: *cash flow*, *present value*, *stochastic [cash flow]*, and *random variable*.

A *cash flow* is a mapping, or function, c from $[0,\infty)$ into the real numbers that describes how much cash has been received up to and including $c(t)$. It should have at most a countably infinite number of discontinuities. Sufficient for our purposes are discrete cash flows, which jump at times t_i and are flat elsewhere. Thus we can imagine a cash flow as a countable set of increments, and denote it as $c = \{(t_i, x_i) : i = 1, 2, \dots\}$, where an ordered pair (t,x) represents the receipt of cash x (dollars) at time t (years) from now. By definition, the present is time zero, and the time coordinates must be nonnegative. In other words, cash flows have no memory of the past.

Present value is a function v from $[0,\infty)$ onto the half-open interval $(0,1]$. It states that the receipt of one dollar t years from now is equivalent to the receipt of $v(t)$ dollars now. Four properties of present value are that $v(0) = 1$, that v strictly decreases, that v is continuous, and that $v(t)$ approaches zero as t approaches infinity. Just as cash flows have no memory of the past, present value looks only forward, not backward.[1]

Theoretically, each economic agent could chose its own present-value function; but practically, prices at which U.S. Treasury securities are traded determine a market-accepted present-value function at any moment of time. For centuries investors have been accustomed to think of present value in terms of yield. However, since yields are not constant by maturity, investors should abandon them in favor of the discount function with its equivalence between one dollar at time t and $v(t)$ now.[2]

The present value of discrete cash flow c is $PV[c] = \sum_i x_i \cdot v(t_i)$, if it exists.[3] It is an operator that assigns a real number to a function; and it implies that cash flow c is equivalent to the receipt of $PV[c]$ now. It is a linear operator in that for cash flows c and d and constants α and β, $PV[(\alpha c + \beta d)(t)] = \alpha PV[c(t)] + \beta PV[d(t)]$. Henceforth we

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[1] Inasmuch as accounting looks backward over an accounting period, one must take care not to foist accounting concepts onto present value. The currently popular "results monitoring" projects have not recognized this.

[2] Halliwell [2001, Appendix A] challenges the notion that money somehow works at the yield. If a five-year STRIP (zero-coupon Treasury security) sells at 85.10 for a yield of 3.29 percent per year and a ten-year STRIP sells at 63.38 for a yield of 4.64 percent per year, is it helpful to assert that ten-year money is working harder than five-year?

[3] For some countably infinite cash flows, the sum does not converge. But the present value of a finite cash flow must exist.

assume that at any given moment every economic agent has a present-value operator (whether its own or the market-accepted), and can value any cash flow. If in the next moment the agent's operator changes, it must recalculate the present values of its cash flows; however, it does not follow that the agent should attempt to adjust its present-value operator for the risk that the operator itself will change.[4]

The adjective '*stochastic*' means 'probabilistic' or 'pertaining to a probability system'. In the 1920s and 30s the Russian mathematician Andrei Nikolaevich Kolmogorov devised the modern theory of probability with its fundamental concept of a probability system. In the words of Pfeiffer [1978, p. 29]:

> A *probability system* (or probability space) consists of the triple:
>
> 1. A *basic space* S of elementary outcomes (elements) ξ
> 2. A *class* E *of events* (a sigma field of subsets of S)
> 3. A *probability measure* $P(\cdot)$ defined for each event A in the class E and having the following properties:
>
> (P1) $P(S) = 1$ (probability of the sure event is unity)
> (P2) $P(A) \geq 0$ (probability of an event is nonnegative)
> (P3) If $\mathrm{A} = \{A_i \colon i \in J\}$ is a countable partition of A (i.e., a mutually exclusive class whose union is A), then
>
> $$P(A) = \sum_{i \in J} P(A_i) \qquad \text{(additivity property)}$$

A simplified version of a discrete probability system will serve our needs. The basic space S will consist of a countable number of elementary outcomes ξ_i, for $i = 1, 2, \ldots$. The class E of events will consist of all the subsets of S, i.e., it will be the power set of S. But we will concern ourselves with the "elementary" events $\{\xi_i\}$, and henceforth will ignore the distinction between elementary events and elementary outcomes, referring to

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[4] A change in one's present-value operator can have profound effects, good as well as bad. On the bad side, it may render one insolvent accountingwise. However, that the operator might change in the next

both indiscriminately as ξ_i. The class of these mutually exclusive ξ_i is a countable partition of *S*, and the probability measure is determined by nonnegative $P(\xi_i)$ that sum to unity.

Each agent has its own probability system, or in our simplified version, its own basic space *S* and probability measure *P*. Many elementary outcomes will appear only in one agent's basic space. Even to compare elements of different spaces may be difficult. But should two agents share an elementary outcome, they are free to assign different probabilities to it. And one agent may know for certain something that another regards as uncertain. Probability systems are as subjective as human beings; it is difficult, as well as unnecessary, for anyone to pontificate whether one system is more accurate or more in tune with reality than another. One can only hope that reality is reasonable enough to reward agents according to how accurately they perceive it.

Two characteristics of probability systems need to be appreciated. First, a probability system is as momentary as a present-value operator. An economic agent is free to change its system at any moment. But for now that system defines all that is, and to require it to consider how it might change would be endlessly circular. And second, a probability system is timeless. For example, if a coin toss were an elementary outcome, it would not matter if the toss had already happened and the agent were just ignorant of the outcome, or if the toss were yet to happen.[5]

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moment has no effect on the value of a cash flow at the present moment. The ultimate inference from this is that present value and solvency are unrelated (Section 8).

[5] So we do not need to resolve the paradox of Schrödinger's cat: Before one opens the box is the cat either 100% dead or 100% alive, or is it suspended in a probability distribution between life and death?

A *stochastic cash flow* C is a mapping from a basic space into the collection of cash flows. In the simplified version, such a mapping assigns to each ξ_i a cash flow c_i. One can transfer probability from the elementary outcomes to the cash flows. For any cash flow c, let $A = \{\xi \in S : C(\xi) = c\}$. Then $P(c) = P(A)$. If no outcome maps to c, then $A = \varnothing$ and $P(c) = 0$. Or if many discrete ξ_i map to c, but the probability of each is zero, then still $P(c) = 0$. In these cases c is an impossible cash flow, according to the probability system under consideration. At the other extreme, if every discrete cash flow with positive probability maps to c, then $P(c) = 1$, and the stochastic cash flow is degenerate. There is as little difference between degenerate stochastic cash flow C and cash flow c that it equals with certainty as there is between degenerate random variable X and the number x that it equals with certainty.

Strictly speaking, a *random variable* is a function from S into the real numbers, which function is also "a measurable function with respect to the class of events E." [Pfeiffer, 1978, p. 376] Our simplified version permits us to ignore the subtleties of measurability; for us a random variable is merely a function from the countable basic space into the real numbers. And the probability that the random variable equals some number is the sum of the probabilities of the elementary outcomes that map to that number.

3. PRESENT VALUE AS A RANDOM VARIABLE

So, at length, we are ready to define the present value of a stochastic cash flow. If C is a mapping from S into cash flows (i.e., a stochastic cash flow), and PV is a mapping from cash flows into the real numbers, then the present value of stochastic cash flow C is the composite mapping $PV \circ C$. Being a mapping from S into the real numbers, it is a random variable.

As a very simple example of a stochastic cash flow, let S have two elementary outcomes: $S = \{\text{H, T}\}$. This basic space would model the outcome of a coin toss. The event class E would equal $\{\varnothing, \{\text{H}\}, \{\text{T}\}, S\}$. A fair coin would have the probability measure $P(\{\text{H}\}) = P(\{\text{T}\}) = 0.5$, with $P(\varnothing) = 0$ and $P(S) = 1$. This triple (S, E, P) satisfies the definition of a probability system. Let two cash flows be the receipts of 120 dollars and 80 dollars one year from now; they can be expressed incrementally as $\{(1,120)\}$ and $\{(1,80)\}$. Let stochastic cash flow C map the heads outcome to the first flow, and the tails outcome to the second. According to the set-theoretic definition of a function or mapping, $C = \{\,(\text{H},\{(1,120)\}),(\text{T},\{(1,80)\})\,\}$. At the end of August 2002, a one-year Treasury STRIP traded at 98.71 dollars (on a par value of 100 dollars). Taking that as time zero, we derive the market-accepted $v(1)$ as 0.9871. Hence, $PV(\{(1,120)\}) = 120 \times 0.9871 = 118.45$ dollars, and similarly, $PV(\{(1,80)\})$ equals 78.97 dollars. And the present value of the stochastic cash flow is the function or mapping $X = PV \circ C$, where $X(\text{H}) = PV(C(\text{H})) = PV(\{(1,120)\}) = 118.45$, and similarly with the tails outcome for a value of 78.97. X satisfies the definition of a random variable, and we are allowed to make the four probability statements:

$$\begin{aligned} P(X = 118.45) &= P(\{\text{H}\}) = 0.5 \\ P(X = 78.97) &= P(\{\text{T}\}) = 0.5 \\ P(X = 118.45 \text{ or } X = 78.97) &= P(S) = 1 \\ P(X \neq 118.45 \text{ and } X \neq 78.97) &= P(\varnothing) = 0 \end{aligned}$$

The cumulative distribution function of X is:

$$F_X(x)=\begin{cases} 0 & x < 78.97 \\ 0.5 & 78.97 \le x < 118.45 \\ 1 & 118.45 \le x \end{cases}$$

Exhibit 1 presents a slightly less simple stochastic cash flow. Ten elementary outcomes determine ten cash flows. Each cash flow consists of one ordered pair (t,x), as in the previous example, and these pairs are graphed on a Cartesian half-plane with present-value isobars. Treasury STRIP prices on 30 August 2002 at various maturities provided the raw data to which a $v(t)$ curve was fitted. So the isobar that stems from $(0, x_0)$ plots the curve $x_0/v(t)$; in other words, the isobars are proportional to one another, and fill the half-plane. The present values of the cash flows fall within the [75,100] interval. Provided that the probability of each elementary outcome is ten percent, Exhibit 2 graphs the cumulative density function of the present value of the stochastic cash flow. We have not yet treated valuation, but already one might suspect that the value of this stochastic cash flow must be greater than seventy-five and less than one hundred.

The third and final example of this section deals with a corporate bond. On 01 October 2001 investors paid just over 747 million dollars for unsecured bonds issued by Tyson Foods. The face amount of the bonds was 750 million dollars, and the coupon was 7.25 percent per year, payable semiannually. Moody's Investors Service rated the bonds as Baa3, the lowest of the investment-grade ratings. Exhibit 3 contains this information, along with Moody's default probabilities and recovery parameters.[6] For example, Moody's estimates that Baa3 bonds have a 1.28% chance of default within three years of issue. The chance of the Tyson Foods bond default over its five-year term is 2.79%. And when a company defaults, holders of its bonds usually recover some of the face amount. Moody's estimates that holders of defaulted senior unsecured bonds recover 44.62% ± 26.32% of the face amount.

The bottom-left part of Exhibit 3 shows the cash flow that the bondholders hope to receive, the full payment of interest and principal. According to Moody's default probabilities their hopes have a 97.21% chance of fulfillment. On the bottom-right side of the exhibit are present-value factors, $v(t)$, based on current STRIP prices. The non-defaulted cash flow has a present value of 618,046,875 (principal) and 247,614,404 (interest) for a total of 865,661,279 dollars. The bottom-middle part of the exhibit shows the value of the cash flow that defaults after five interest payments and returns at maturity forty percent of the principal. The present value of this cash flow is 247,218,750 (principal) and 130,372,559 (interest) for a total of 377,591,309 dollars. The exhibit is part of a spreadsheet that simulates default probabilities and recovery rates. The default probability is a uniform (0,1) random variable, such as the 1.00% in the exhibit, which is compared with the Baa3 default probabilities. And the recovery factor (e.g., 40.00%) is modeled as a beta random variable whose parameters α and β are matched to the mean and standard deviation of Moody's recovery rates. The spreadsheet simulated 10,000 cash flows and calculated their present values. Exhibit 4

[6] Moody's revises these probabilities and parameters annually, and one can download them from the website riskcalc.moodysrms.com/us/research/defrate.asp.

graphs[7] the distribution of the present value, along with reference lines for the mean ($E[X]$ = 854,166,433) and the purchase price ($q[X]$ = 747,187,500). Therefore, the expected profit is 106,978,933 dollars. The present value is at its maximum of 865,661,279 dollars past the ninety-seventh percentile, and thereafter declines in a fairly straight line effectively to zero. The probability that the bondholders will lose money, i.e., $P(PV(X) < 747{,}187{,}500)$ is 2.73%. Nearly all the defaults constitute a loss of money.

These three examples illustrate the concept of the present value of a stochastic cash flow as a random variable. It differs from the common concept of present value as a number. The common concept would discount a stochastic cash flow at a rate of return that is greater than the risk-free rate. Typical investors, considering the Tyson Foods bond at the time of issue, would know that the present value of its cash flow, if it does not default, is 865,661,279 dollars. And they would know that discounting the non-defaulted cash flow at 3.87 percent per year results in this dollar amount. Now if the bond does not default, its stochastic cash flow is degenerate and equivalent to a simple cash flow; hence the correctness of the equation $PV(X|\text{no default}) = 865{,}661{,}279$ dollars. This, indeed, is the mode of the present-value distribution (97.21% probable according to Moody's statistics). But does it do justice to the remaining 2.79% for one to decide somehow to discount the non-defaulted cash flow at 7.48%, and offer the resulting amount of 747,187,500 dollars? If not, then Donald Mango [2003, footnote 8] is correct to write that "the method of risk-adjusted discounting ... represents an example of 'overloading an operator,' piling additional functional burden onto what

[7] Lee [1988] introduced actuaries to this form of graphing a probability distribution, wherein quantiles of the distribution are plotted against its percentiles, a form now commonly called a Lee diagram.

should be a single purpose operator."

We believe that risk-adjusted discounting, with its concept of present value as a number, short-circuits the stochasticity of present value, is a legacy from pre-probabilistic, deterministic ages, and may be responsible for many of the woes in the past two decades of the insurance and reinsurance industry.[8] The conclusion of this section is that deriving the present value of a stochastic cash flow as a random variable is necessary for valuing the cash flow. More pointedly, if the random variables of the present values of two stochastic cash flows are equal, then the values of the two flows must also be equal.

4. THE UTILITY OF A DISTRIBUTION OF PRESENT VALUE

The value of stochastic cash flow C depends on the random variable X that maps from elementary outcomes ξ_i of basic space S to the present values of its outcome-dependent cash flows $PV[c_i]$. The cumulative distribution function of X, viz., $F_X(x)$, is $P(X \leq x) = \sum_{PV[c_i] \leq x} P(\xi_i)$. The cumulative distribution function of a random variable maps from the real numbers into the unit interval [0,1]. Though closely associated with its random variable, it lacks information about the basic space. Hence, from this distribution alone, one would not know how X relates to other random variables. However, an economic agent could evaluate the sum of its stochastic cash flows, i.e., its

[8] Some think that the deterministic concept approximates the probability concept, perhaps as Newtonian physics approximates special relativity. However, the deficiency is much more glaring, rather like that of Aristotelian physics in comparison with Newtonian. Appendix A critiques risk-adjusted discounting.

total (present-valued) stochastic wealth W, from the distribution of W. Such an agent would have to express preferences; for example, whether a normally distributed wealth of mean 600 and standard deviation 100 is preferable to one of mean 550 and standard deviation 50. So an economic agent values not stochastic cash flows themselves, but rather the effect of such cash flows on its total stochastic wealth. Let W represent an agent's current stochastic wealth. Let X be a stochastic cash flow offered at price q. If the agent purchases X, its resulting wealth will be $W+X-q$. The agent needs a decision-making operator U that assigns real numbers to distributions of stochastic wealth, such that W_1 is preferable to W_2 if and only if $U(W_1) > U(W_2)$.

Karl Borch [1961, p. 248] briefly recounts the history of utility theory, particularly how it fell out of favor with late nineteenth-century economists, who deemed it too complex to be workable. Around 1900 the indifference theory of Vilfredo Pareto, which seemed to circumvent utility functions, found favor with many. But in the 1940s Von Neumann and Morgenstern [1972] resurrected utility theory by proving that every decision-making operator U that possesses certain reasonable properties implies a utility function.[9] Borch himself offers a proof [1961, pp. 249-251] that every operator U that satisfies two axioms implies a utility function. The first axiom, which is hardly disputable, is that U constitutes "a complete preference ordering over the set of all probability distributions." The second axiom is that if U deems two distributions as equivalent, it will also deem equivalent their linear combinations with a third distribution. He discusses criticism of the second axiom, but concludes, "... this general criticism does not concern ...

[9] See also Debreu [1987, §4.6] for a proof of the existence of an n-dimensional utility function for an economic agent that can form preferences about baskets of n goods.

insurance where the only events considered are payment of different amounts of money." [1961, p. 254]

So we will proceed on the assumption that an economic agent estimates both the present value of its stochastic wealth W and the present value X of the stochastic cash flow. Moreover, the agent has a utility function u, so that its decision whether to purchase the cash flow at price q is determined by the relation $E[u(W+X-q)] \sim E[u(W)]$. To those who argue that estimating random variables and utility functions is too much to ask of economic agents, there is a two-part rejoinder. First, economic decision-making is not an easy problem. But rather than to cut the problem down to the size of the economic agents, we ought to build the agents up to the size of the problem. And second, the capital-allocation alternatives have grown so complicated that one rightly wonders whether it be any less difficult to apply them than to apply utility theory.

The properties of a realistic utility function are well known.[10] The utility of wealth w, $u(w)$, should be strictly increasing, twice differentiable, and concave downward. These properties imply increasing utility and diminishing marginal utility, viz., that $u'(w) > 0$ and $u''(w) < 0$. To such functions applies Jensen's inequality, $E[u(W)] \leq u(E[W])$, with equality if and only if W is a degenerate random variable. Most treatments of utility theory consider only these three properties; however, it is reasonable to add that

[10] Some actuarial references favorable to utility theory are Bowers [1986], Bühlmann [1980], Gerber [1979 and 1998], Halliwell [1999 and 2001], Longley-Cook [1998], Mango [2003], Panjer [1998], Schnapp [2001], Sundt [1991], and Van Slyke [1995 and 1999]. Economic works, e.g., Debreu [1987], Dixit [1990], Duffie [1990], and Von Neumann [1972], invariably begin with a utility-theoretic foundation.

$\lim_{w \to \infty} u'(w) = 0$, from which it also follows that $\lim_{w \to \infty} u''(w) = 0$. Furthermore, since stochastic cash flows can be much larger than an agents' stochastic wealth, one cannot set lower and upper bounds on $W+X-q$. Hence, $u(w)$ should exist for all real w. Candidates for utility functions are either quadratic, power, logarithmic, or exponential. But of these four types, only the exponential function is appropriate for all real numbers. For the power and logarithmic functions are not defined for zero and negative numbers, and the quadratic function decreases after its vertex.

The decision whether to purchase stochastic cash flow X at price q hinges on the comparison $E[u(W + X - q)] \sim E[u(W)]$. For a small enough q (perhaps negative), the left side of the equation is greater, and the purchase is desirable. But as q increases, the left side decreases and there should be a unique q^* at which both sides are equal. At a price greater than q^* the purchase is undesirable. So it is true of stochastic cash flows in general what Borch writes about insurance losses:

> In insurance a basic assumption is that there will always exist a unique amount of money which is the lowest premium at which a company will undertake to pay a claim with a known probability distribution. This assumption establishes an equivalence between certain and uncertain events. [1961, p. 249]

Borch writes from the insurance standpoint of receiving a minimum positive premium p for paying positive losses L, from which standpoint the decision hinges on the comparison $E[u(W - L + p)] \sim E[u(W)]$. But paying a positive loss corresponds to receiving a negative payoff, and receiving a positive premium to paying a negative price. So "the lowest premium at which the company will undertake to pay a claim" from Borch's insurance standpoint is equivalent to the price q^* "greater than which the purchase is undesirable" from our investment standpoint. It augurs well for a valuation theory that one may treat liabilities as negative assets, and assets as negative liabilities.

For one whose wealth is the real number *w*, the decision to purchase hinges on the comparison $E[u(w+X-q)] \sim u(w)$. And due to Jensen's inequality:

$$\begin{aligned} E[u(w+X-q)] &\leq u(E[w+X-q]) \\ &\leq u(w+E[X]-q) \end{aligned}$$

If *q* equals *E*[*X*], the inequality becomes $E[u(w+X-E[X])] \leq u(w)$, with equality if and only if *X* is degenerate. Thus an agent whose wealth is deterministic cannot improve its expected utility by purchasing at expected value. A non-degenerate stochastic cash flow must sell at less than its expected present value for the expected utility of the agent to increase. This is not always the case when wealth *W* is truly stochastic, since *X* might countervail *W*.

The *q** that solves the equation $E[u(W+X-q)] = E[u(W)]$ satisfies the two principles of Appendix A. As per the first principle, it must lie within the minimum and maximum bounds of *X*. In particular, if *X* is degenerate, *q** must equal *X* with probability one. As per the second, for any real number *k*, $E[u(W+(X+k)-(q+k))] = E[u(W)]$ if and only if $E[u(W+X-q)] = E[u(W)]$. Therefore, a change in the level of *X* can be offset only by the same change in *q*.

The strengths of this version of utility theory notwithstanding, it has two defects that may explain why many sympathizers remain unconvinced. First, the comparison $E[u(W+X-q)] \sim E[u(W)]$ allows one to calculate the unique *q** at which both sides are equal, at which price the economic agent merely conserves its expected utility. But a ceiling or maximum price is not the same as a bidding price. Knowing not to pay more than 100 dollars for something is not quite the same as knowing what to pay for it. And second, these ceiling prices are not linear; for example, the ceiling price for twice X is not twice the ceiling price for X. The next section will remove these defects.

5. OPTIMAL UTILITY AND INDIVIDUAL EQUILIBRIUM

To determine whether $E[u(W+X-q)]>E[u(W)]$ is to concentrate on the price q. Perhaps we should concentrate not on the price q at which to buy the whole X, but rather on the amount of X to buy at the price q. It may not be realistic to buy a fraction of a house, or a negative amount of land; but stochastic cash flows are ideally scaleable. The expected utility of an agent that purchases θ units of X at price q per unit is $f(\theta, q)=E[u(W+\theta X-\theta q)]$. The derivative with respect to q is $\frac{\partial f}{\partial q}=-\theta \cdot E[u'(W+\theta X-\theta q)]$. Since u' is positive, the expectation is positive, and the sign of the derivative is opposite to that of θ. If θ is positive, a decrease in price q increases expected utility f. And if θ is negative, an increase in price q increases f. Hence, a buyer (with positive θ) seeks as low a price as possible, whereas a seller (with negative θ) seeks as high a price as possible. Nothing more than this obvious truth is to be gained from concentrating on price.

A more subtle and fruitful perspective is to treat q as given and to determine how much of X to purchase. The derivative of the expected utility with respect to θ is

$\frac{\partial f}{\partial \theta} = E[u'(W + \theta X - \theta q)(X - q)]$. Appendix B proves that if $P(X > q)$ and $P(X < q)$ are both greater than zero (i.e., that $X - q$ has both upside and downside potential), then f as a function of θ looks like a concave downward parabola. In symbols, $\lim_{\theta \to \pm\infty} f(\theta, q) = -\infty$. And f strictly increases up to a maximum, after which it strictly decreases. Hence, an economic agent that has little influence on price turns its attention to buying the unique amount that optimizes its expected utility. What determines price is left to Section 6; but for now one might reasonably expect prices to be less erratic and more reflective of value when the concern of agents is how much to buy or to sell, rather than to negotiate the most advantageous deals.

Exhibit 5 graphs $f(\theta)$ for various values of q. In this example $W = 0$, X is a fifty-percent chance of receiving 100 dollars, and the agent's utility is exponential, viz., $u(w) = -e^{-0.1w}$. Not unexpectedly, all the curves intersect at $\theta = 0$, for the price of X is irrelevant if the agent has none; in symbols, $f(0, q) = E[u(W)] = E[-e^0] = -1$. As long as $0 < q < 100$ the curves rise from negative infinity, attain a maximum, and fall back to negative infinity. If $0 < q < E[X] = 50$, the risk-averse agent will optimize its expected utility with a long position ($\theta > 0$). If $E[X] < q < 100$, a short position ($\theta < 0$) is maximal; at $q = 50$ the agent neither buys nor sells. If q were 0 (a "can't lose" situation), or even negative (a "must win" situation), the agent would buy an infinite amount of X. Similarly, if q were 100 dollars (a "can't win" situation) or greater (a "must lose" situation) the agent would sell an infinite amount. In addition to four curves with maxima, the graph shows curves for the "must lose" situation of $q = 110$ and for the "can't lose" situation of $q = 0$. Due to the fact that this agent cannot distinguish X from $100 - X$, the family of curves is symmetric about the vertical line $\theta = 0$, i.e., $f(-\theta, 100 - q) = f(\theta, q)$.

The total differential of f is:

$$\begin{aligned} df &= \frac{\partial f}{\partial \theta} d\theta + \frac{\partial f}{\partial q} dq \\ &= E[u'(W + \theta X - \theta q)(X - q)]d\theta - \theta E[u'(W + \theta X - \theta q)]dq \end{aligned}$$

Since u' is everywhere positive, $E[u'(W + \theta X - \theta q)] > 0$. Therefore, an increase in price (a positive dq) causes a change in f whose sign is opposite to that of θ. In other words, on the positive or long side of the θ-axis high-q curves lie below low-q curves. Correspondingly, on the negative or short side of the θ-axis high-q curves lie above low-q curves. As one can see in the exhibit, to increase price causes the expected utility of long positions to decrease and the expected utility of short positions to increase

From this point we restrict our attention to those values of q for which both Prob[$X < q$] and Prob[$X > q$] are greater than zero, for which values $f(\theta)$ has a maximum. Amount θ at price q is optimal if and only if:

$$\frac{\partial f}{\partial \theta} = E[u'(W + \theta X - \theta q)(X - q)] = 0$$

One would like to know how the optimal amount varies with price; in particular, one would expect it to vary inversely with price. The total differential of this partial derivative is:

$$\begin{aligned} d\left(\frac{\partial f}{\partial \theta}\right) &= \frac{\partial E[u'(W + \theta X - \theta q)(X - q)]}{\partial \theta} d\theta + \frac{\partial E[u'(W + \theta X - \theta q)(X - q)]}{\partial q} dq \\ &= E\left[u''(W + \theta X - \theta q)(X - q)^2\right] d\theta - E[u''(W + \theta X - \theta q)\theta(X - q) + u'(W + \theta X - \theta q)]dq \end{aligned}$$

Optimal ($d\theta$, dq) combinations maintain the total differential at zero:

$$0 = E\left[u''(W + \theta X - \theta q)(X - q)^2\right]d\theta - E\left[u''(W + \theta X - \theta q)\theta(X - q) + u'(W + \theta X - \theta q)\right]dq$$

Because u'' is everywhere negative the coefficient of $d\theta$ is negative. And because u' is everywhere positive, the second term within the coefficient of dq is positive. If the whole coefficient of dq is positive, $d\theta$ and dq will vary inversely. It may not be positive in all cases, even with the fact that $E[u'(W + \theta X - \theta q)(X - q)] = 0$. But in the case of exponential utility, $u''(w) = -a^2 e^{-aw} = -a \cdot ae^{-aw} = -a \cdot u'(w)$. Consequently:

$$\begin{aligned} E[u''(W + \theta X - \theta q)\theta(X - q)] &= E[-a \cdot u'(W + \theta X - \theta q)\theta(X - q)] \\ &= (-a\theta)E[u'(W + \theta X - \theta q)(X - q)] \\ &= (-a\theta) \cdot 0 \\ &= 0 \end{aligned}$$

Then the whole coefficient of dq is positive. In behavioral terms, with an increase of q the agent deems a decrease of θ to be optimal. But even without exponential utility, for optimal θ to vary inversely with q should be usual; real agents buy less at higher prices, and more at lower.

An economic agent can optimize its utility simultaneously for many stochastic cash flows. Let the present values of the cash flows be X_j for $j = 1, 2, \ldots$, and let them be offered at prices q_j. So the agent seeks to maximize $f(\Theta) = E\left[u\left(W + \sum_j \theta_j X_j - \sum_j \theta_j q_j\right)\right]$.

The partial derivatives $\frac{\partial f}{\partial \theta_k} = E\left[u'\left(W + \sum_j \theta_j X_j - \sum_j \theta_j q_j\right)(X_k - q_k)\right]$ must be zero at the optimal values θ^*. Letting W^* denote the optimal portfolio $W + \sum_j \theta_j^* X_j - \sum_j \theta_j^* q_j$, we must have for all k, $E[u'(W^*)(X_k - q_k)] = 0$. This means that the prices of the stochastic cash flows with respect to which an agent has optimized its expected utility, even though the agent passively accepted them, satisfy the equations:

$$\begin{aligned} q_j &= \frac{E[u'(W^*)X_j]}{E[u'(W^*)\quad]} \\ &= E[\Psi X_j] \end{aligned}$$

A portfolio optimized with respect to cash flows X_j is also in equilibrium with respect to them; in other words, the agent wishes neither to buy nor to sell amounts of X_j. The agent can express the value of each cash flow as $q_j = E[\Psi X_j]$. This may not be true of every stochastic cash flow; but it is true of the flows with respect to which the portfolio is optimized.

The random variable Ψ is equal to $u'(W^*)/E[u'(W^*)]$, and $E[\Psi]=1$. Ψ is called variously a state-price random variable (Halliwell [2001] and Panjer [1998]), a deflator (Christofides and Smith [2001]), and a Radon-Nikodym derivative (Gerber and Pafumi [1998]). In terms of the probability system, Ψ maps from elementary outcome ξ_i to:

$$\frac{u'(W^*(\xi_i))}{\sum_j P(\xi_j)\cdot u'(W^*(\xi_j))}$$

Because of diminishing marginal utility u' will be less positive for the favorable outcomes of W^* than it will be for the unfavorable. Therefore $E[\Psi X] = \sum_i P(\xi_i)\Psi(\xi_i)X(\xi_i)$ reweights the present value of a stochastic cash flow, giving more weight to the outcomes at which the portfolio is less valuable.[11] Hence, if X tends to vary with W^*, $E[\Psi X]$ will tend to be

[11] So $P(\xi)\cdot\Psi(\xi)$ qualifies as a probability measure (Section 2). That it equals zero if and only if $P(\xi) = 0$ makes it an equivalent measure. Wang [2001] refers to $P(\xi)\cdot\Psi(\xi)$ as distorted probability.

less than $E[X]$. If it tends to vary against W^*, $E[\Psi X]$ will tend to be greater.

A perfectly optimized portfolio would be in equilibrium with respect to all stochastic cash flows. Equipoised and wishing to buy or to sell nothing, its economic agent would value every cash flow as $q_X = E[u'(W^*)X]/E[u'(W^*)] = E[\Psi X]$. This formula satisfies the principles of Appendix A. For first, it handles correctly the degenerate cash flow:

$$q_k = E[\Psi k] = E[\Psi]\cdot k = 1\cdot k = k$$

And second, as a weighted average it stays within any bounds. Moreover, it is linear:

$$q_{\Sigma\alpha_i X_i} = E[\Psi(\sum \alpha_i X_i)] = \sum \alpha_i E[\Psi X_i] = \sum \alpha_i q_{X_i}$$

The linearity of valuation is paradoxical. Valuation should be linear, and indeed is linear, if an agent is in equilibrium. However, if the agent is in equilibrium, it neither buys nor sells. But an agent that buys or sells believes that the result will restore equilibrium, and buys or sells into the equilibrium price. In other words, the valuation of economic agents anticipates equilibrium.

6. PRICE DETERMINATION AND MARKET EQUILIBRIUM

Until now we have thought of economic agents as having no influence on price. But consider a market of $n > 1$ agents.[12] Since at any given moment stochastic cash flow C, with present value X, can have just one price, the i^{th} agent seeks to maximize its expected utility $f_i(\theta_i) = E[u_i(W_i + \theta_i X_{(i)} - \theta_i q)]$, the optimal amount being θ_i^*. In this

[12] There is no need to distinguish between large and small markets (as per Wang [2001, §2]), a distinction that continuity of size renders arbitrary.

formulation each agent has its own utility function, current wealth, and optimum. Moreover, each has its own estimate of the properties of *X*, which is indicated by a subscript in parentheses, $X_{(i)}$. But really this means that each agent has its own probability system. The only quantity common to all is the price *q*. Sellers belong as much as buyers to the market, selling being just the buying of a negative amount and vice versa.

Whether an agent regards itself as a price-maker, a price-taker, or something in between, for any given *q*, the agent should purchase the amount θ that maximizes its expected utility. But in a market they collaborate by setting *q* such that their optimal θ* amounts clear. Normally to clear is for net buying to be zero, i.e., $\sum_i \theta_i^* = 0$. However, it could be otherwise, e.g., $\sum_i \theta_i^* = 1$, as in an auction. There is one and only one price *q* at which optimal amounts θ* clear. For at higher prices, the expected utilities are maximized at lesser θ values, which means that sellers want to sell more than buyers want to buy. Conversely, at lower prices, the utilities are maximized at greater θ values, and buyers want to buy more than sellers want to sell.

In mathematical terms, a market of *n* agents is a mechanism for solving a system of *n*+1 equations in *n*+1 variables. The *n*+1 equations consist of the *n* equations that maximize the expected utilities (or that set their first derivatives to zero) and the clearing equation $\sum_{i=1}^{n} \theta_i^* = 0 \text{ (or 1)}$. And the *n*+1 variables are the *n* values of θ and the one price *q*.

Essentially, price is a device whereby the agents of a market maximize their expected utilities. Most agents claim to have little or no influence on price, and great or total influence on quantity. But if the market solves a system of simultaneous equations, then in overall equilibrium the random wealth W^* of each agent is more than acceptable; as far as cash flow X is concerned it is optimal. The agents collaborate, mostly unawares, to arrive at an equilibrium in which each does as well as possible.[13]

The remaining exhibits develop this idea in a three-agent market. The three agents are A, B, and C, or Abel, Baker, and Charlie for realism. At first, let us assume that the wealth of each is deterministic at 100,000 dollars, and that their utility functions are the same at $u(x) \propto -e^{-0.000005x}$. An agent at this wealth and with this utility would regard a gain of 11,000 dollars and a loss of 10,000 dollars as offsetting.

Exhibit 6 describes how these agents would price stochastic cash flow X, which they all believe to be a fifty-percent chance of gaining 100,000 dollars. Four parameters determine the exhibit, one price q and three amounts θ, one for each agent. These parameters are the shaded cells of the topmost table of the exhibit. However, the θ amount for Charlie is constrained so that the three amounts total to one. In other words, stochastic cash flow X must be completely auctioned off to Abel, Baker, and Charlie. So really, the free parameters are the unit price q and the θ amounts of Abel and Baker.

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[13] Such an overall equilibrium is known as a Pareto optimum (Gerber, 1979, Chapter 7), technically defined as a state in which one cannot do better unless another does worse.

The next table simply lists the marginal utilities of the agents, which here are the same. The next three tables take q and the θ amounts and calculate each agent's expected marginal utility, i.e., $E[u'(W+\theta X-\theta q)(X-q)]$. This appears in the last cell of the "Mean" row of each agent's table, which is the sumproduct of the columns labeled "Probability" and "$u'(\ )(X\text{-}q)$". The expected marginal utilities are fed into the last column of the topmost table, and their summary statistic in the "Total" row is their root mean square, i.e., the square root of the average of their squares.

The Excel Solver add-in seeks q, Abel's θ, and Baker's θ so as to minimize the root mean square. According to theory, there is one, and only one, equilibrium at which all three expected marginal utilities are zero, which happens if and only if their root mean square is zero. The solver deemed 5.959E-03 as the minimum, with individual marginal utilities 1.931E-03, 9.495E-03, 3.555E-03. Being six or seven orders of magnitude less than the state marginal utilities, these amounts are effectively zero. So Abel, Baker, and Charlie buy equal shares (33.33%) of X, which makes sense since they have the same utility and the same outlook as to Probability, W, and X. And together they buy all of X at a price of 45,843 dollars. At a higher price they would not buy it all, and at a lower price they would want to buy too much. It doesn't matter here from where X enters the financial universe of Abel, Baker, and Charlie. There does not have to be a fourth agent who owns all of X. One may posit an infinitely risk-averse quasi-agent, such as "Luck" or "Possibility," which will unload X at any price with upside potential.

In Exhibit 7 Abel is less risk-averse than Baker and Charlie; his exponential parameter $a=2.500\text{E}-06$ is half theirs. They all have the same outlook; but Abel has twice the appetite for risk. This implies that the price must rise from 45,843 to discourage overbuying. The solver arrived at the price of 46,879 dollars, at which price Abel takes half and Baker and Charlie take quarters. Abel, in effect, functions like two agents: If a fourth agent were added to Exhibit 6 having the same parameters and outlook as Abel, Baker, and Charlie, the solver would arrive at the price of 46,879 and each would have a quarter share.

Exhibit 8 takes a step into the reality of differing opinions. Charlie continues to believe that *X* is a fifty-percent chance of gaining 100,000 dollars. But Abel assesses it as a fifty-percent chance of gaining 80,000, and Baker as a forty-percent chance of gaining 100,000. All have the same utility; but to Abel and Baker stochastic cash flow *X* is less attractive than it was in Exhibit 6. Therefore, it is no surprise that the equilibrium comes at the lower price of 39,353 dollars, at which Abel's and Baker's shares are only 8.09% and 5.41%. Charlie rates *X* highly, and buys 86.50% of it. This example shows that the market model developed in this paper does not require the economic agents to have the same assessment of risk. Each agent must be free to create its own probability system.

Finally, Exhibit 9 treats the spreading of risk within a closed system. Charlie owns *X* (in addition to his deterministic wealth of 100,000 dollars). And Charlie's θ amount is constrained so that the three amounts total to zero. Otherwise, the exhibit is the same as Exhibit 6. Not unexpectedly, the equilibrium is the same; the price is 45,843 dollars, and each agent ends up with a third of *X*.

This market-tempered utility theory generalizes to any number of agents and any risk-averse utility functions. Moreover, the probability system of each agent, in which its wealth W and the cash flow X are related, is arbitrary. Each agent is free to act on its own beliefs, whether accurate or not. Even as to the issue of present value, mentioned in Section 2, each agent is free to make its own judgments. But this model countenances no difference between large and small markets; it holds as much for a market of two agents as for a market of millions. A market would seem to be no more and no less than the sum of its parts, i.e., a number of agents each seeking to maximize its own expected utility. It is agents that move their markets, not markets their agents.

7. REINSURANCE PRICING

It is instructive to apply this valuation theory to reinsurance. Suppose that an insurance company asks reinsurers to assume a catastrophic risk, e.g., its exposure to hurricane losses in excess of a certain threshold. We will assume that the insurer wants, or is constrained, to reinsure this risk at any reasonable cost. Exhibit 10 shows how a market of two reinsurers might determine the price. The exhibit is set up for insurance losses L and premiums p, rather than for payoffs X and prices q. But Section 4 explained how losses are equivalent to negative payoffs, and premiums to negative prices.

Reinsurer A is freshly capitalized at one billion dollars, which it has placed in cash and government securities. After examining the underwriting information, it estimates the probability of losing fifty million dollars to be four percent, and that of losing one hundred million to be two percent. So the expected loss is four million dollars. Reinsurer B already has an insurance and investment portfolio; in fact, it estimates the distribution of the present value of its wealth as twenty-five percent at 1.1 billion dollars, fifty percent at 1.0 billion, and twenty-five percent at 0.9 billion. So its size is comparable to that of Reinsurer A, and we've assumed both their utilities to be exponential with *a*-parameter 5.000E-09.

Though Reinsurer B agrees with Reinsurer A that the expected loss is four million dollars, it believes the probability of losing twenty-five million dollars to be one percent, that of losing fifty million dollars to be two and a half percent, and that of losing one hundred million to be two and a half percent. The "Probability" column of Reinsurer B in the exhibit is the outer product of (25%, 50%, and 25%) and (94%, 4%, 2%). If loss column "L" consisted of three blocks of (0, 50, 100) million dollars, the hurricane loss would be independent of the stochastic wealth of Reinsurer B. If this were the case, exponential utility would allow Reinsurer B to value this risk on its own (Appendix C). Both reinsurers would have the same assessment of the risk, and each would take a fifty percent share at the total premium of 4,781,718 dollars.[14]

However, Reinsurer B does not deem the risk as independent; the tendency is for the loss to be greater when the current stochastic wealth is less. In fact, the correlation coefficient between W and L is –8.2 percent. Perhaps Reinsurer B has already assumed some of the same hurricane risk from other insurers. In any case, it is not as eager to assume the new risk as is Reinsurer A. If fact, a total premium of 5,501,545 dollars is needed for a fifty percent share of the risk to maximize the expected utility of

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[14] Each reinsurer receives its share of the premium. The θ factors in the fifth and sixth columns of each reinsurer's calculation take this into account.

Reinsurer B, versus the 4,781,718 for Reinsurer A. The undesirable covariance increases the quote of Reinsurer B by more than 700,000 dollars. But the reinsurers together will assume the whole risk for a total premium of 5,106,791 dollars, Reinsurer A signing the slip for 68.29 percent and Reinsurer B for 31.71 percent.

One often hears about erratic swings in the reinsurance market. Exhibit 11 suggests how this valuation theory might help to moderate such swings. Suppose that there is a third reinsurer, Reinsurer C, freshly capitalized just like Reinsurer A. But this reinsurer is dour about the hurricane risk, estimating a ten percent chance of losing fifty million dollars, and a three percent chance of losing one hundred million. So it expects eight million dollars of loss, twice as much as what the others expect. Therefore, it sits on the sidelines, lamenting the overcapitalized and soft reinsurance market while Reinsurers A and B offer 5.1 million dollars. Reinsurer C would be content with an equal share of the risk, but a total premium of 8.9 million dollars would be needed to maximize its expected utility at a one-third share (and 10.8 million at a full share).

Why not let Reinsurer C sign the placement slip for a negative share? This is the equivalent to short selling in other markets; so why not here? Implicitly the insurance company is on the slip at a –100 percent share. The short positions on the slip can lose at most the premiums that they pay; it is the long ones that are "naked" to the losses. In fact, since losses and premiums are inverses of payoffs and prices, it is hard to define here what is long and what is short. According to the exhibit, Reinsurer C expresses its belief in cheap reinsurance by taking a –90.91 percent share. The effect of its contrariness is to raise the price from 5,106,791 to 6,077,896 dollars. Instantly the market hardens by nineteen percent, and everyone (other than, perhaps, the insurance company) is happy. One might have qualms about a reinsurer's having more than a 100 percent share; but this is less likely to happen in a market of more than three.

Though it is not uncommon for direct-writing reinsurers to cooperate and to take shares of risks, reinsurance brokers would have the advantage to implement this particular strategy of negative shares, even as they have advantages to foster cooperation among reinsurers. However, at present reinsurers are not prepared for market pricing, for few understand that the premium for a share of a deal should not equal the share of the premium for the whole deal.[15]

8. IMPLICATIONS

Before concluding, we will here draw out implications for two important and relevant topics, solvency and investment. As to solvency, risk-adjusted discounting has misled many to elevate it from the status of a constraint to that of a valuation method. Solvency is important, especially for insurance and reinsurance companies; so important, in fact, that they ought to decline deals that jeopardize it no matter how attractively priced they are. But the insurance industry, along with its regulators and analysts, has adopted the method of the banking industry, as expressed in the so-called "Basle Accord" (Basle Committee, 1988). This method is concerned with how a company's net worth might deteriorate from one (annual) balance sheet to the next. Capital is charged against, or

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[15] Again, risk-adjusted discounting is the culprit. Rates of return are scaleable; the ROE of half the deal at half the price is the same as the ROE of the whole deal at the whole price.

allocated to, elements of wealth that are subject to various risks. A comparison of net worth with this overall "risk-based capital" determines whether the company is financially strong or weak.

This has led many, both in banking and insurance, to cost-of-capital pricing. One calculates how much capital to allocate to a deal and subtracts the cost of this capital from the deal's expected value. Now, to be fair, the solvency regulations refrain from saying how to price banking and insurance products. But lack of vice is not virtue. For regulators do believe cost-of-capital pricing to be a proper application of their solvency regulations. In fact, they can't imagine how pricing could be done otherwise, and they are gratified to see their industries following in their trains. This busies many technical minds with constructing ideal risk measures, according to which one may allocate enough, but not too much, capital to risky deals. As Glenn Meyers [2002] expresses it, "We have to balance the cost of an insolvency with the cost of holding capital."

Both the regulation and the application are flawed, the flaw in both concerning time. First, the regulation countenances only accounting items and how they might change from statement to statement. But cash, not capital, is the *prima materia* of the financial universe. To stress the current balance sheet with an array of one-year assaults tells something about financial strength; but it does not plumb to the depth of cash. Better is to estimate the probability for a company's cash account to remain positive throughout its runoff. A company highly certain to have enough cash on hand to quit its obligations should be deemed stronger than a company highly certain to have a positive net worth one year hence. This superior criterion of strength requires peering into the cash level.

Second, nearly everyone blithely assumes that money works, in particular, that it works at a certain rate per time period. So one hundred dollars working at ten percent per year should be paid ten dollars at the end of the year.[16] But if a risk were resolved in a very short time, days or even hours, capital allocated to it should be paid pro-rata as to time. Unlike repairmen, who charge by the hour or fraction thereof, capital does not bill by the whole year.[17] The whole picture of insurance needs revamping. Fussing over regulation and its effect upon accounting statements evades economics.[18] Insurance is not the exposing of capital to loss and the pooling of risks to optimize such exposure. Rather, it is a transaction in which a stochastic cash flow is exchanged to the benefit of the expected utilities of both parties (Bowers, 1986, Chapter 1, aptly titled "The *Economics* of Insurance").

As to the second topic, investment is a species of valuing stochastic cash flows. An insurance or reinsurance company ought always to revise its estimate of the present value of the sum of its stochastic cash flows. To keep negligible the probability that this present-value random variable is negative (perhaps more accurately, the probability that its run-off cash account would ever be negative) the company must limit both its insurance and investment activities. Investment neither piggybacks upon nor

[16] But who gets paid, the money or its owner? If the money works for its owner, is it a slave? Appendix A argues that for money to work is nonsensical and misleading language. Obviously it is a figure of speech. But if none bothers (or is able) to translate it into literal language, how can the figure be harmless?

[17] "The is no natural unit of time." (Halliwell, 1999, Appendix E) In the same section he teases out the ramifications to capital allocation of doing business on other planets. Venus might become the Bermuda of the next century, for its shorter year may afford reinsurers accounting advantages!

[18] One can improve wealth on paper without improving real wealth. The fault is not just with the present accounting rules; it will always exist, even after the eagerly awaited convergence of IAS and GAAP in 2005. One should make sound economic decisions and let the accounting chips fall where they may.

supercharges underwriting. Since each crowds out the other, the business of insurance should be to underwrite well, not to underwrite to generate funds to invest well.

Insurance deals are perfect specimens of stochastic cash flows; insurance is the ideal setting in which to apply this theory.[19] Investing in equities is far more complicated. For the stock investor puts money not into a project, but into a corporation whose employees will put money into projects indefinitely. Since one bets not on a deal, but on persons who will bet on deals, the equity investor is at one remove from the stochastic cash flows. It is not just a matter of buying in, possibly receiving a few dividend checks, and selling out; it is not even as simple as projecting a dividend stream, as per the Gordon dividend-growth model. To value a stock is to value the human management of a stochastic perpetuity. (No wonder that investment theory eschews stochastic cash flows!) In companies that understand this theory and the near idealness of its application to insurance chief actuaries will be kings; chief financial officers will be charged with corporate reporting and cash management. More than forty years ago Karl Borch predicted the ascendance of actuaries: "The traditional approach implies that the actuary should play a rather modest part in the management of his company. ... In the light of these theories [for decision making under uncertainty] it appears that the actuary should take a broader view of his duties." [1961, pp. 245f.]

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[19] More accurately, it will be the ideal setting, if insurers and reinsurers should get back to basics, one of which is to underwrite exposures free of moral and morale hazards. According to Rob Jones of Standard & Poor's, "Reinsurers realize that they must focus on underwriting performance rather than investment portfolios." ("Reinsurers Must Get Back To Basics," *Reactions Rendez Vous Reporter*, September 11, 2002, p. 12, www.rvs-monte-carlo.com/docs/reaction/Reaction 11-09-2002.pdf.

9. CONCLUSION

From taking seriously the meaning of 'present' in 'present value' our theory of valuation derives. Present value really is value now, not value on the horizon of time, not value as of the next accounting statement. The present is right this moment, not after the next news flash, not even after a double take. Anything else, however cleverly concealed, is value in the future, to which the present is linked by capital working at percent per year. How strange that this understanding of 'present' is as countercultural to 1990s finance as such slogans as "Live for the moment!" were to 1960s society! But perhaps it will catch on in this decade.

The belief that value now must impound what it might later become confuses subject and object. "The price of an asset or a deal is different from the cash flow – as different as subject is from object." (Halliwell, 2001, §6) Hence, many cannot tell the difference between keen appraisers and savvy traders. Theorists construct models in which present price depends on future price, the more knowable on the less knowable. Predicting the future replaces valuing the present. This misconception is twin to that of subsuming the value of risk into the value of time. Recognizing this may well cause a financial revolution, as Oakley Van Slyke (1995, p. 587) writes:

> The theory of finance, both as it is and as it will be after the coming revolution, is by its nature prescriptive. The theory of finance suggests how a decision-maker should make decisions among alternative courses of action. The theory of finance is not descriptive. It does not show why people or institutions behave as they actually do.[20]

[20] Similarly Borch [1961, §5.2]: "Shackle does not consider his theory as normative in the sense that it states how rational businessmen should make decisions. All he claims is that his theory describes, or explains how businessmen actually reach their decisions." Which is more important, a description of how something is done or a description of how it ought to be done? Rationality has to do with oughtness.

One can enumerate risks of all sorts, e.g., that payoffs will be too small or too large, or too soon or too late, that interest rates will go up or down, that courts will be too lax or too stern, that the company comptroller will embezzle millions of dollars, that a virus will infect the company database, that an asteroid will strike the earth. These may or may not be legitimate concerns for the management of an insurance or reinsurance company. But valuation impounds into the price of a stochastic cash flow the distribution of the probability of its present value. Hence, valuation countenances only the risks that affect that distribution.

Present value is a random variable. An economic agent must know the present value W of its current stochastic cash flow. Then it must estimate the present value X of the proposed deal (e.g., as in Exhibit 2), as well as its joint distribution with W. Utility theory is the bridge from random variables to values. An agent should know how much θ of the deal to purchase at price q, i.e., the amount that maximizes its expected utility $f(\theta) = E[u(W + \theta X - \theta q)]$. Interaction of agents with one another will produce a clearing price, at which each agent purchases an optimal amount. The analytics of all this may seem too complex and difficult a task. However, insurance cash flows are perfect specimens of stochastic cash flows, and actuaries are fortunate to work in such an ideal laboratory. Furthermore, capital-allocation programs have grown at least as complex and difficult. Perhaps actuaries disenchanted with them will examine present value and utility, and give the finance world something to emulate.

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Exhibit 1

Stochastic Cash Flow on Present-Value Coordinate System

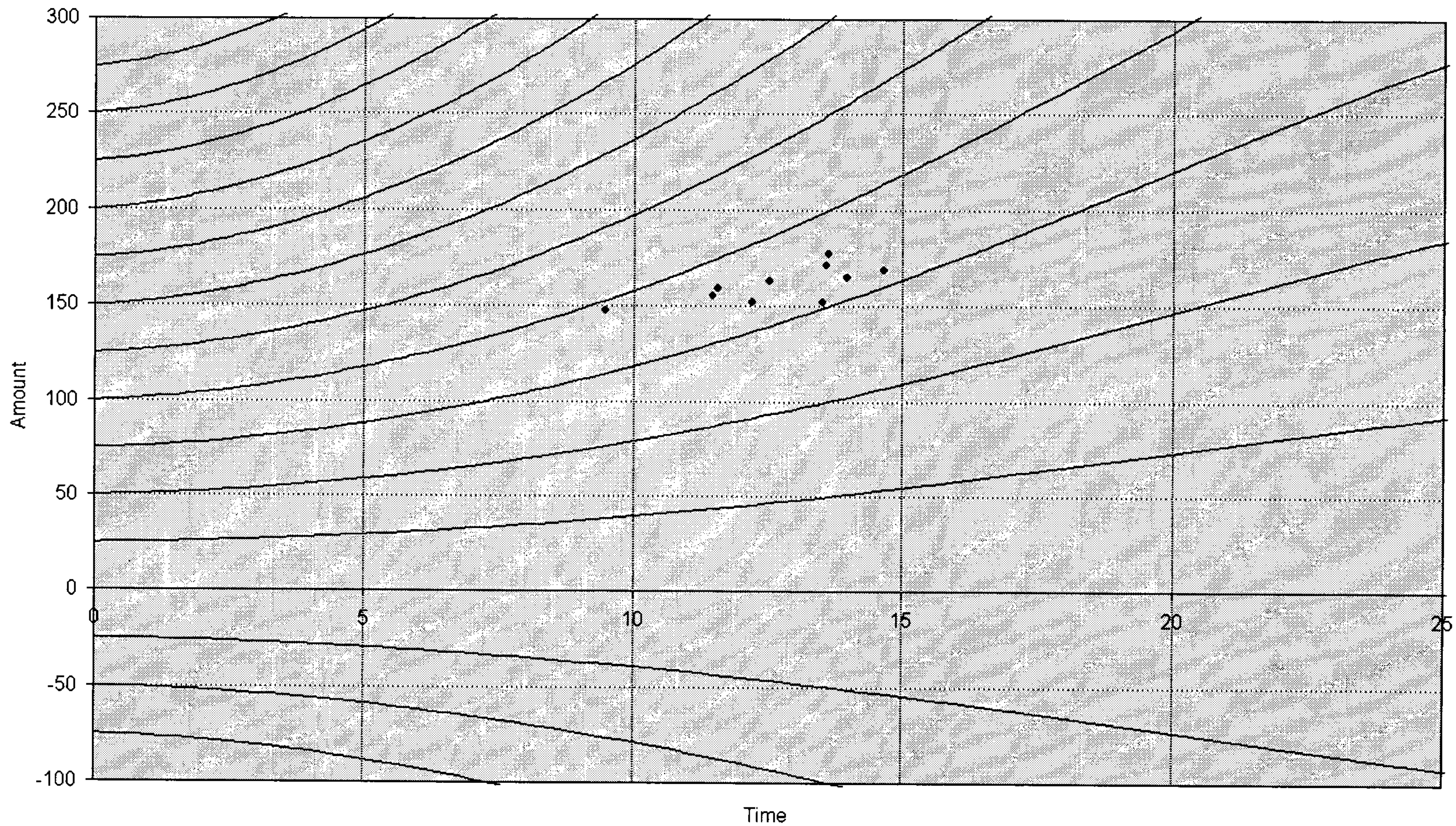


Exhibit 2

Cumulative Distribution of Present Value

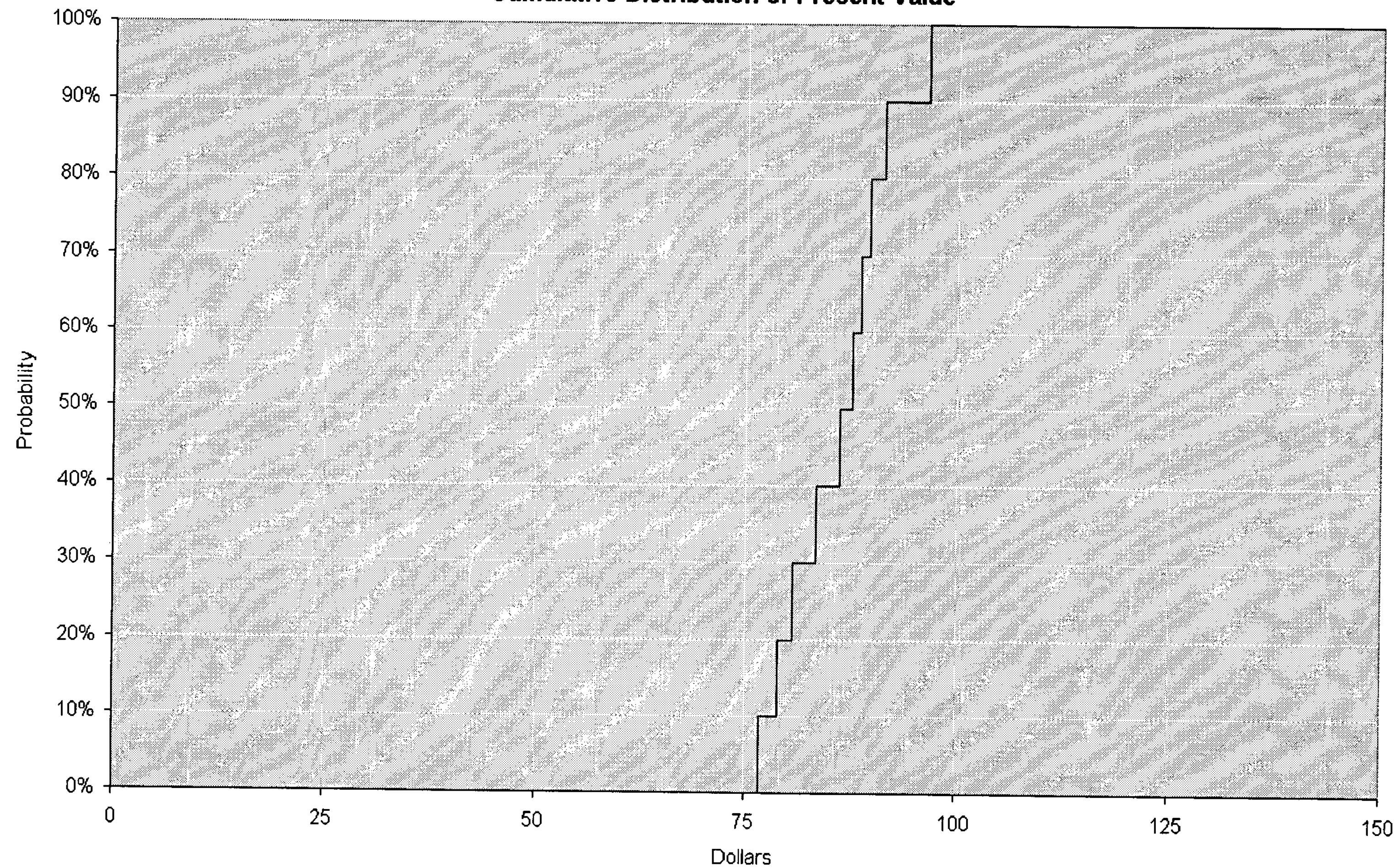


Exhibit 3

Issuer	TYSON FOODS INC
Issue Date	01 Oct 2001
Maturity	01 Oct 2006
Price	747,187,500
Face Amount	750,000,000
Coupon	7.25%
Frequency	semi-annual
Moody's Rating	Baa3
Structure	senior unsecured

Recovery Parameters

μ	44.62%
σ	26.32%
α	1.145
β	1.422
$B(\alpha, \beta)$	40.00%
Recovery	300,000,000

Without Default

t	Date	Default Probability	Actual	Principal	Interest
0.0	01 Oct 2001	0.00%	1		0
0.5	01 Apr 2002	0.00%	1		27,187,500
1.0	01 Oct 2002	0.00%	1		27,187,500
1.5	01 Apr 2003	0.00%	1		27,187,500
2.0	01 Oct 2003	0.00%	1		27,187,500
2.5	01 Apr 2004	0.00%	1		27,187,500
3.0	01 Oct 2004	0.00%	1		27,187,500
3.5	01 Apr 2005	0.00%	1		27,187,500
4.0	01 Oct 2005	0.00%	1		27,187,500
4.5	01 Apr 2006	0.00%	1		27,187,500
5.0	01 Oct 2006	0.00%	1	750,000,000	27,187,500
Present Value				618,046,875	247,614,404

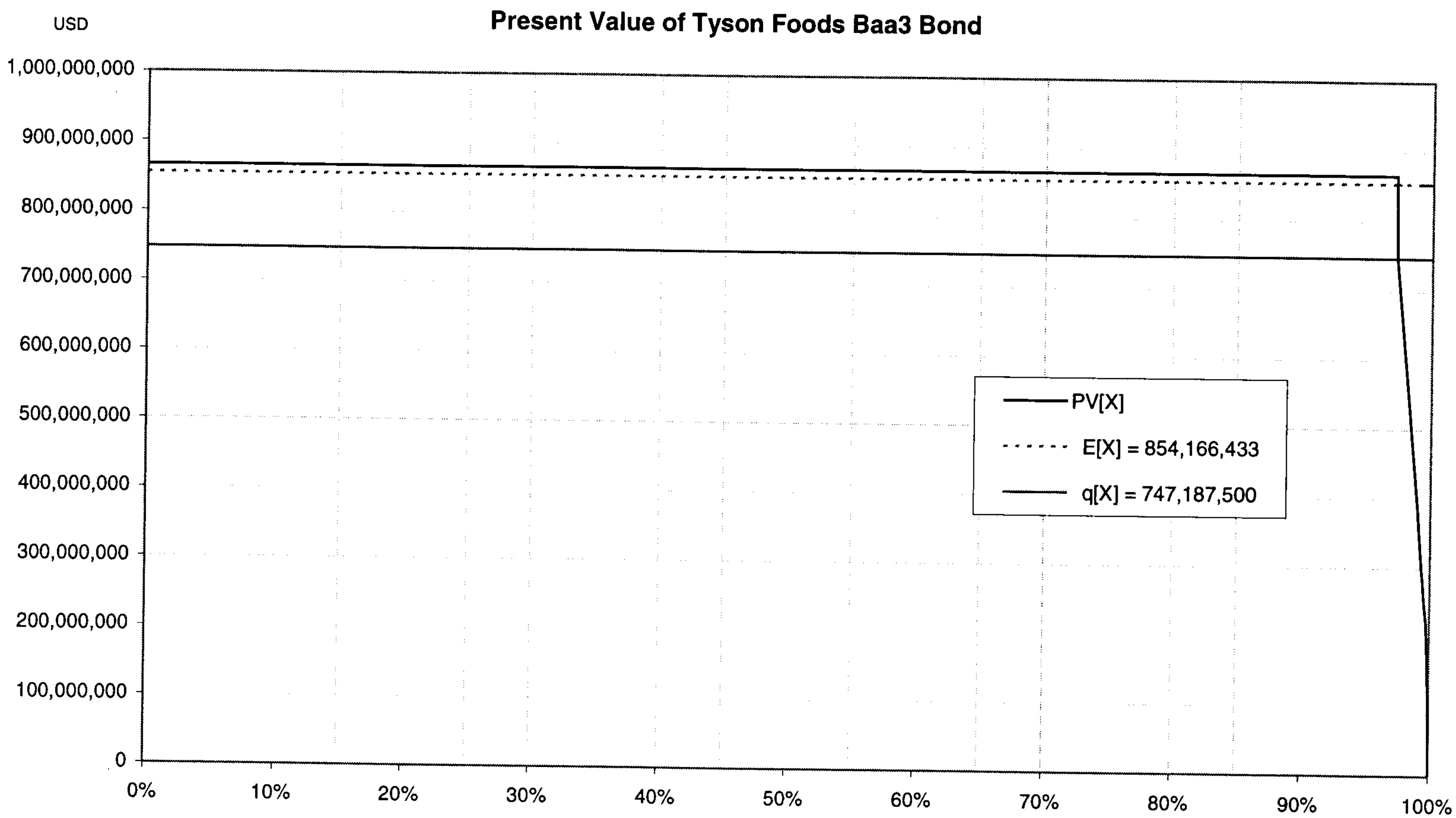
Simulated Default

Default Probability	Actual	Principal	Interest
0.00%	1		0
0.20%	1		27,187,500
0.31%	1		27,187,500
0.48%	1		27,187,500
0.75%	1		27,187,500
0.98%	1		27,187,500
1.28%	0		0
1.68%	0		0
2.21%	0		0
2.48%	0		0
2.79%	0	300,000,000	0
1.00%		247,218,750	130,372,559

@ 01 October 2001

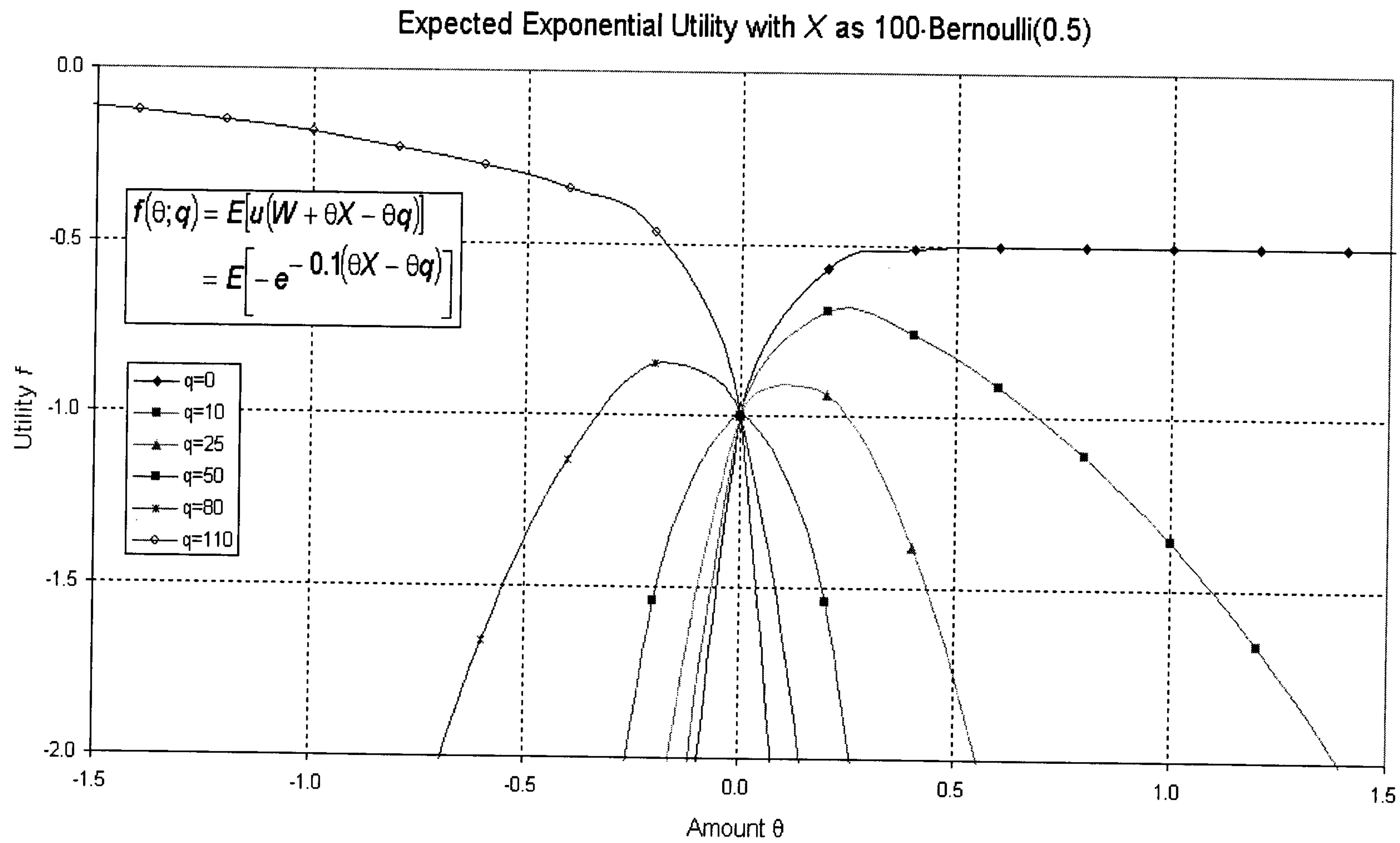
STRIP Prices	$v(t)$
100 00/32	1.000
98 27/32	0.988
97 22/32	0.977
96 02/32	0.960
94 14/32	0.944
92 18/32	0.925
90 12/32	0.904
88 06/32	0.882
86 02/32	0.860
84 07/32	0.842
82 13/32	0.824

Exhibit 4



Present Value of Tyson Foods Baa3 Bond
USD
1,000,000,000
900,000,000
800,000,000
700,000,000
600,000,000
500,000,000
400,000,000
300,000,000
200,000,000
100,000,000
0
0%
10%
20%
30%
40%
50%
60%
70%
80%
90%
100%
PV[X]
E[X] = 854,166,433
q[X] = 747,187,500

Exhibit 5



Expected Exponential Utility with X as 100·Bernoulli(0.5)
$f(\theta;q) = E[u(W + \theta X - \theta q)]$
$= E\left[-e^{-0.1(\theta X - \theta q)}\right]$
q=0
q=10
q=25
q=50
q=80
q=110
Utility f
Amount θ
0.0
-0.5
-1.0
-1.5
-2.0
-1.5
-1.0
-0.5
0.0
0.5
1.0
1.5

Exhibit 6

Three Identical Agents

Agent	Price q	θ	E[u'( )(X-q)]
Abel	45,843	33.33%	1.931E-03
Baker	45,843	33.33%	9.495E-03
Charlie	45,843	33.33%	3.555E-03
Total	45,843	100.00%	5.959E-03

Agent	a	$u'(x)$
Abel	5.000E-06	exp(-5.000E-06 * x)
Baker	5.000E-06	exp(-5.000E-06 * x)
Charlie	5.000E-06	exp(-5.000E-06 * x)

Abel	Probability	W	X	X-q	W+θX-θq	u'(W+θX-θq)	u'( )(X-q)
State 1	50%	100,000	0	-45,843	84,719	6.547E-01	-3.001E+04
State 2	50%	100,000	100,000	54,157	118,052	5.542E-01	3.001E+04
State 3							
State 4							
State 5							
State 6							
State 7							
State 8							
State 9							
State 10							
Mean	100%	100,000	50,000	4,157	101,386		1.931E-03

Baker	Probability	W	X	X-q	W+θX-θq	u'(W+θX-θq)	u'( )(X-q)
State 1	50%	100,000	0	-45,843	84,719	6.547E-01	-3.001E+04
State 2	50%	100,000	100,000	54,157	118,052	5.542E-01	3.001E+04
State 3							
State 4							
State 5							
State 6							
State 7							
State 8							
State 9							
State 10							
Mean	100%	100,000	50,000	4,157	101,386		9.495E-03

Charlie	Probability	W	X	X-q	W+θX-θq	u'(W+θX-θq)	u'( )(X-q)
State 1	50%	100,000	0	-45,843	84,719	6.547E-01	-3.001E+04
State 2	50%	100,000	100,000	54,157	118,052	5.542E-01	3.001E+04
State 3							
State 4							
State 5							
State 6							
State 7							
State 8							
State 9							
State 10							
Mean	100%	100,000	50,000	4,157	101,386		3.555E-03

Exhibit 7

One Agent Less Risk-Averse

Agent	Price *q*	θ	E[*u*'( )(*X*-*q*)]
Abel	46,879	50.00%	2.005E-03
Baker	46,879	25.00%	1.425E-02
Charlie	46,879	25.00%	1.500E-02
Total	46,879	100.00%	1.200E-02

Agent	*a*	*u*'(*x*)
Abel	2.500E-06	exp(-2.500E-06 * x)
Baker	5.000E-06	exp(-5.000E-06 * x)
Charlie	5.000E-06	exp(-5.000E-06 * x)

Abel	Probability	*W*	*X*	*X*-*q*	*W*+θ*X*-θ*q*	*u*'(*W*+θ*X*-θ*q*)	*u*'( )(*X*-*q*)
State 1	50%	100,000	0	-46,879	76,560	8.258E-01	-3.871E+04
State 2	50%	100,000	100,000	53,121	126,561	7.288E-01	3.871E+04
State 3							
State 4							
State 5							
State 6							
State 7							
State 8							
State 9							
State 10							
Mean	100%	100,000	50,000	3,121	101,560		2.005E-03

Baker	Probability	*W*	*X*	*X*-*q*	*W*+θ*X*-θ*q*	*u*'(*W*+θ*X*-θ*q*)	*u*'( )(*X*-*q*)
State 1	50%	100,000	0	-46,879	88,280	6.431E-01	-3.015E+04
State 2	50%	100,000	100,000	53,121	113,280	5.676E-01	3.015E+04
State 3							
State 4							
State 5							
State 6							
State 7							
State 8							
State 9							
State 10							
Mean	100%	100,000	50,000	3,121	100,780		1.425E-02

Charlie	Probability	*W*	*X*	*X*-*q*	*W*+θ*X*-θ*q*	*u*'(*W*+θ*X*-θ*q*)	*u*'( )(*X*-*q*)
State 1	50%	100,000	0	-46,879	88,280	6.431E-01	-3.015E+04
State 2	50%	100,000	100,000	53,121	113,280	5.676E-01	3.015E+04
State 3							
State 4							
State 5							
State 6							
State 7							
State 8							
State 9							
State 10							
Mean	100%	100,000	50,000	3,121	100,780		1.500E-02

Exhibit 8

Agents of Differing Opinions

Agent	Price q	θ	$E[u'(\,)(X-q)]$
Abel	39,353	8.09%	1.431E-02
Baker	39,353	5.41%	3.093E-03
Charlie	39,353	86.50%	6.819E-03
Total	39,353	100.00%	9.327E-03

Agent	a	$u'(x)$
Abel	5.000E-06	exp(-5.000E-06 * x)
Baker	5.000E-06	exp(-5.000E-06 * x)
Charlie	5.000E-06	exp(-5.000E-06 * x)

Abel	Probability	W	X	$X-q$	$W+\theta X-\theta q$	$u'(W+\theta X-\theta q)$	$u'(\,)(X-q)$
State 1	50%	100,000	0	-39,353	96,816	6.163E-01	-2.425E+04
State 2	50%	100,000	80,000	40,647	103,289	5.966E-01	2.425E+04
State 3							
State 4							
State 5							
State 6							
State 7							
State 8							
State 9							
State 10							
Mean	100%	100,000	40,000	647	100,052		1.431E-02

Baker	Probability	W	X	$X-q$	$W+\theta X-\theta q$	$u'(W+\theta X-\theta q)$	$u'(\,)(X-q)$
State 1	60%	100,000	0	-39,353	97,872	6.130E-01	-2.412E+04
State 2	40%	100,000	100,000	60,647	103,280	5.967E-01	3.619E+04
State 3							
State 4							
State 5							
State 6							
State 7							
State 8							
State 9							
State 10							
Mean	100%	100,000	40,000	647	100,035		3.093E-03

Charlie	Probability	W	X	$X-q$	$W+\theta X-\theta q$	$u'(W+\theta X-\theta q)$	$u'(\,)(X-q)$
State 1	50%	100,000	0	-39,353	65,959	7.191E-01	-2.830E+04
State 2	50%	100,000	100,000	60,647	152,461	4.666E-01	2.830E+04
State 3							
State 4							
State 5							
State 6							
State 7							
State 8							
State 9							
State 10							
Mean	100%	100,000	50,000	10,647	109,210		6.819E-03

Exhibit 9

Agents as a Secondary Market

Agent	Price q	θ	$E[u'(\,)(X-q)]$
Abel	45,843	33.33%	1.436E-02
Baker	45,843	33.33%	8.972E-03
Charlie	45,843	-66.67%	5.156E-03
Total	45,843	0.00%	1.022E-02

Comment
Charlie owns X

Agent	a	$u'(x)$
Abel	5.000E-06	exp(-5.000E-06 * x)
Baker	5.000E-06	exp(-5.000E-06 * x)
Charlie	5.000E-06	exp(-5.000E-06 * x)

Abel	Probability	W	X	$X-q$	$W+\theta X-\theta q$	$u'(W+\theta X-\theta q)$	$u'(\,)(X-q)$
State 1	50%	100,000	0	-45,843	84,719	6.547E-01	-3.001E+04
State 2	50%	100,000	100,000	54,157	118,052	5.542E-01	3.001E+04
State 3							
State 4							
State 5							
State 6							
State 7							
State 8							
State 9							
State 10							
Mean	100%	100,000	50,000	4,157	101,386		1.436E-02

Baker	Probability	W	X	$X-q$	$W+\theta X-\theta q$	$u'(W+\theta X-\theta q)$	$u'(\,)(X-q)$
State 1	50%	100,000	0	-45,843	84,719	6.547E-01	-3.001E+04
State 2	50%	100,000	100,000	54,157	118,052	5.542E-01	3.001E+04
State 3							
State 4							
State 5							
State 6							
State 7							
State 8							
State 9							
State 10							
Mean	100%	100,000	50,000	4,157	101,386		8.972E-03

Charlie	Probability	W	X	$X-q$	$W+\theta X-\theta q$	$u'(W+\theta X-\theta q)$	$u'(\,)(X-q)$
State 1	50%	100,000	0	-45,843	130,562	5.206E-01	-2.386E+04
State 2	50%	200,000	100,000	54,157	163,895	4.407E-01	2.386E+04
State 3							
State 4							
State 5							
State 6							
State 7							
State 8							
State 9							
State 10							
Mean	100%	150,000	50,000	4,157	147,229		5.156E-03

Exhibit 10

Reinsurance Pricing

Agent	Premium p	θ	E[u'( )(-L+p)]
Reinsurer A	5,106,791	68.29%	4.276E-03
Reinsurer B	5,106,791	31.71%	-5.540E-03
Total	5,106,791	100.00%	4.948E-03

Agent	a	$u'(x)$
Reinsurer A	5.000E-09	exp(-5.000E-09 * x)
Reinsurer B	5.000E-09	exp(-5.000E-09 * x)

Reinsurer A	Probability	W	L	$-L+p$	$W-\theta L+\theta p$	$u'(W-\theta L+\theta p)$	$u'(\,)(-L+p)$
State 1	94.0%	1,000,000,000	0	5,106,791	1,003,487,660	6.621E-03	3.381E+04
State 2	4.0%	1,000,000,000	50,000,000	-44,893,209	969,340,385	7.854E-03	-3.526E+05
State 3	2.0%	1,000,000,000	100,000,000	-94,893,209	935,193,109	9.317E-03	-8.841E+05
State 4							
State 5							
State 6							
State 7							
State 8							
State 9							
State 10							
Mean	100%	1,000,000,000	4,000,000	1,106,791	1,000,755,878		4.276E-03

Reinsurer B	Probability	W	L	$-L+p$	$W-\theta L+\theta p$	$u'(W-\theta L+\theta p)$	$u'(\,)(-L+p)$
State 1	23.5%	1,100,000,000	0	5,106,791	1,101,619,131	4.054E-03	2.070E+04
State 2	1.0%	1,100,000,000	25,000,000	-19,893,209	1,093,692,769	4.218E-03	-8.390E+04
State 3	0.5%	1,100,000,000	50,000,000	-44,893,209	1,085,766,407	4.388E-03	-1.970E+05
State 4	47.0%	1,000,000,000	0	5,106,791	1,001,619,131	6.684E-03	3.413E+04
State 5	2.0%	1,000,000,000	50,000,000	-44,893,209	985,766,407	7.235E-03	-3.248E+05
State 6	1.0%	1,000,000,000	100,000,000	-94,893,209	969,913,682	7.832E-03	-7.432E+05
State 7	23.5%	900,000,000	0	5,106,791	901,619,131	1.102E-02	5.627E+04
State 8	1.0%	900,000,000	100,000,000	-94,893,209	869,913,682	1.291E-02	-1.225E+06
State 9	0.5%	900,000,000	100,000,000	-94,893,209	869,913,682	1.291E-02	-1.225E+06
State 10							
Mean	100%	1,000,000,000	4,000,000	1,106,791	1,000,350,913		-5.540E-03

	Probability	W	L	$-L+p$	$W-\theta L+\theta p$	$u'(W-\theta L+\theta p)$	$u'(\,)(-L+p)$
State 1							
State 2							
State 3							
State 4							
State 5							
State 6							
State 7							
State 8							
State 9							
State 10							
Mean	0%	0	0	0	0		0.000E+00

Exhibit 11

Reinsurance Pricing with a Short Position

Agent	Premium p	θ	$E[u'(\,)(-L+p)]$
Reinsurer A	6,077,896	116.45%	-1.891E-02
Reinsurer B	6,077,896	74.45%	6.643E-04
Reinsurer C	6,077,896	-90.91%	-9.633E-03
Total	6,077,896	100.00%	1.226E-02

Agent	a	$u'(x)$
Reinsurer A	5.000E-09	exp(-5.000E-09 * x)
Reinsurer B	5.000E-09	exp(-5.000E-09 * x)
Reinsurer C	5.000E-09	exp(-5.000E-09 * x)

Reinsurer A	Probability	W	L	$-L+p$	$W-\theta L+\theta p$	$u'(W-\theta L+\theta p)$	$u'(\,)(-L+p)$
State 1	94.0%	1,000,000,000	0	6,077,896	1,007,077,996	6.504E-03	3.953E+04
State 2	4.0%	1,000,000,000	50,000,000	-43,922,104	948,850,639	8.702E-03	-3.822E+05
State 3	2.0%	1,000,000,000	100,000,000	-93,922,104	890,623,282	1.164E-02	-1.093E+06
State 4							
State 5							
State 6							
State 7							
State 8							
State 9							
State 10							
Mean	100%	1,000,000,000	4,000,000	2,077,896	1,002,419,808		-1.891E-02

Reinsurer B	Probability	W	L	$-L+p$	$W-\theta L+\theta p$	$u'(W-\theta L+\theta p)$	$u'(\,)(-L+p)$
State 1	23.5%	1,100,000,000	0	6,077,896	1,104,525,231	3.995E-03	2.428E+04
State 2	1.0%	1,100,000,000	25,000,000	-18,922,104	1,085,911,753	4.385E-03	-8.297E+04
State 3	0.5%	1,100,000,000	50,000,000	-43,922,104	1,067,298,274	4.813E-03	-2.114E+05
State 4	47.0%	1,000,000,000	0	6,077,896	1,004,525,231	6.587E-03	4.004E+04
State 5	2.0%	1,000,000,000	50,000,000	-43,922,104	967,298,274	7.935E-03	-3.485E+05
State 6	1.0%	1,000,000,000	100,000,000	-93,922,104	930,071,318	9.558E-03	-8.977E+05
State 7	23.5%	900,000,000	0	6,077,896	904,525,231	1.086E-02	6.601E+04
State 8	1.0%	900,000,000	100,000,000	-93,922,104	830,071,318	1.576E-02	-1.480E+06
State 9	0.5%	900,000,000	100,000,000	-93,922,104	830,071,318	1.576E-02	-1.480E+06
State 10							
Mean	100%	1,000,000,000	4,000,000	2,077,896	1,001,547,075		6.643E-04

Reinsurer C	Probability	W	L	$-L+p$	$W-\theta L+\theta p$	$u'(W-\theta L+\theta p)$	$u'(\,)(-L+p)$
State 1	87.0%	1,000,000,000	0	6,077,896	994,474,668	6.927E-03	4.210E+04
State 2	10.0%	1,000,000,000	50,000,000	-43,922,104	1,039,928,982	5.519E-03	-2.424E+05
State 3	3.0%	1,000,000,000	100,000,000	-93,922,104	1,085,383,296	4.397E-03	-4.129E+05
State 4							
State 5							
State 6							
State 7							
State 8							
State 9							
State 10							
Mean	100%	1,000,000,000	8,000,000	-1,922,104	1,001,747,358		-9.633E-03

APPENDIX A

The Imposture of Risk-Adjusted Discounting

One widely used and highly respected textbook on finance opens with the definitive and programmatic statement:

> To calculate present value, we discount expected payoffs by the rate of return offered by equivalent investment alternatives in the capital market. The rate of return is often referred to as the discount rate, hurdle rate, or opportunity cost of capital. [Brealey and Myers, 2002, p. 15]

That the payoffs (here assumed to be positive, or at least nonnegative) may be stochastic is unimportant. As long as one knows which investments are equivalent to the one in question, and at what rate of return these equivalent investments are discounted, one needs only to discount the "expected payoffs." On the same page the authors repeat this statement, explicitly mentioning risky investments:

> Here we can invoke a second basic financial principle: *A safe dollar is worth more than a risky one*. Most investors avoid risk when they can do so without sacrificing return. However, the concepts of present value and the opportunity cost of capital still make sense for risky investments. It is still proper to discount the payoff by the rate of return offered by an equivalent investment. But we have to think of *expected* payoffs and *expected* rates of return on other investments.

One can hardly disagree with the second basic principle; if the present values of the expected cash flows of a riskless investment and a risky investment are equal, the risky investment should be worth less. But the authors see no way to make the risky investment worth less other than to discount it at a greater rate of return, a return equal to that "offered by an equivalent investment." One should a least wonder how to tell which investments are equivalent.[21] And if there are equivalents, is there one that is

[21] Why do the authors even suggest that not all investments are equivalent when they intend to argue that the Capital Asset Pricing Model renders them all β-equivalent?

standard and priced by itself? If there is no standard, how do we avoid circular reasoning, viz., that A should be discounted at 15% because of B and that B should be discounted at 15% because of A? But even apart from the matter of circular reasoning, we can here show that discounting expected payoffs at risk-adjusted rates of return leads to inconsistencies. This will help us to see that conceiving present value as a random variable is necessary for consistently valuing stochastic cash flows.

Consider again the simplest stochastic cash flow of Section 3, the receipt of either 120 or 80 dollars one year from now, depending on a coin toss. The source of this example [D'Arcy, 1999, p. 23] assumed a risk-free discount rate of seven percent per year, and a risk-adjusted discount rate of twelve. The expected flow of 100 dollars would be discounted at twelve percent per year to yield a present value of 89.29 dollars. Though this seems reasonable, inconsistencies appear when one generalizes the problem to time *t*. The general present-value formula is $\$100/(1.12)^t$; but for $t > 4.89$ years this will be less than the present value of the tails-dependent cash flow $\$80/(1.07)^t$. Wishing to avoid this inconsistency, many would resort to making the risk-adjusted discount rate to vary with *t*, i.e., $\$100/(1+r(t))^t$ However, as *t* increases, the discount rate must decrease, approaching the risk-free lower bound of seven percent per year. And as *t* approaches zero, $\$100/(1+r(t))^t$ approaches \$100 (unless $r(t)$ approaches infinity), and the expected profit vanishes. Moreover, if the discount rate had to vary with time, equivalent investment alternatives would be fewer and harder to identify.

Finance textbooks assume that the cash flows to be discounted are nonnegative.

Normally this is true of investments, especially of stocks and bonds. But stochastic cash flows are more comprehensive. We could think of insured losses as negative payoffs, and the coin-toss flow would be the receipt of either –120 or –80 dollars one year from now. An insurer would like to "pay" less than the present value of –100 dollars, and this would require discounting at a rate less than the risk-free seven percent per year.[22] But as with the example with positive amounts, one would have to take care not to pierce the envelope, especially its lower bound of $-\$120/(1.07)^t$; and the problem of discounting too much for large *t* and too little for small *t* persists.

Inconsistency appears in yet another form. What is to prevent us from decomposing the \$120/\$80 cash flow into a degenerate cash flow of *x* dollars one year from now and a stochastic cash flow of either \$120–*x* or \$80–*x* one year from now, depending on the coin toss? It is hard to imagine why the "\$120–*x* or \$80–*x*" flow would not be equivalent to the same investments as those to which the "\$120 or \$80" flow is equivalent. Then it too would be discounted at twelve percent per year. But the *x* dollars are now discounted at seven percent per year, and the present value depends on *x*:

$$PV = x/(1.07)^t + (\$100 - x)/(1 + r(t))^t$$

Should a change in the level of the stochastic cash flow place the asset into a different class of equivalents, and justify a risk-adjusted rate that now depends not just on *t*, but also on *x*, i.e., $r(t,x)$? Especially troublesome is the case of $x = \$100$, the expected

[22] So according to Robert Bustic [1988, p. 149]: "The risk-adjusted interest rate [for loss reserves], being lower than yield rates available in the market, falls between the two extremes of not discounting (a zero interest rate) and discounting with a market rate." On p. 169 he estimates the risk-adjusted rate to be about four percent per year lower than the risk-free rate. That U.S. Treasury rates might become so low that one would discount loss reserves at a negative rate was unimaginable to him at the time.

payoff, in which case $PV = \$100/(1.07)^t$. This clearly violates the second principle of Brealey and Myers, for risky dollars are now worth as much as safe ones.

The only answer to this form of inconsistency is to forbid decomposing cash flows, or more accurately, to insist that each stochastic cash flow has one and only one suitable decomposition. This actually is the genesis of capital allocation. If stochastic cash flow C has negative outcomes, one allocates enough capital k to offset its negative outcomes, or at least to render them insignificant. Then one discounts the combination of the degenerate cash flow $-k$ and (the expected payoff of) the stochastic cash flow $k+C$.[23] But which degenerate cash flow k is the suitable one? In particular, if C has no lower bound, how much capital is required to render the remaining negative outcomes insignificant? Only to true believers are these inconsistencies trivial. The rest, not knowing a better way, may acquiesce to risk-adjusted discounting and capital allocation; but they suspect that such practices are arbitrary.

On two principles the foregoing argument has been based. The first principle is that the value of a stochastic cash flow must lie within the minimum and maximum values of its outcome-dependent cash flows. In other words, if random variable X is the present value of a stochastic cash flow, as defined in Section 3, and if for some real number a $P(X < a) = 0$, then the value of the stochastic cash flow must be greater than or equal to a. Similarly, if for some b $P(X > b) = 0$, the value must be less than or equal to b. In the

[23] Some practitioners discount the expected flows separately, C at the risk-free rate and the allocated capital at some ROE rate. This only underscores the absence of theory from risk-adjusted discounting.

extreme, if random variable X is degenerate, i.e., for some a $P(X = a) = 1$, the value of the stochastic cash flow must be a. For example, if all ten outcomes of Exhibit 1 lay on the \$100 isobar, the cash flow would have to be worth 100 dollars.

Some disagree with this inference, arguing that the several points extending over the isobar create a timing risk that no single point has. However, the risk that affects value resides not in the timing of the outcomes, but rather in their present value. And the present value is always 100 dollars. Others argue that if the present-value operator, or the coordinate system, changes, the ten outcomes will lie on different isobars. Though this is true, it ignores what was said in Section 2, viz., that the present-value operator is momentary and does not second-guess itself. Moreover, if this argument were valid, it would prove too much. For it would work even if the cash flow consisted of just one point on the \$100 isobar, for instance, the point (5, \$117.69). One might just as well argue that due to "interest-rate risk" this one point ought to be worth somewhat less than 100 dollars.

Donald Mango has coined a phrase that poignantly addresses this issue:

> ... involves discounting cash flows at a default-free rate. Scenario analysis is built upon the premise that possible, realizable, plausible outcomes can be generated and analyzed. For the entire process to work, each generated scenario is "**conditionally certain**": given the scenario occurs, its outcome is certain. Where it is not, the entire practice of simulation modeling would be undermined by "meta-uncertainty."
>
> Uncertainty for the contract in total is represented in the distribution across all modeled scenarios, and the probability weights assigned to those scenarios. In other words, *uncertainty is reflected between the scenarios, not within them.* Given conditional certainty, scenario cash flows can be discounted at a default-free rate. [2003, p. 355f.]

Conditional certainty is a striking concept. For our basic space S, consisting of a countable number of elementary outcomes ξ_i, the Theorem of Total Probability states:

$$P(A) = \sum_i P(A|\xi_i)P(\xi_i)$$

We can analogize from the probability of event *A* to the value of stochastic cash flow *C*, using present value as the conditional operator:

$$V(C) = \sum_i PV(C|\xi_i)P(\xi_i)$$

But this would be the expectation of the present value. One must adjust the probabilities $P(\xi_i)$ to obtain a risk-adjusted value for *C*, a value that will lie between any minimum and maximum since it is a weighted average.[24]

The second principle is that to change the level of a stochastic cash flow is to change its value by the amount of the level. In symbols, $V(k+C) = k+V(C)$. Philosophically this means that no point on the real continuum is special, not even zero.[25] Negative one million dollars is merely one million dollars less than zero, and two million dollars less than one million. It may mean bankruptcy to an accountant,[26] but for valuation it is just another real number. The principle means also that we can decompose cash flows. In fact, we could generalize this principle into linearity.[27] For stochastic cash flows *C* and *D*, and constants α and β, $V[\alpha C + \beta D] = \alpha V[C] + \beta V[D]$. A non-linear valuation operator enables arbitrage; nevertheless, finance textbooks are devoted more to risk-adjusted

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[24] Adjusted probabilities reappear in the discussion of Section 5 concerning state prices.
[25] As mentioned in Section 2, this renders valuation independent of solvency.
[26] Some argue that negative wealth is not possible with bankruptcy laws. However, these are positive laws, and something *de jure humano* is not worthy of the name "theory." Moreover, bankruptcy law does not make everyone's wealth the greater of zero and book value. One whose book value is negative must file for bankruptcy, turn his assets over to a court for liquidation to his creditors, and possibly suffer long-term consequences (e.g., ineligibility for credit, imprisonment). Furthermore, creditors and society at large bear the debt that remains after liquidation. So bankruptcy laws recognize the reality of negative wealth.
[27] Paradoxically, linearity obtains only in an equilibrium (Section 5). Though the generalization is important, for the purpose of this appendix it is unnecessary.

discounting than to the avoidance of arbitrage. Risk-adjusted discounting violates these two principles, and additionally, cannot build realistic risk loads into short-duration risks.

Finally, we comment about the notions of rate of return and cost of capital. For centuries individuals and banks have lent money "for so many years at such an interest rate per year, payable so many times per year." So everyone became accustomed to measuring the cost of money in percent per year, and to imagining that borrowed money worked at that rate of return. In Section 2 we argued for the present-value function $v(t)$. Rather than say, for example, that the cost of dollars is five percent per year, we should say that the cost of one dollar t years from now is $v(t)$ dollars today. The cost of money later is not percent per year, but rather an amount of money now. But even if one insists on cost of capital as percent per year, one cannot apply it to stochastic cash flows. Money loaned out at five percent per year is money that is eventually repaid. But capital allocated to a risky project is capital that stands ready to be sacrificed. Even if it were proper to say, "My money works at five percent per year while it's away from me," it would not necessarily be proper to say, "My money works at fifteen percent per year while it's risking its life."[28]

[28] For similar opinions that a stochastic cash flow cannot be valued by adjusting the discount rate see Halliwell [2001, §3], Schnapp [2001], and Van Slyke [1995 and 1999]. Schnapp [2001, §8] writes, " ... the certainty equivalent price for future outcomes can be obtained by discounting the uncertain outcomes to present value and then determining the certainty equivalent price." So too Van Slyke: "The cost of uncertainty in the future is treated as a real cost, and is discounted at only the time value of money indicated by the currency markets." [1995, p. 617] and " ... a distinction between the time value of money and the cost of risk. The time value of money is recognized by replacing all outcomes that may be realized at future times with their equivalent values in current dollars." [1999, p. 140]

APPENDIX B

Properties of the Expected Utility Function

Here follows a proof of the claim in Section 5 that if the net cash flow, i.e., the cash flow minus its price, or $X - q$, has both upside and downside potential, then the graph of the equation $f(\theta) = E[u(W + \theta X - \theta q)]$ looks like a concave downward parabola (provided that u is a risk-averse utility function). Hence, $f(\theta)$ has one and only one maximum. This holds true regardless of the joint distribution of W and X.

The second derivative is $f''(\theta) = E\left[u''(W + \theta X - \theta q)(X - q)^2\right]$. As a risk-averse utility function, u'' is negative. Therefore, unless $\text{Prob}[X = q] = 1$ (contrary to the downside-upside assumption), f'' is everywhere negative. This establishes that f is concave downward, or equivalently, that f' strictly decreases. Hence, there can be at most one critical value, a value θ at which $f'(\theta) = 0$, at which value f is a maximum. But not established is that there must be at least one critical point. It will be proven that $\lim_{\theta \to \pm\infty} f'(\theta) = \mp\infty$. The continuity of f' will then guarantee a zero.

For the proof that $\lim_{\theta \to +\infty} f'(\theta) = -\infty$, one may assume that θ is positive. Since the stochastic cash flow has downside potential at price q, there exists some $q_0 < q$ such that $F_X(q_0) = \text{Prob}[X \le q_0] > 0$. And since it has upside potential, $1 - F_X(q) = \text{Prob}[X > q] > 0$. And the probability of the middle interval, $q_0 < X \le q$, is greater than or equal to zero. So the first step is to express $f'(\theta)$ in terms of conditional probability:

$$\begin{aligned} f'(\theta) &= E[u'(W+\theta X-\theta q)(X-q)] \\ &= E[u'(W+\theta X-\theta q)(X-q)|X \le q_0]F_X(q_0) \\ &+ E[u'(W+\theta X-\theta q)(X-q)|q_0 < X \le q](F_X(q)-F_X(q_0)) \\ &+ E[u'(W+\theta X-\theta q)(X-q)|q < X](1-F_X(q)) \end{aligned}$$

In the middle interval, $q_0 < X \le q$, $u'(W+\theta X-\theta q)(X-q) \le 0$. For u' is everywhere positive, and in that interval $X-q$ is less than or equal to zero; hence, the product is less than or equal to zero. And in the upper interval, $X-q>0$. Since θ is positive, $\theta X-\theta q>0$ and $W+\theta X-\theta q>W$. u' is positive, but strictly decreasing. Hence, $u'(W+\theta X-\theta q)<u'(W)$, and $u'(W+\theta X-\theta q)(X-q)<u'(W)(X-q)$. Therefore:

$$\begin{aligned} f'(\theta) &= E[u'(W+\theta X-\theta q)(X-q)|X \le q_0]F_X(q_0) \\ &+ E[u'(W+\theta X-\theta q)(X-q)|q_0 < X \le q](F_X(q)-F_X(q_0)) \\ &+ E[u'(W+\theta X-\theta q)(X-q)|q < X](1-F_X(q)) \\ &\le E[u'(W+\theta X-\theta q)(X-q)|X \le q_0]F_X(q_0) \\ &+ E[u'(W+\theta X-\theta q)(X-q)|q < X](1-F_X(q)) \\ &< E[u'(W+\theta X-\theta q)(X-q)|X \le q_0]F_X(q_0) \\ &+ E[u'(W)(X-q)|q < X](1-F_X(q)) \end{aligned}$$

The next step concerns the lower interval. Since here $X \le q_0$, $X-q \le q_0-q$. u' is positive, so the following are true:

$$\begin{aligned} u'(W+\theta X-\theta q)(X-q) &\le u'(W+\theta X-\theta q)(q_0-q) \\ E[u'(W+\theta X-\theta q)(X-q)|X \le q_0] &\le E[u'(W+\theta X-\theta q)|X \le q_0](q_0-q) \quad . \\ &= E[-u'(W+\theta X-\theta q)|X \le q_0](q-q_0) \end{aligned}$$

However, $-u'$ is an increasing, but concave downward function. It meets the conditions of Jensen's inequality (Section 4):

$$\begin{aligned} E[u'(W+\theta X-\theta q)(X-q)|X\le q_0] &\le E[-u'(W+\theta X-\theta q)|X\le q_0](q-q_0) \\ &\le -u'(E[(W+\theta X-\theta q)|X\le q_0])(q-q_0) \\ &= -u'(E[W|X\le q_0]+\theta E[X-q|X\le q_0])(q-q_0) \end{aligned}$$

So finally:

$$\begin{aligned} f'(\theta) &< E[u'(W+\theta X-\theta q)(X-q)|X\le q_0]F_X(q_0) \\ &\quad + E[u'(W)(X-q)|q<X](1-F_X(q)) \\ &< -u'(E[W|X\le q_0]+\theta E[X-q|X\le q_0])(q-q_0)F_X(q_0) \\ &\quad + E[u'(W)(X-q)|q<X](1-F_X(q)) \end{aligned}$$

θ appears only once on the right side of the inequality. Its coefficient, $E[X-q|X\le q_0]$, is negative. So as θ approaches ∞, the argument of $-u'$ approaches $-\infty$. Since $-u'$ is an increasing, concave downward function, it must approach $-\infty$. And its coefficient, $(q-q_0)F_X(q_0)$, is positive. Therefore, $\lim_{\theta\to+\infty} f'(\theta)=-\infty$.

Duality makes the proof of the opposite limit simple. Since X has both upside and downside potential at price q, $-X$ has both upside and downside potential at price $-q$. What is a limit as $\theta\to-\infty$ with cash flow X at price q can be expressed as a limit as $-\theta=\zeta\to+\infty$ with cash flow $-X$ at price $-q$:

$$\begin{aligned} \lim_{\theta\to-\infty} f'(\theta) &= \lim_{\theta\to-\infty} E[u'(W+\theta X-\theta q)(X-q)] \\ &= \lim_{\zeta\to+\infty} E[u'(W+(-\zeta)X-(-\zeta)q)(X-q)] \\ &= \lim_{\zeta\to+\infty} E[u'(W+\zeta(-X)-\zeta(-q))((-X)-(-q))]\cdot(-1) \\ &= -\infty\cdot(-1) \\ &= +\infty \end{aligned}$$

Since f' is continuous and strictly decreasing, but ranges over all the real numbers, it must have one and only one zero. So f has one and only one maximum.

APPENDIX C

Exponential Utility

Is there an ideal form of utility? Section 4 stated commonly accepted properties, viz., that $u(w)$ should be increasing, twice differentiable, and concave downward. One could argue that quadratic, power, and logarithmic functions have these properties. However, the quadratic function has a maximum utility at its vertex; utilities of greater wealth are undefined. And power and logarithmic functions are undefined for negative wealth. Arguments in Appendix A to the contrary, many appeal to bankruptcy law for putting a floor of zero under the wealth of an economic agent. But even here power and logarithmic functions fail, because utility distributions could have probability masses at $u(0) = -\infty$. The ideal utility function would be defined for all real numbers, and its first derivative would approach zero as wealth approached positive infinity. It is hard to devise any function with all these properties other than the exponential.

Hans Gerber [1979, p. 70] proposes five desirable properties for premium calculation. One of these properties is additivity, viz., that the premium for the combination of two independent risks should equal the sum of the premiums for the single risks. He demonstrates that "the principle of zero utility" has this property if and only if the utility function is linear or exponential. Gerber's principle is equivalent to our formulation of utility in Section 4. However, in Section 5 we recommended a principle of maximal utility, according to which an economic agent whose stochastic wealth W is in equilibrium should value stochastic cash flow X according to the formula:

$$q_X = \frac{E[u'(W)X]}{E[u'(W)\ ]}$$

Being linear in X, this formula is additive for any utility function. Moreover, the formula holds even when the risks are not independent, which exposes the weakness that Gerber's principle is not in general additive.

In the case of exponential utility the formula above becomes:

$$q_X = \frac{E\left[-e^{-aW}X\right]}{E\left[-e^{-aW}\ \right]} = \frac{E\left[e^{-aW}X\right]}{E\left[e^{-aW}\ \right]}$$

Hans Bühlmann [1980, p. 58] notes that when X and $W–X$ are independent:

$$\begin{aligned} q_X &= \frac{E\left[e^{-aW}X\right]}{E\left[e^{-aW}\ \right]} \\ &= \frac{E\left[e^{-aX}X\right]}{E\left[e^{-aX}\ \right]} \end{aligned}$$

The right side of the last equation is known as the Esscher transform of X. This form is especially attractive to insurance, since it allows "quantum" cash flows, i.e., cash flows independent of the rest of the portfolio, to be valued on their own. This suggests that the ideal utility function should allow quantum cash flows to be valued on their own; otherwise, one would have to know potentially everything in order to value anything. Unfortunately, we have not been able to prove that only exponential utility allows for this.

However, there is a related property. If X is independent of $W–X$, then for any constant k it is also independent of $k+W–X$. The principle is that the level of wealth should be irrelevant to equilibrium; in other words, if W is an equilibrium wealth, then so too should be $k+W$. This comports with the second principle of Appendix A, viz., that no point on the real continuum is special, not even zero. This will offend many "solvency-minded" persons, but even in everyday matters value is unrelated to one's financial condition. Asking a valuation formula to depend on k is like asking a shopkeeper to charge lower prices to the poor than to the rich. Therefore, we recommend that the valuations of an ideal utility function should be invariant to level; in symbols, for all k, W, and X:

$$\frac{E[u'(k+W)X]}{E[u'(k+W)\ \ ]}=\frac{E[u'(W)X]}{E[u'(W)\ \ ]}$$

As we are about to see, this recommendation makes demands on the form of u.

To begin, this equation cannot be true for all X unless:

$$\frac{u'(k+W)}{E[u'(k+W)]}=\frac{u'(W)}{E[u'(W)]}$$

And one can substitute tW for W, for arbitrary real number t. Hence, u must ensure for all k, t, and W the equality:

$$\frac{u'(k+tW)}{E[u'(k+tW)]}=\frac{u'(tW)}{E[u'(tW)]}$$

Now differentiate the equation with respect to t:

$$\frac{E[u'(k+tW)]u''(k+tW)W-E[u''(k+tW)W]u'(k+tW)}{E[u'(k+tW)]^2}=\frac{E[u'(tW)]u''(tW)W-E[u''(tW)W]u'(tW)}{E[u'(tW)]^2}$$

In particular, this equation is true for $t=0$:

$$\frac{E[u'(k)]u''(k)W-E[u''(k)W]u'(k)}{E[u'(k)]^2}=\frac{E[u'(0)]u''(0)W-E[u''(0)W]u'(0)}{E[u'(0)]^2}$$

And its simplified form is:

$$\frac{u''(k)(W-E[W])}{u'(k)}=\frac{u''(0)(W-E[W])}{u'(0)}$$

But this can not hold for all W unless for all k:

$$\frac{u''(k)}{u'(k)} = \frac{u''(0)}{u'(0)}$$

Only linear and exponential functions solve this equation, which expresses constant absolute risk aversion (Longley-Cook [1998, p. 90]). And by standardizing $u(0)$ and $u'(0)$ (Gerber [1979, p. 68] and Halliwell [1999, §6]) the linear function becomes the limit as a approaches zero of the exponential function $u(w) = \left(1 - e^{-aw}\right)/a$.

The ideal form of utility is (linear-)exponential. It is hard to devise other functions that possess the commonly accepted properties. Exponential utility allows for independent risks to be valued on their own, and perhaps it alone has this distinction. Finally, it alone renders valuation independent of the arbitrary level of an agent's stochastic wealth.

APPENDIX D

Cumulants and the Esscher Transform

In accordance with Appendix C, the formula for the value of a quantum cash flow (a cash flow that is independent from the rest of an agent's portfolio) can be transformed:

$$\begin{aligned} q_X &= \frac{E\left[Xe^{-aX}\right]}{E\left[\ e^{-aX}\right]} \\ &= -\frac{d}{da}\ln E\left[e^{-aX}\right] \\ &= -\frac{d}{da}\ln M_X(-a) \\ &= -\frac{d}{da}\psi_X(-a) \end{aligned}$$

The function $\psi_X(a)$, the logarithm of the moment generating function, is called the cumulant generating function (Daykin [1994, p. 23] and Halliwell [1999, Appendix C]), not to be confused with the state-price random variable Ψ of Section 5. Its derivatives evaluated at zero are called the cumulants of X; they are the κ_i of the Maclaurin-series expansion $\psi_X(t) = \sum_{i=1}^{\infty} \frac{\kappa_i t^i}{i!}$. The first three cumulants are the mean, the variance, and the skewness; the fourth cumulant is the kurtosis with an adjustment:

$$\begin{aligned} \kappa_1 &= E[X] = \mu \\ \kappa_2 &= E\left[(X-\mu)^2\right] = \sigma^2 \\ \kappa_3 &= E\left[(X-\mu)^3\right] \\ \kappa_4 &= E\left[(X-\mu)^4\right] - 3\sigma^4 \end{aligned}$$

All the cumulants of the normal distribution beyond the second are zero. The cumulants of a sum of independent random variables equal the sums of the variables' cumulants.

So the value of a quantum cash flow can be expressed in terms of its cumulants:

$$
\begin{aligned}
q_X &= -\frac{d}{da}\psi_X(-a) \\
&= -\frac{d}{da}\sum_{i=1}^{\infty}(-1)^i \frac{\kappa_i a^i}{i!} \\
&= \sum_{i=1}^{\infty}(-1)^{i-1}\frac{\kappa_i a^{i-1}}{(i-1)!} \\
&= \kappa_1 - \kappa_2 a + \frac{\kappa_3}{2}a^2 - \frac{\kappa_4}{6}a^3 + \ldots \\
&= \mu - \sigma^2 a + \frac{\kappa_3}{2}a^2 - \frac{\kappa_4}{6}a^3 + \ldots
\end{aligned}
$$

From this follows the formula for the expected profit:

$$
\begin{aligned}
\pi_X &= E[X] - q_X \\
&= \mu - q_X \\
&= \sigma^2 a - \frac{\kappa_3}{2}a^2 + \frac{\kappa_4}{6}a^3 - \ldots
\end{aligned}
$$

As a first-order approximation, $\pi_X \approx \sigma^2 a$, which holds exactly for normally distributed X.

One can generalize the cumulant generating function and its Maclaurin series to two random variables:

$$
\begin{aligned}
\psi_{X,W}(s,t) &= \ln E\left[e^{sX+tW}\right] \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\kappa_{ij}}{i!\,j!}s^i t^j
\end{aligned}
$$

The derivative of this function with respect to s, evaluated at $(0, -a)$, is the general formula for value:

$$\frac{\partial}{\partial s}\psi_{X,W}(s,t) = \frac{E\left[Xe^{sX+tW}\right]}{E\left[\ e^{sX+tW}\right]}$$
$$= \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\frac{\kappa_{ij}}{(i-1)!\,j!}s^{i-1}t^{j}$$

$$\left.\frac{\partial}{\partial s}\psi_{X,W}(s,t)\right|_{\substack{s=0\\t=-a}} = \frac{E\left[Xe^{-aW}\right]}{E\left[\ e^{-aW}\right]}$$
$$= \sum_{j=0}^{\infty}\frac{\kappa_{1j}}{j!}(-a)^{j}$$

Somewhat tediously, the formulae for the following cumulants can be derived:[29]

$$\begin{aligned}
\kappa_{10} &= E[X] = \mu_X \\
\kappa_{11} &= E[(X-\mu_X)(W-\mu_W)] = Cov[X,W] = \sigma_{X,W} \\
\kappa_{12} &= E\left[(X-\mu_X)(W-\mu_W)^2\right] \\
\kappa_{13} &= E\left[(X-\mu_X)(W-\mu_W)^3\right] - 3\sigma_{X,W}\sigma_W^2
\end{aligned}$$

By replacing X with W one obtains the cumulants of W in keeping with the earlier formulae. The formula for expected profit is:

$$\begin{aligned}
\pi_X &= E[X] - q_X \\
&= E[X] - \left(\kappa_{10} - \kappa_{11}a + \frac{\kappa_{12}}{2}a^2 - \frac{\kappa_{13}}{6}a^3 + \ldots\right) \\
&= E[X] - \left(E[X] - Cov[X,W]a + \frac{\kappa_{12}}{2}a^2 - \frac{\kappa_{13}}{6}a^3 + \ldots\right) \\
&= Cov[X,W]a - \frac{\kappa_{12}}{2}a^2 + \frac{\kappa_{13}}{6}a^3 - \ldots
\end{aligned}$$

Though beautiful, the formula is often impractical, not just because high-order cumulants are unfamiliar, but mainly because it may converge slowly. In the case of Reinsurer A of Exhibit 10, q_X is nowhere near the total premium of 5.1 million dollars after four terms,

---

[29] Kozik and Larson [2001, 58-63] derive similar formulae, but in terms of rates of return. Though much can be learned from their derivation, rate of return is an accounting notion that introduces an arbitrary time horizon (viz., one year). Our present-value perspective (Sections 2 and 3), which keeps to monetary units (dollars) and impounds all the future into the present moment, avoids the risk-adjusted discounting implicit in the rate-of-return approach (Appendix A).

i.e., after the κ_{13} term. Since the order of the magnitude of the j^{th} term of the series for q_X is $\kappa_{1_j} a^j \approx W^j a^j \approx (1{,}000{,}000{,}000 \times 5.000\text{E}-09)^j = 5^j$, the factorial in the denominator does not begin to dampen the series before the fifth term.

Nevertheless, the formula provides the general first-order approximation, $\pi_X \approx \sigma_{X,W} a$, which is exact when *X* and *W* are normally distributed. That expected profit, or risk load, is approximately proportional to covariance is true to the spirit of the Capital Asset Pricing Model. Since we can express the approximation with the correlation coefficient as $\pi_X \approx \rho_{X,W} \sigma_X \sigma_W a$, one may say that risk load is proportional to standard deviation (Kreps [1990, p. 198]). And if *X* is independent of *W*, the risk load should be zero. However, one must distinguish the independence of *X* from *W* and the independence of *X* from *W–X*. In the latter case,

$$
\begin{aligned}
Cov[X,W] &= Cov[X, X + W - X] \\
&= Cov[X,X] + Cov[X, W - X] \\
&= Var[X]
\end{aligned}
$$

Then the risk load is proportional to variance. From the premise that *X* should be small relative to *W* some[30] infer that the covariance and the risk load should be effectively zero. The premise itself is debatable; however, the proper conclusion is merely that π_X should be small relative to π_W. It certainly is not negligible relative to σ_X^2

[30] E.g., Wang [2001, §2]: "For an insurance market in which any individual risk is negligible relative to the aggregate risk, under the assumption that *X* and *Z–X* are independent, the equilibrium premium for *X* equals the expected loss without risk loading."