Grouping Loss Distributions by Tail Behavior Part II: Continuous Families

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Abstract: *This three part paper addresses the task of modelling the right hand tail of a severity distribution. In Part I the excess ratio function is used to define a discrete sequence of loss distributions with related moments and similar tail behavior. Part II extends this to continuous one-parameter families and provides some examples. Part III provides the main result: that under some reasonable conditions, each such family has a limiting distribution which is exponential. The paper then exploits this to 1) group loss distributions based on tail behavior and 2) promote the choice of (mixed) exponentials to model tail behavior.*

Remark 56 *This is Part II of a three part paper. We assume familiarity with Part I and continue our numbering from Part I.*

4 Continuous Families of Distributions

While we introduced taking the coderived distribution as a discrete process, we use Proposition 46 to generalize our definitions:

Definition 57 *For any SLDFn F and positive $c \in \mathbb{R}$ with $\mu_F^{(c)} < \infty$, the* ***c-th coderived loss distribution function*** *of F is the LDFn G with survival function*

$$S_G(x) = \frac{\int_x^\infty (y-x)^c f_F(y)dy}{\mu_F^{(c)}}$$

which we denote as $G = \widetilde{F}^{[c]}$. For $c < 0$, the ***c-th coderived loss distribution function*** *of F is the LDFn G, if such exists, satisfying*

$$\widetilde{G}^{[-c]} = F.$$

The set $\widetilde{F}^{[\mathbb{R}]} = \left\{ \widetilde{F}^{[c]} | c \in \mathbb{R} \text{ such that } \mu_F^{(c)} < \infty \right\}$ is called the ***coderived orbit*** *of the loss distribution function F.*

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Remark 58 *It follows from this calculation or from Proposition 46 that this agrees with the earlier definition of coderived loss distribution when $c \in \mathbb{Z}$. Indeed, under either definition we trivially have, for any loss SLDFn F,*

$$\begin{aligned} \widetilde{F}^{[0]} &= F \\ \widetilde{F}^{[1]} &= \widetilde{F} \end{aligned}$$

and

$$\widetilde{\widetilde{F}^{[c]}} = \widetilde{F}^{[c+1]} \text{ for every } c \in \mathbb{R}.$$

For any SLDFn F and $c > 0$, the c-th coderived SLDFn $\widetilde{F}^{[c]}$ exists $\Leftrightarrow \mu_F^{(c)} < \infty$. This is consistent with the original construction $S_{\widetilde{F}}(x) = \frac{S_F(x)}{\mu_F}$. Consequently we chose to use the formulation

$$S_{\widetilde{F}^{[c]}}(x) = \frac{\int_x^\infty (y-x)^c f_F(y)dy}{\mu_F^{(c)}}$$

in the definition. For $c < 0$ it is sometimes useful to try the following formula

$$S_{\widetilde{F}^{[c]}}(x) = \lim_{M\to\infty} \frac{\int_x^M (y-x)^c f_F(y)dy}{\int_0^M y^c f_F(y)dy}.$$

For example in the case that F is a mixture of exponentials, $\mu_F^{(c)} < \infty$ only for $c > -1$, but $\widetilde{F}^{[c]}$ exists for every $c \in \mathbb{R}$ and in that special case the latter formula works for every $c \in \mathbb{R}$.

Proposition 59 *For any SLDFn F and positive constants $a, c \in \mathbb{R}$ with $\mu_F^{(c)} < \infty$:*

1. $S_{\widetilde{F}^{[c]}}(x) = \frac{\int_0^\infty z^c f_F(x+z)dz}{\mu_F^{(c)}} = \frac{c\int_x^\infty (y-x)^{c-1} S_F(y)dy}{\mu_F^{(c)}} = \frac{c\int_0^\infty z^{c-1} S_F(x+z)dz}{\mu_F^{(c)}}$
2. $f_{\widetilde{F}^{[c]}}(x) = \frac{c\int_x^\infty (y-x)^{c-1} f_F(y)dy}{\mu_F^{(c)}} = \frac{c\int_0^\infty z^{c-1} f_F(x+z)dz}{\mu_F^{(c)}}$
3. $\lambda_{\widetilde{F}^{[c]}}(x) = \frac{\int_x^\infty (y-x)^{c-1} f_F(y)dy}{\int_x^\infty (y-x)^{c-1} S_F(y)dy} = \frac{\int_0^\infty z^{c-1} f_F(x+z)dz}{\int_0^\infty z^{c-1} S_F(x+z)dz}$

Proof. The substitution $z \mapsto x - y$ will be used routinely to change the lower limit of integration between $y = x$ and $z = 0$. By Proposition 11 we have

$$\int_x^\infty (y-x)^c f_F(y)dy = c\int_x^\infty (y-x)^{c-1} S_F(y)dy$$

and the rest is straightforward calculation. For Item 1

$$\begin{aligned} S_{\widetilde{F}^{[c]}}(x) &= \frac{\int_x^\infty (y-x)^c f_F(y)dy}{\mu_F^{(c)}} \\ &= \frac{c\int_x^\infty (y-x)^{c-1} S_F(y)dy}{\mu_F^{(c)}}. \end{aligned}$$

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For Item 2 we differentiate under the integral

$$\begin{aligned} f_{\widetilde{F}^{[c]}}(x) &= -\frac{dS_{\widetilde{F}^{[c]}}}{dx}(x) \\ &= -\frac{d}{dx}\left(\frac{\int_x^\infty (y-x)^c f_F(y)dy}{\mu_F^{(c)}}\right) \\ &= -\left(\frac{\int_x^\infty \frac{d}{dx}\left((y-x)^c\right) f_F(y)dy}{\mu_F^{(c)}}\right) \\ &= \frac{c\int_x^\infty (y-x)^{c-1} f_F(y)dy}{\mu_F^{(c)}} \end{aligned}$$

and Item 3 follows from Items 1 and 2. ■

Letting **B** denote the beta function; we will make use of the following results from calculus

$$\begin{aligned} \int_a^b (x-a)^p (b-x)^q \, dx &= (b-a)^{p+q+1} \mathbf{B}(p+1,q+1) \text{ where } p>-1, q>-1 \text{ and } b>a \\ \Gamma(c)\Gamma(1-c) &= \frac{\pi}{\sin c\pi} \text{ where } 0<c<1. \end{aligned}$$

Proposition 60 *If F is a nonvanishing SLDFn and $c \in (0,1)$, then:*

$$\mu_F^{(c)}\mu_{\widetilde{F}^{[c]}}^{(-c)} = \frac{c\pi}{\sin c\pi} \text{ and } S_F(x) = \frac{\int_x^\infty (y-x)^{-c} f_{\widetilde{F}^{[c]}}(y)dy}{\mu_{\widetilde{F}^{[c]}}^{(-c)}}.$$

Proof. Let $G = \widetilde{F}^{[c]}$,so that $\widetilde{G}^{[-c]} = F$. We have

$$\frac{\int_x^\infty (y-x)^{-c} f_G(y)dy}{\mu_G^{(-c)}} = \frac{\int_x^\infty (y-x)^{-c} f_{\widetilde{F}^{[c]}}(y)dy}{\mu_G^{(-c)}}$$

$$\begin{aligned} &= \frac{\int_x^\infty (y-x)^{-c}\left(c\int_y^\infty (z-y)^{c-1} f_F(z)dz\right)dy}{\mu_F^{(c)}\mu_G^{(-c)}} \\ &= \frac{c}{\mu_F^{(c)}\mu_G^{(-c)}}\int_x^\infty\int_y^\infty (y-x)^{-c}(z-y)^{c-1} f_F(z)dzdy \\ &= \frac{c}{\mu_F^{(c)}\mu_G^{(-c)}}\int_x^\infty\int_x^z (y-x)^{-c}(z-y)^{c-1} f_F(z)dydz \end{aligned}$$

Letting **B** denote the beta function and noting that $c \in (0,1) \Rightarrow -c > -1$ and $c-1 > -1$

$$\frac{\int_x^\infty (y-x)^{-c} f_G(y)dy}{\mu_G^{(-c)}} = \frac{c}{\mu_F^{(c)}\mu_G^{(-c)}}\int_x^\infty f_F(z)\left(\int_x^z (y-x)^{-c}(z-y)^{c-1}\,dy\right)dz$$

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$$
\begin{aligned}
&= \frac{c}{\mu_F^{(c)}\mu_G^{(-c)}}\int_x^\infty f_F(z)\left((z-x)^{-c+(c-1)+1}\mathbf{B}(-c+1,(c-1)+1)\right)dz \\
&= \frac{c\mathbf{B}(1-c,c)}{\mu_F^{(c)}\mu_G^{(-c)}}\int_x^\infty f_F(z)\left((z-x)^0\right)dz \\
&= \frac{c\mathbf{B}(1-c,c)}{\mu_F^{(c)}\mu_G^{(-c)}}\int_x^\infty f_F(z)dz \\
&= \frac{c\frac{\pi}{\sin c\pi}}{\mu_F^{(c)}\mu_G^{(-c)}}\int_x^\infty f_F(z)dz \\
&= \frac{c\pi S_F(x)}{\mu_F^{(c)}\mu_G^{(-c)}\sin c\pi}
\end{aligned}
$$

and letting $x=0$ in the equality, it follows that

$$
\begin{aligned}
1 &= \frac{\int_0^\infty y^{-c}f_G(y)dy}{\mu_G^{(-c)}} = \frac{c\pi S_F(0)}{\mu_F^{(c)}\mu_G^{(-c)}\sin c\pi} = \frac{c\pi}{\mu_F^{(c)}\mu_G^{(-c)}\sin c\pi} \\
&\Rightarrow\ \mu_F^{(c)}\mu_{\widetilde{F}^{[c]}}^{(-c)} = \mu_F^{(c)}\mu_G^{(-c)} = \frac{c\pi}{\sin c\pi}
\end{aligned}
$$

and further that for every $x\in[0,\infty)$

$$
S_F(x) = S_{\widetilde{G}^{[-c]}}(x) = \frac{\int_x^\infty (y-x)^{-c} f_G(y)dy}{\mu_G^{(-c)}} = \frac{\int_x^\infty (y-x)^{-c} f_{\widetilde{F}^{[c]}}(y)dy}{\mu_{\widetilde{F}^{[c]}}^{(-c)}}
$$

as required. ■

The following result generalizes Proposition 44 and shows that with the exception of instances when the coderived distribution fails to exist, the additive group of reals acts on the set of SLDFns under this definition. This vindicates our use of the term "orbit" and gives credence to the view that this is the "correct" way to extend the definition of coderived variable from discrete to continuous.

Proposition 61 *For any SLDFn F and positive constants $c,d\in\mathbb{R}$ with $\mu_F^{(c+d)}<\infty$, letting $\mathbf{B}$ denote the beta function:*

1. $\mu_{\widetilde{F}^{[c]}}^{(d)} = \frac{(c+d+1)\mathbf{B}(d+1,c+1)\mu_F^{(c+d)}}{\mu_F^{(c)}}$
2. $\widetilde{\widetilde{F}^{[c]}}^{[d]} = \widetilde{F}^{[c+d]}$
3. $\mu_{\widetilde{F}^{[c]}} = \frac{\mu_F^{(c+1)}}{(c+1)\mu_F^{(c)}}$
4. $\left(CV_{\widetilde{F}^{[c]}}\right)^2 = \frac{2(c+1)\mu_F^{(c)}\mu_F^{(c+2)}}{(c+2)\left(\mu_F^{(c+1)}\right)^2} - 1$

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Proof. Note first that by definition

$$\mu_F^{(c+d)} S_{\widetilde{F}^{[c+d]}}(x) = \int_x^\infty (y-x)^{c+d} f_F(y)dy.$$

On the other hand, we have

$$\mu_{\widetilde{F}^{[c]}}^{(d)} S_{\widetilde{\widetilde{F}^{[c]}}^{[d]}}(x) = \int_x^\infty (y-x)^d f_{\widetilde{F}^{[c]}}(y)dy$$

$$\begin{aligned}
&= \int_x^\infty (y-x)^d \left(\frac{c\int_y^\infty (z-y)^{c-1} f_F(z)dz}{\mu_F^{(c)}} \right) dy \\
&= \frac{c}{\mu_F^{(c)}} \int_x^\infty \int_y^\infty (y-x)^d (z-y)^{c-1} f_F(z)dzdy \\
&= \frac{c}{\mu_F^{(c)}} \int_x^\infty \int_x^z (y-x)^d (z-y)^{c-1} f_F(z)dydz
\end{aligned}$$

$$\begin{aligned}
&= \frac{c}{\mu_F^{(c)}} \int_x^\infty f_F(z) \left(\int_x^z (y-x)^d (z-y)^{c-1} dy \right) dz \\
&= \frac{c}{\mu_F^{(c)}} \int_x^\infty f_F(z) \left((z-x)^{c+d} \mathbf{B}(d+1,c) \right) dz \\
&= \frac{c\mathbf{B}(d+1,c)}{\mu_F^{(c)}} \int_x^\infty (z-x)^{c+d} f_F(z)dz
\end{aligned}$$

$$\begin{aligned}
&= \frac{\Gamma(d+1)c\Gamma(c)}{\mu_F^{(c)}\Gamma(c+d+1)} \int_x^\infty (z-x)^{c+d} f_F(z)dz \\
&= \frac{\Gamma(d+1)\Gamma(c+1)}{\mu_F^{(c)}\Gamma(c+d+1)} \int_x^\infty (z-x)^{c+d} f_F(z)dz
\end{aligned}$$

$$\begin{aligned}
&= \frac{(c+d+1)\Gamma(d+1)\Gamma(c+1)}{\mu_F^{(c)}(c+d+1)\Gamma(c+d+1)} \int_x^\infty (z-x)^{c+d} f_F(z)dz \\
&= \frac{(c+d+1)\Gamma(d+1)\Gamma(c+1)}{\mu_F^{(c)}\Gamma(c+1+d+1)} \int_x^\infty (z-x)^{c+d} f_F(z)dz \\
&= \frac{(c+d+1)\mathbf{B}(d+1,c+1)}{\mu_F^{(c)}} \int_x^\infty (z-x)^{c+d} f_F(z)dz.
\end{aligned}$$

Letting $x = 0$ we have

$$\begin{aligned}
\mu_{\widetilde{F}^{[c]}}^{(d)} &= \mu_{\widetilde{F}^{[c]}}^{(d)} S_{\widetilde{\widetilde{F}^{[c]}}^{[d]}}(0) = \frac{(c+d+1)\mathbf{B}(d+1,c+1)}{\mu_F^{(c)}} \int_0^\infty z^{c+d} f_F(z)dz \\
&= \frac{(c+d+1)\mathbf{B}(d+1,c+1)}{\mu_F^{(c)}} \mu_F^{(c+d)}
\end{aligned}$$

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which proves Item1. For Item 2 the above equations imply

$$\begin{aligned}\mu^{(d)}_{\widetilde{F}^{[c]}} S_{\widetilde{\widetilde{F}^{[c]}}^{[d]}}(x) &= \frac{(c+d+1)\mathbf{B}(d+1,c+1)}{\mu_F^{(c)}} \int_x^\infty (z-x)^{c+d} f_F(z)dz \\ &= \frac{(c+d+1)\mathbf{B}(d+1,c+1)}{\mu_F^{(c)}} \left(\mu_F^{(c+d)} S_{\widetilde{F}^{[c+d]}}(x)\right)\end{aligned}$$

which by Item 1 gives

$$\begin{aligned}S_{\widetilde{\widetilde{F}^{[c]}}^{[d]}}(x) &= \frac{(c+d+1)\mathbf{B}(d+1,c+1)\mu_F^{(c+d)} S_{\widetilde{F}^{[c+d]}}(x)}{\mu^{(d)}_{\widetilde{F}^{[c]}} \mu_F^{(c)}} = S_{\widetilde{F}^{[c+d]}}(x) \\ \Rightarrow \quad \widetilde{\widetilde{F}^{[c]}}^{[d]} &= \widetilde{F}^{[c+d]}.\end{aligned}$$

And since

$$\begin{aligned}\mathbf{B}(2,c+1) &= \frac{\Gamma(2)\Gamma(c+1)}{\Gamma(c+3)} = \frac{\Gamma(c+1)}{(c+2)\Gamma(c+2)} \\ &= \frac{\Gamma(c+1)}{(c+2)(c+1)\Gamma(c+1)} = \frac{1}{(c+2)(c+1)}\end{aligned}$$

we see that Item 3 is just the case $d = 1$ of Item 1

$$\begin{aligned}\mu_{\widetilde{F}^{[c]}} &= \mu^{(1)}_{\widetilde{F}^{[c]}} = \frac{(c+2)\mathbf{B}(2,c+1)\mu_F^{(c+1)}}{\mu_F^{(c)}} \\ &= \frac{(c+2)\mu_F^{(c+1)}}{(c+2)(c+1)\mu_F^{(c)}} = \frac{\mu_F^{(c+1)}}{(c+1)\mu_F^{(c)}}.\end{aligned}$$

Finally, we have by Proposition 46 and Part 3

$$\begin{aligned}\left(CV_{\widetilde{F}^{[c]}}\right)^2 &= 2\left(\frac{\mu_{\widetilde{\widetilde{F}^{[c]}}}}{\mu_{\widetilde{F}^{[c]}}}\right) - 1 = 2\left(\frac{\mu_{\widetilde{F}^{[c+1]}}}{\mu_{\widetilde{F}^{[c]}}}\right) - 1 \\ &= 2\left(\frac{\frac{\mu_F^{(c+2)}}{(c+2)\mu_F^{(c+1)}}}{\frac{\mu_F^{(c+1)}}{(c+1)\mu_F^{(c)}}}\right) - 1 \\ &= \frac{2(c+1)\mu_F^{(c)}\mu_F^{(c+2)}}{(c+2)\left(\mu_F^{(c+1)}\right)^2} - 1\end{aligned}$$

and the proof is complete. ■

Proposition 62 *For any non-vanishing loss SLDFn F and positive constant c with $\mu_F^{(c)} < \infty$:*

$$\tau_F = \tau_{\widetilde{F}^{[c]}}.$$

Proof. We have from l'Hôpital

$$\lim_{x\to\infty}\frac{S_F(x)}{S_{\widetilde{F}^{[c]}}(x)} = \lim_{x\to\infty}\frac{-f_F(x)}{-f_{\widetilde{F}^{[c]}}(x)} = \lim_{x\to\infty}\frac{f_F(x)}{f_{\widetilde{F}^{[c]}}(x)}$$

whence

$$\begin{aligned} 1 &= \left(\lim_{x\to\infty}\frac{S_F(x)}{S_{\widetilde{F}^{[c]}}(x)}\right)\left(\lim_{x\to\infty}\frac{f_F(x)}{f_{\widetilde{F}^{[c]}}(x)}\right)^{-1} \\ &= \left(\lim_{x\to\infty}\frac{S_F(x)}{S_{\widetilde{F}^{[c]}}(x)}\right)\left(\lim_{x\to\infty}\left(\frac{f_F(x)}{f_{\widetilde{F}^{[c]}}(x)}\right)^{-1}\right) \\ &= \left(\lim_{x\to\infty}\frac{S_F(x)}{S_{\widetilde{F}^{[c]}}(x)}\right)\left(\lim_{x\to\infty}\frac{f_{\widetilde{F}^{[c]}}(x)}{f_F(x)}\right) \end{aligned}$$

$$\begin{aligned} &= \lim_{x\to\infty}\frac{S_F(x)}{S_{\widetilde{F}^{[c]}}(x)}\frac{f_{\widetilde{F}^{[c]}}(x)}{f_F(x)} \\ &= \lim_{x\to\infty}\frac{S_F(x)}{f_F(x)}\frac{f_{\widetilde{F}^{[c]}}(x)}{S_{\widetilde{F}^{[c]}}(x)} \\ &= \lim_{x\to\infty}\frac{\lambda_{\widetilde{F}^{[c]}}(x)}{\lambda_F(x)} \end{aligned}$$

$$\Rightarrow \tau_F = \lim_{x\to\infty}\lambda_F(x) = \lim_{x\to\infty}\lambda_{\widetilde{F}^{[c]}}(x) = \tau_{\widetilde{F}^{[c]}}$$

as required. ■

The relation

$$F \sim G \quad \Leftrightarrow \quad \text{there exists } c\in\mathbb{R} \text{ such that } G = \widetilde{F}^{[c]} \Leftrightarrow G \in \widetilde{F}^{[\mathbb{R}]}$$

defines an equivalence relation on the class of SLDFns

$$F = \widetilde{F}^{(0)} \Rightarrow F \sim F$$

$$F \sim G \Rightarrow \text{there exists } c\in\mathbb{R} \text{ such that } G = \widetilde{F}^{[c]} \Rightarrow F = \widetilde{G}^{[-c]} \Rightarrow G \sim F$$

$$F \sim G, G \sim H \Rightarrow \text{there exist } c, d\in\mathbb{R} \text{ such that } G = \widetilde{F}^{[c]}, H = \widetilde{G}^{[d]}$$

$$\Rightarrow H = \widetilde{G}^{[d]} = \widetilde{\left(\widetilde{F}^{[c]}\right)}^{d} = \widetilde{F}^{[c+d]} \Rightarrow F \sim H.$$

We just observed that the real-valued mapping $F \mapsto \tau_F$ is constant on equivalence classes, i.e., orbits. In this regard we make the:

Definition 63 *For any SLDFn F and real number c set*

$$\Gamma_F(c) = \lim_{x\to\omega_F}\frac{\int_x^\infty (y-x)^{c-1} f_F(y)dy}{S_F(x)}.$$

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Remark 64 *Note that for the exponential distribution $F(x) = 1 - e^{-x}$ we have*

$$\begin{aligned}
\Gamma_F(c) &= \lim_{x\to\omega_F} \frac{\int_x^\infty (y-x)^{c-1} e^{-y} dy}{e^{-x}} = \lim_{x\to\infty} \int_x^\infty (y-x)^{c-1} e^{-(y-x)} dy \\
&= \lim_{x\to\infty} \int_0^\infty z^{c-1} e^{-z} dz = \lim_{x\to\infty} \Gamma(c) = \Gamma(c)
\end{aligned}$$

which helped prompt the choice of notation.

Proposition 65 *For any SLDFn F and positive constants $a, c \in \mathbb{R}$ with $\mu_F^{(c)} < \infty$:*

1. $L_{\widetilde{F}^{[c]}}(t) = \frac{c\mu_F^{(c-1)}}{t\mu_F^{(c)}} \left(1 - L_{\widetilde{F}^{[c-1]}}(t)\right)$
2. $\mu_{\widetilde{F}^{[c-1]}} = \frac{\mu_F^{(c)}}{c\mu_F^{(c-1)}}$
3. $\tau_{F_a} = a\tau_F$
4. $\Gamma_{F_a}(c) = a^{1-c}\Gamma_F(c)$.

Proof. For Item 1

$$\begin{aligned}
L_{\widetilde{F}^{[c]}}(t) &= E_{\widetilde{F}^{[c]}}\left[e^{-tX}\right] \\
&= \int_0^\infty e^{-tx} f_{\widetilde{F}^{[c]}}(x) dx \\
&= \int_0^\infty e^{-tx} \left(\frac{c\int_x^\infty (y-x)^{c-1} f_F(y) dy}{\mu_F^{(c)}} \right) dx \\
&= \frac{c}{\mu_F^{(c)}} \int_0^\infty e^{-tx} \left(\int_x^\infty (y-x)^{c-1} f_F(y) dy \right) dx \\
&= \frac{c}{\mu_F^{(c)}} \int_0^\infty u dv
\end{aligned}$$

where

$$\begin{aligned}
u &= \int_x^\infty (y-x)^{c-1} f_F(y) dy = \mu_F^{(c-1)} S_{\widetilde{F}^{[c-1]}} \text{ and } v = -\frac{e^{-tx}}{t} \\
\text{and so } du &= -\mu_F^{(c-1)} f_{\widetilde{F}^{[c-1]}}.
\end{aligned}$$

We have

$$L_{\widetilde{F}^{[c]}}(t) = \frac{c}{\mu_F^{(c)}} \left([uv]_0^\infty - \int_0^\infty v du \right)$$

$$
\begin{aligned}
&= \frac{c}{\mu_F^{(c)}} \left(\left[-\frac{e^{-tx}}{t} \int_x^\infty (y-x)^{c-1} f_F(y) dy \right]_0^\infty - \frac{\mu_F^{(c-1)}}{t} \int_0^\infty e^{-tx} f_{\widetilde{F}^{[c-1]}}(x) dx \right) \\
&= \frac{c}{\mu_F^{(c)}} \left(\frac{1}{t} \int_0^\infty y^{c-1} f_F(y) dy - \frac{\mu_F^{(c-1)}}{t} L_{\widetilde{F}^{[c-1]}}(t) \right) \\
&= \frac{c}{\mu_F^{(c)}} \left(\frac{\mu_F^{(c-1)}}{t} - \frac{\mu_F^{(c-1)}}{t} L_{\widetilde{F}^{[c-1]}}(t) \right) \\
&= \frac{c\mu_F^{(c-1)}}{t\mu_F^{(c)}} \left(1 - L_{\widetilde{F}^{[c-1]}}(t)\right)
\end{aligned}
$$

For Item 2, invoke Item 1 and Proposition 39 applied to the LDFn $\widetilde{F}^{[c-1]}$, noting that for any LDFn G, $0 < L_G(1) < 1$

$$
\begin{aligned}
L_{\widetilde{F}}(t) &= \frac{1 - L_F(t)}{\mu t} \quad \text{for } t > 0 \\
\frac{c\mu_F^{(c-1)}}{t\mu_F^{(c)}} \left(1 - L_{\widetilde{F}^{[c-1]}}(t)\right) &= L_{\widetilde{F}^{[c]}}(t) = \frac{1}{t\mu_{\widetilde{F}^{[c-1]}}} \left(1 - L_{\widetilde{F}^{[c-1]}}(t)\right)
\end{aligned}
$$

$$
\Rightarrow \quad \frac{c\mu_F^{(c-1)}}{\mu_F^{(c)}} = \frac{1}{\mu_{\widetilde{F}^{[c-1]}}}
$$

$$
\Rightarrow \quad \mu_{\widetilde{F}^{[c-1]}} = \frac{\mu_F^{(c)}}{c\mu_F^{(c-1)}}.
$$

Item 3 follows from Proposition 48

$$
\tau_{F_a} = \lim_{x \to \omega_{F_a}} \lambda_{F_a}(x) = \lim_{x \to a\omega_F} a\lambda_F(ax) = a \lim_{x \to \omega_F} \lambda_F(x) = a\tau_F.
$$

And for Item 4

$$
\Gamma_{F_a}(c) = \lim_{x \to \omega_{F_a}} \frac{\int_x^{\omega_{F_a}} (y-x)^{c-1} f_{F_a}(y) dy}{S_{F_a}(x)}
$$

$$
\begin{aligned}
&= \lim_{x \to a\omega_F} \frac{\int_x^{a\omega_F} (y-x)^{c-1} a f_F(ay) \, dy}{S_F(ax)} \\
&= \lim_{x \to a\omega_F} \frac{a^{1-c} \int_x^{a\omega_F} (ay-ax)^{c-1} f_F(ay) \, ady}{S_F(ax)} \\
&= a^{1-c} \lim_{ax \to \omega_F} \frac{\int_{ax}^{\omega_F} (z-ax)^{c-1} f_F(z) dz}{S_F(z)} \\
&= a^{1-c} \lim_{y \to \omega_F} \frac{\int_y^{\omega_F} (z-y)^{c-1} f_F(z) dz}{S_F(z)} \\
&= a^{1-c} \Gamma_F(c)
\end{aligned}
$$

as required. ■

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Proposition 66 *If F is an SLDFn with $0 < \tau_F < \infty$ and a any positive constant, then:*

$$F_a \in \widetilde{F}^{(\mathbb{R})} \quad \Leftrightarrow \quad a = 1.$$

Proof. The $\Leftarrow$ direction is trivial. For $\Rightarrow$

$$\begin{aligned} F_a &\in \widetilde{F}^{(\mathbb{R})} \Rightarrow \tau_F = \tau_{F_a} = a\tau_F \\ 0 &< \tau_F < \infty \Rightarrow a = 1 \end{aligned}$$

as required. ■

What really prompted the notation are Items 5 and 6 of the following:

Proposition 67 *For any non-vanishing SLDFn F with finite mean and $c \in \mathbb{R}$ with $\mu_F^{(c)} < \infty$:*

1. $\Gamma_F(1) = 1$
2. $\tau_F > 0 \Rightarrow \Gamma_F(2) = \frac{1}{\tau_F}$
3. $\Gamma_F(c) = \lim_{x\to\infty} \frac{\int_0^\infty z^{c-1} f_F(z+x)dz}{S_F(x)}$
4. $\Gamma_{\widetilde{F}}(c) = \Gamma_F(c)$
5. $\tau_F \Gamma_F(c+1) = c\Gamma_F(c)$
6. $\tau_F > 0$ and $c \in \mathbb{Z} \Rightarrow \Gamma_F(c) = \tau_F^{1-c}\Gamma(c)$

Proof. We clearly have

$$\Gamma_F(1) = \lim_{x\to\omega_F} \frac{\int_x^{\omega_F} (y-x)^0 f_F(y)dy}{S_F(x)} = \lim_{x\to\omega_F} \frac{\int_x^{\omega_F} f_F(y)dy}{S_F(x)} = \lim_{x\to\omega_F} \frac{S_F(x)}{S_F(x)} = \lim_{x\to\omega_F} 1 = 1$$

verifying Item 1. When $\tau_F > 0$, we have from l'Hôpital and Proposition 22

$$\begin{aligned} \Gamma_F(2) &= \lim_{x\to\omega_F} \frac{\int_x^{\omega_F} (y-x)^1 f_F(y)dy}{S_F(x)} = \lim_{x\to\omega_F} \frac{\mu_F R_F(x)}{S_F(x)} \\ &= \mu_F \lim_{x\to\omega_F} \frac{R_F(x)}{S_F(x)} = \mu_F \lim_{x\to\omega_F} \frac{\frac{-S_F(x)}{\mu_F}}{-f_F(x)} \\ &= \lim_{x\to\omega_F} \frac{S_F(x)}{f_F(x)} = \lim_{x\to\omega_F} \frac{1}{\lambda_F(x)} = \frac{1}{\tau_F} \end{aligned}$$

proving Item 2. For Item 3, just use the change of variable $z = y - x$. For Item 4, we have, using Item 3

$$\begin{aligned} \Gamma_{\widetilde{F}}(c) &= \lim_{x\to\omega_F} \frac{\int_0^{\omega_F} z^{c-1} f_{\widetilde{F}}(z+x)dz}{S_{\widetilde{F}}(x)} \\ &= \lim_{x\to\omega_F} \frac{\int_0^{\omega_F} z^{c-1} \frac{S_F(z+x)}{\mu_F} dz}{R_F(x)} \end{aligned}$$

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$$
\begin{aligned}
&= \frac{1}{\mu_F} \lim_{x\to\omega_F} \frac{\int_0^{\omega_F} z^{c-1} S_F(z+x)dz}{R_F(x)} \\
&= \frac{1}{\mu_F} \lim_{x\to\omega_F} \frac{\frac{d}{dx}\int_0^{\omega_F} z^{c-1} S_F(z+x)dz}{\frac{d}{dx}R_F(x)} \\
&= \frac{1}{\mu_F} \lim_{x\to\omega_F} \frac{\int_0^{\omega_F} z^{c-1} \frac{d}{dx}\left(S_F(z+x)\right)dz}{\frac{-S_F(x)}{\mu_F}}
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{x\to\infty} \frac{-\int_0^{\infty} z^{c-1} f_F(z+x)\frac{d(z+x)}{dx}dz}{-S_F(x)} \\
&= \lim_{x\to\omega_F} \frac{\int_0^{\omega_F} z^{c-1} f_F(z+x)dz}{S_F(x)} \\
&= \Gamma_F(c)
\end{aligned}
$$

which establishes Item 4. For Item 5 we have

$$
\begin{aligned}
\tau_F(c)\Gamma_F(c+1) &= \tau_{\widetilde{F}[c]}\Gamma_F(c+1) \\
&= \lim_{x\to\omega_F} \lambda_{\widetilde{F}[c]}(x)\Gamma_F(c+1) \\
&= \lim_{x\to\omega_F} \frac{f_{\widetilde{F}[c]}(x)}{S_{\widetilde{F}[c]}(x)}\Gamma_F(c+1)
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{x\to\omega_F} \frac{c\int_x^{\omega_F}(y-x)^{c-1} f_F(y)dy}{\int_x^{\omega_F}(y-x)^{c} f_F(y)dy} \lim_{x\to\omega_F} \frac{\int_x^{\omega_F}(y-x)^{c} f_F(y)dy}{S_F(x)} \\
&= c\lim_{x\to\omega_F} \frac{\int_x^{\omega_F}(y-x)^{c-1} f_F(y)dy}{\int_x^{\omega_F}(y-x)^{c} f_F(y)dy} \frac{\int_x^{\omega_F}(y-x)^{c} f_F(y)dy}{S_F(x)} \\
&= c\lim_{x\to\omega_F} \frac{\int_x^{\omega_F}(y-x)^{c-1} f_F(y)dy}{S_F(x)} = c\Gamma_F(c).
\end{aligned}
$$

And finally, for Item 6 note that the formula holds for $c = 1$ and $c = 2$. by Items 1 and 2. Define $\Gamma^*(c) = \tau_F^{c-1}\Gamma_F(c)$,then by Item 5

$$
\begin{aligned}
\Gamma^*(c+1) &= \tau_F^{c+1-1}\Gamma_F(c+1) \\
&= \tau_F^{c}\frac{c\Gamma_F(c)}{\tau_F} = c\tau_F^{c-1}\Gamma_F(c) \\
&= c\Gamma^*(c)
\end{aligned}
$$

and so Γ^* and Γ satisfy the same recurrence formula and agree on 1 and 2, whence $\Gamma^* = \Gamma$ on $\mathbb{Z}$, as required. ■

Corollary 68 *If F and G are SLDFns with finite means and $\tau_F\tau_G > 0$, then*

$$
\Gamma_F(c) = \Gamma_G(c) \text{ for every } c \in \mathbb{Z} \quad \Leftrightarrow \quad \tau_F = \tau_G.
$$

Corollary 69 *If F is an SLDFn with finite mean and $\tau_F > 0$, then*

$$\tau_F = 1 \Leftrightarrow \Gamma_F(n) = n! \text{ for every } n \in \mathbb{N}.$$

Proposition 70 *If F is a non-vanishing SLDFn, then for any $c > 1$ such that $\mu_F^{(c-1)} < \infty$, we have:*

$$\Gamma_F(c) = (c-1) \lim_{x \to \omega_F} \int_0^{\omega_F} z^{c-2} \left(\frac{S_F(x+z)}{S_F(x)} \right) dz.$$

Proof. By Proposition 11 we have

$$\int_x^{\omega_F} (y-x)^{c-1} f_F(y)dy = (c-1) \int_x^{\omega_F} (y-x)^{c-2} S_F(y)dy$$

from which we find that

$$\begin{aligned}
\Gamma_F(c) &= \lim_{x \to \omega_F} \frac{\int_x^{\omega_F} (y-x)^{c-1} f_F(y)dy}{S_F(x)} \\
&= \lim_{x \to \omega_F} \frac{(c-1) \int_x^{\omega_F} (y-x)^{c-2} S_F(y)dy}{S_F(x)} \\
&= (c-1) \lim_{x \to \omega_F} \int_x^{\omega_F} (y-x)^{c-2} \left(\frac{S_F(y)}{S_F(x)} \right) dy \\
&= (c-1) \lim_{x \to \omega_F} \int_0^{\omega_F} z^{c-2} \left(\frac{S_F(x+z)}{S_F(x)} \right) dz
\end{aligned}$$

as required. ■

Proposition 71 *If F is a non-vanishing SLDFn with $\tau_F > 0$ and is such that for every $c > 0$ we have $\mu_F^{(c)} < \infty$, then:*

$$\lim_{x \to \infty} \frac{f_{\widetilde{F}^{[c]}}(x)}{f_F(x)} = \frac{\Gamma_F(c+1)}{\mu_F^{(c)}} \quad \text{for every } c > 0.$$

Proof. We have

$$f_{\widetilde{F}^{[c]}}(x) = \frac{c \int_x^{\infty} (y-x)^{c-1} f_F(y)dy}{\mu_F^{(c)}}$$

which implies that

$$\begin{aligned}
\frac{f_{\widetilde{F}^{[c]}}(x)}{f_F(x)} &= \frac{\frac{c \int_x^{\infty} (y-x)^{c-1} f_F(y)dy}{\mu_F^{(c)}}}{f_F(x)} \\
&= \frac{\frac{c \int_x^{\infty} (y-x)^{c-1} f_F(y)dy}{S_F(x)}}{\mu_F^{(c)} \frac{f_F(x)}{S_F(x)}} \\
&= \frac{\frac{c \int_x^{\infty} (y-x)^{c-1} f_F(y)dy}{S_F(x)}}{\mu_F^{(c)} \lambda_F(x)}
\end{aligned}$$

and recalling the definition $\Gamma_F(c) = \lim_{x\to\infty} \frac{\int_x^\infty (y-x)^{c-1} f_F(y)dy}{S_F(x)}$, we find from Proposition 67 that

$$\begin{aligned}
\lim_{x\to\infty} \frac{f_{\widetilde{F}^{(c)}}(x)}{f_F(x)} &= \frac{c}{\mu_F^{(c)}} \lim_{x\to\infty} \frac{\frac{\int_x^\infty (y-x)^{c-1} f_F(y)dy}{S_F(x)}}{\lambda_F(x)} \\
&= \frac{c}{\mu_F^{(c)}} \frac{\lim_{x\to\infty} \frac{\int_x^\infty (y-x)^{c-1} f_F(y)dy}{S_F(x)}}{\lim_{x\to\infty} \lambda_F(x)} \\
&= \frac{c\Gamma_F(c)}{\mu_F^{(c)}\tau_F} = \frac{\tau_F \Gamma_F(c+1)}{\mu_F^{(c)}\tau_F} = \frac{\Gamma_F(c+1)}{\mu_F^{(c)}}
\end{aligned}$$

as required. ■

Proposition 72 *If F is an SLDFn with finite mean, then there exist unique $a, b, c \in \mathbb{R} \cup \{\infty\}$, $a \le b \le 0, 1 \le c$ such that:*

$$\begin{aligned}
(a,c) &= \left\{ x \in \mathbb{R} - \{a,c\} \mid \text{there exists SLDFn } G \text{ such that } G = \widetilde{F}^{[x]} \right\} \\
(b,c) &= \left\{ x \in \mathbb{R} - \{b,c\} \mid \mu_F^{(x)} < \infty \right\}.
\end{aligned}$$

Proof. It is clear from the above that both sets are connected subsets of $\mathbb{R}$ containing $(0,1)$ and that they share a right hand endpoint c. It is also clear from what has been shown that $a \le b$. The rest follows from Proposition 23. ■

Proposition 73 *For any SLDFns F and G with $\mu_F^{(n)}$, $\mu_G^{(n)} < \infty$ for every $n \in \mathbb{N}$ and $\tau_F \tau_G > 0$, letting $\mathbf{B}$ denote the beta function:*

$$G = \widetilde{F_a}^{[c]} \text{ for some positive constants } a, c \in \mathbb{R}$$

$$\Leftrightarrow$$

$$a = \frac{\tau_G}{\tau_F} \text{ and there exists } c > 0 \text{ such that } \tau_G^k \mu_G^{(k)} = \frac{\tau_F^k (c+k+1)\mathbf{B}(k+1, c+1)\mu_F^{(c+k)}}{\mu_F^{(c)}} \text{ for every } k \in \mathbb{N}.$$

Proof. Suppose that $G = \widetilde{F_a}^{[c]}$ for some positive constants $a, c \in \mathbb{R}$, then

$$\begin{aligned}
G &= \left(\widetilde{F_a}\right)^{[c]} = \left(\widetilde{F}^{[c]}\right)_a \\
&\Rightarrow \tau_G = \tau_{\widetilde{F_a}^{[c]}} = \tau_{\left(\widetilde{F}^{[c]}\right)_a} = a\tau_{\widetilde{F}^{[c]}} = a\tau_F \\
&\Rightarrow a = \frac{\tau_G}{\tau_F}
\end{aligned}$$

and we have for every $k \in \mathbb{N}$

$$\mu_G^{(k)} = \mu_{\widetilde{F_a}^{[c]}}^{(k)} = \frac{(c+k+1)\mathbf{B}(k+1, c+1)\mu_{F_a}^{(c+k)}}{\mu_{F_a}^{(c)}}$$

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$$
\begin{aligned}
&= \frac{a^c(c+k+1)\mathbf{B}(k+1,c+1)\mu_F^{(c+k)}}{a^{c+k}\mu_F^{(c)}} \\
&= \frac{(c+k+1)\mathbf{B}(k+1,c+1)\mu_F^{(c+k)}}{a^k\mu_F^{(c)}} \\
&= \frac{(c+k+1)\mathbf{B}(k+1,c+1)\mu_F^{(c+k)}}{\left(\frac{\tau_G}{\tau_F}\right)^k\mu_F^{(c)}}
\end{aligned}
$$

$$
\Rightarrow \tau_G^k\mu_G^{(k)} = \frac{\tau_F^k(c+k+1)\mathbf{B}(k+1,c+1)\mu_F^{(c+k)}}{\mu_F^{(c)}} \text{ for every } k \in \mathbb{N}
$$

which establishes the $\Rightarrow$ direction. Conversely, letting $a = \frac{\tau_G}{\tau_F} > 0$ those same equations imply that

$$
\tau_G^k\mu_G^{(k)} = \frac{\tau_F^k(c+k+1)\mathbf{B}(k+1,c+1)\mu_F^{(c+k)}}{\mu_F^{(c)}} \text{ for every } k \in \mathbb{N}
$$

$$
\begin{aligned}
\Rightarrow\quad & \mu_G^{(k)} = \mu_{\widetilde{F_a}^{[c]}}^{(k)} \text{ for every } k \in \mathbb{N} \\
\Rightarrow\quad & L_G = L_{\widetilde{F_a}^{[c]}} \Rightarrow G = \widetilde{F_a}^{[c]} = \widetilde{F}_a^{[c]}
\end{aligned}
$$

and the proof is complete. ■

This suggests that one way to decompose the set of all SLDFns F with $0 < \tau_F < \infty$ is into disjoint $\sim$invariant subsets of "coordinated half planes" of the form

$$
(0,\infty)\widetilde{F}^{[\mathbb{R}]} = \left\{(a,c) \longleftrightarrow \widetilde{F_a}^{[c]} | a \in (0,\infty), c \in \mathbb{R}\right\}.
$$

Such a plane is akin to an orbit under the affine-like action of the direct product $(0,\infty)\times\mathbb{R}$ of the multiplicative group of positive reals by the additive group of reals (subgroup of a Borel subgroup of $SL_2(\mathbb{R})$). The above Proposition provides one approach for determining when two SLDFns "lie on the same plane". Note that while there are infinitely many equations to check, mathematical induction should often apply to make this doable. Also, you may need to swap roles of F and G to deal with the possibility of $c < 0$. From knowledge of moments $\mu_F^{(c)}$ as c varies for some empirical data, the above formulas show how to pick (a, c) to match the first two moments (first solve for c to match the CV

$$
\frac{2\,(c+1)\,\mu_F^{(c)}\mu_F^{(c+2)}}{(c+2)\left(\mu_F^{(c+1)}\right)^2} = \left(CV_{\widetilde{F}^{[c]}}\right)^2 + 1 = \frac{\mu_{\widetilde{F}^{[c]}}^{(2)}}{\left(\mu_{\widetilde{F}^{[c]}}^{(1)}\right)^2}
$$

and then determine a as the scalar adjustment to match the mean). We will soon see how to quantify the difference in the thickness of the tail between any

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two elements of such a plane. We will see that for loss variables F and G in different planes, we need only be able to compare one pair of representatives from the two planes to be able to compare any two elements in the union of the two planes, including, of course, F and G.

The real-valued mapping $F \mapsto \tau_F$ defined on the set of SLDFns is constant on equivalence classes, i.e., orbits. The main result of this paper is to specify the possible structures for $\widetilde{F}^{[\mathbb{R}]}$ as they relate with the ultimate settlement rate τ_F and other metrics for the "thickness" of the tail, as that concept is defined later. This part of the paper concludes with some examples. In the next and final part we will see that the structure of $\widetilde{F}^{(n)}$ becomes more "monotone", "smooth", and "tail-like" as n increases and make mathematically precise what that statement means.

5 Examples

This section presents some simple examples.

Example 74 *Uniform density: let F be uniformly distributed on the finite interval $[a, b]$ where $0 \leq a < b$. The following are well-known and readily verified*

$$F(x) = \left\{ \begin{array}{cc} 0 & x \leq a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & b \leq x \end{array} \right\}$$

$$f_F(x) = \left\{ \begin{array}{cc} 0 & x < a \\ \frac{1}{b-a} & a < x < b \\ 0 & b < x \end{array} \right\}$$

$$S_F(x) = \left\{ \begin{array}{cc} 1 & x \leq a \\ \frac{b-x}{b-a} & a \leq x \leq b \\ 0 & b \leq x \end{array} \right\}$$

$$\lambda_F(x) = \left\{ \begin{array}{cc} 0 & x < a \\ \frac{1}{b-x} & a < x < b \\ \infty & x \geq b \end{array} \right\}$$

$$\begin{aligned} \alpha_F &= a \qquad \omega_F = b \\ \mu_F &= \frac{b+a}{2} \\ M_F(t) &= \frac{e^{bt} - e^{at}}{(b-a)\,t} \qquad t > 0 \end{aligned}$$

Propositions 25, 38, and 46 lead to

$$f_{\widetilde{F}}(x) = \left\{ \begin{array}{cc} \frac{2}{a+b} & x < a \\ \frac{2(b-x)}{b^2-a^2} & a \leq x \leq b \\ 0 & x > b \end{array} \right\}$$

$$S_{\widetilde{F}}(x) = \left\{ \begin{array}{cc} \frac{a+b-2x}{a+b} & x \leq a \\ \frac{(b-x)^2}{b^2-a^2} & a \leq x \leq b \\ 0 & x \geq b \end{array} \right\}$$

$$M_{\widetilde{F}}(t) = 2\frac{e^{bt} - e^{at} - (b-a)t}{(b^2 - a^2)\,t^2} \quad t > 0$$

Observe that

$$M_F(t) = \frac{e^{bt} - e^{at}}{(b-a)\,t} = \frac{\sum_{k=0}^{\infty} \frac{(bt)^k - (at)^k}{k!}}{(b-a)\,t}$$

$$\begin{aligned} &= \sum_{k=0}^{\infty} \frac{\left(b^k - a^k\right) t^{k-1}}{(b-a)\,k!} \\ &= \sum_{k=1}^{\infty} \frac{\left(\frac{b^{(k-1)+1} - a^{(k-1)+1}}{((k-1)+1)(b-a)}\right) t^{k-1}}{(k-1)!} \\ &= \sum_{k=0}^{\infty} \frac{\left(\frac{b^{k+1} - a^{k+1}}{(k+1)(b-a)}\right) t^k}{k!} \end{aligned}$$

$$\Rightarrow \mu_F^{(k)} = \frac{b^{k+1} - a^{k+1}}{(k+1)\,(b-a)} = \frac{\sum_{j=0}^{k} b^j a^{k-j}}{k+1}$$

And so from Propositions 27 and 40

$$\begin{aligned} \mu_{\widetilde{F}}^{(k)} &= \frac{\mu_F^{(k+1)}}{(k+1)\,\mu_F} = \frac{\frac{b^{k+2} - a^{k+2}}{(k+2)(b-a)}}{(k+1)\left(\frac{b+a}{2}\right)} \\ &= 2\frac{b^{k+2} - a^{k+2}}{(k+2)\,(k+1)\,(b^2 - a^2)} \end{aligned}$$

$$\mu_{\widetilde{F}[k]} = \frac{\mu_F^{(k+1)}}{(k+1)\,\mu_F^{(k)}} = \frac{\frac{b^{k+2} - a^{k+2}}{(k+2)(b-a)}}{(k+1)\left(\frac{b^{k+1} - a^{k+1}}{(k+1)(b-a)}\right)} = \frac{b^{k+2} - a^{k+2}}{(k+2)\,(b^{k+1} - a^{k+1})}.$$

In particular

$$\mu_{\widetilde{F}} = \frac{b^3 - a^3}{3\,(b^2 - a^2)}.$$

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Notice that for $a > 0$

$$\begin{aligned}
\mu_{\widetilde{F}^{[k]}} &= \frac{b^{k+2} - a^{k+2}}{(k+2)\left(b^{k+1} - a^{k+1}\right)} \\
&= \frac{b\left(\frac{b}{a}\right)^{k+1} - a}{(k+2)\left(\left(\frac{b}{a}\right)^{k+1} - 1\right)} \\
&= \frac{b\left(\left(\frac{b}{a}\right)^{k+1} - 1\right) + b - a}{(k+2)\left(\left(\frac{b}{a}\right)^{k+1} - 1\right)} \\
&= \frac{b}{k+2} + \frac{b-a}{(k+2)\left(\left(\frac{b}{a}\right)^{k+1} - 1\right)} \\
&\Rightarrow \lim_{k\to\infty} \mu_{\widetilde{F}^{[k]}} = 0
\end{aligned}$$

and we see that as k increases, the LDFns $\widetilde{F}^{[k]}$ become concentrated at the value 0. On the other hand, if b is the maximum loss, then $S_{\widetilde{F}^{[k]}}(b-\epsilon) > 0$ for every $k \in \mathbb{N}$, $\epsilon > 0$. More generally we have:

Example 75 *Consider the case when the PDF has finite support and is bounded away from 0, i.e., there exist $a, b, \alpha, \beta \in \mathbb{R}$ with $0 \le a < b$ and $0 < \alpha \le \beta$ such that $f_F(x) = 0$ for $x \notin [a,b]$ and $\alpha \le f_F(x) \le \beta$ $x \in [a,b]$. In this case we have, for $c > 0$*

$$\frac{\alpha\left(b^{c+1} - a^{c+1}\right)}{c+1} = \alpha\int_a^b x^c dx \le \mu_F^{(c)} \le \beta\int_a^b x^c dx = \frac{\beta\left(b^{c+1} - a^{c+1}\right)}{c+1}$$

$$\begin{aligned}
\Rightarrow \mu_{\widetilde{F}^{[c]}} &= \frac{\mu_F^{(c+1)}}{(c+1)\,\mu_F^{(c)}} \\
&\le \frac{\frac{\beta\left(b^{c+2} - a^{c+2}\right)}{c+2}}{(c+1)\left(\frac{\alpha\left(b^{c+1} - a^{c+1}\right)}{c+1}\right)} \\
&= \frac{\beta}{\alpha(c+2)}\frac{b^{c+2} - a^{c+2}}{b^{c+1} - a^{c+1}} \\
&= \frac{\beta}{\alpha(c+2)}\frac{a+b}{a+b}\frac{b^{c+2} - a^{c+2}}{b^{c+1} - a^{c+1}} \\
&= \frac{\beta(a+b)}{\alpha(c+2)}\frac{b^{c+2} - a^{c+2}}{b^{c+2} - a^{c+2} + ab(b^c - a^c)} \\
&< \frac{\beta(a+b)}{\alpha(c+2)}\frac{b^{c+2} - a^{c+2}}{b^{c+2} - a^{c+2}} \\
&= \frac{\beta(a+b)}{\alpha(c+2)}
\end{aligned}$$

$$\Rightarrow \lim_{c\to\infty} \mu_{\widetilde{F}^{[c]}} = 0$$

and again, as one would expect from the uniform density example, we see that as c increases the LDFns $\widetilde{F}^{[c]}$ become concentrated at the value 0.

Example 76 *Exponential Distribution: Let F have an exponential density with mean $\mu > 0$. We have:*

$$\mu_F = \sigma_F = \mu \quad CV_F = 1 \quad \alpha_F = 0, \quad \omega_F = \infty, \quad \lambda_F = \frac{1}{\mu} = \tau_F$$

$$\mu_F^{(k)} = \mu^k k! \quad L_F(t) = \frac{1}{1+\mu t},\ t > -\frac{1}{\mu}$$

$$f_{\widetilde{F}}(x) = \frac{S_F(x)}{\mu} = \left(\frac{1}{\mu}\right) e^{-\frac{x}{\mu}} = f_F(x)$$

$$\Rightarrow \widetilde{F} = F \Rightarrow \widetilde{F}^{[n]} = F \quad \textit{for every } n \in \mathbb{N}.$$

The converse also holds

$$\begin{aligned} \widetilde{F} &= F \\ &\Rightarrow \widetilde{(F_\mu)} = \widetilde{F}_\mu = F_\mu. \end{aligned}$$

Letting $G = F_\mu$, we have $G = \widetilde{G}$ and $\mu_G = \mu_{F_\mu} = \frac{\mu_F}{\mu} = 1$. Define $g(x) = S_G(-x)$ for $x < 0$, then

$$\begin{aligned} \frac{dg}{dx} &= \frac{d\left(S_G(-x)\right)}{dx} = \frac{dS_G}{d(-x)}\frac{d(-x)}{dx} = \left(-f_G\left(-x\right)\right)(-1) \\ &= f_G\left(-x\right) = f_{\widetilde{G}}\left(-x\right) = \frac{S_G(-x)}{\mu_G} = S_G(-x) = g(x) \end{aligned}$$

$$\frac{dg}{dx} = g(x),\ g(0) = 1 \Rightarrow g(x) = e^x$$

$$\Rightarrow S_G(-y) = g(y) = e^y$$

$$\Rightarrow S_F(x) = S_{G_{\frac{1}{\mu}}}(x) = S_G\left(\frac{x}{\mu}\right) = g\left(-\frac{x}{\mu}\right) = e^{-\frac{x}{\mu}}$$

and F has an exponential density with mean $\mu > 0$. More generally we have for $c > 0$

$$S_F(x) = e^{-\frac{x}{\mu}}$$

$$\Rightarrow S_{\widetilde{F}^{[c]}}(x) = \frac{\int_x^\infty (y-x)^c f_F(y)dy}{\mu_F^{(c)}}$$

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$$= \frac{\int_0^\infty z^c f_F(z+x)dz}{\mu_F^{(c)}} = \frac{\int_0^\infty z^c \frac{e^{-\frac{z+x}{\mu}}}{\mu}dz}{\mu_F^{(c)}}$$

$$= \frac{e^{-\frac{x}{\mu}} \int_0^\infty z^c \frac{e^{-\frac{z}{\mu}}}{\mu}dz}{\mu_F^{(c)}} = \frac{e^{-\frac{x}{\mu}}\mu_F^{(c)}}{\mu_F^{(c)}}$$

$$= e^{-\frac{x}{\mu}} = S_F(x)$$

$$\Rightarrow \widetilde{F}^{[c]} = F.$$

We have established

$$S_F(x) = e^{-\frac{x}{\mu}} \Leftrightarrow \widetilde{F}^{[\mathbb{R}]} = \{F\}.$$

Note too that for any $c \geq 0$

$$F^{>c}(x) = 1 - \frac{S_F(x+c)}{S_F(c)} = 1 - \frac{e^{-\frac{x+c}{\mu}}}{e^{-\frac{c}{\mu}}} = 1 - \frac{e^{-\frac{c}{\mu}}e^{-\frac{x}{\mu}}}{e^{-\frac{c}{\mu}}} = 1 - e^{-\frac{x}{\mu}} = F(x)$$

$$\Rightarrow F^{>c} = F$$

with the converse again being true, i.e., this too characterizes the exponential distribution. Indeed for any SLDFn G *:*

$$G^{>c} = G \quad \textit{for every } c \geq 0$$

$$\Rightarrow \omega_G = \omega_{G^{>1}} = \omega_G - 1 \Rightarrow \omega_G = \infty$$

$$\Rightarrow 1 - S_G(x) = G(x) = G^{>c}(x) = 1 - \frac{S_G(x+c)}{S_G(c)} \quad \textit{for every } x, c \geq 0$$

$$\Rightarrow S_G(x) = \frac{S_G(x+c)}{S_G(c)} \quad \textit{for every } x, c \geq 0$$

$$\Rightarrow S_G(x+y) = S_G(x)S_G(y) \quad \textit{for every } x, y \geq 0$$

$$\Rightarrow f_G(x+y) = -\frac{dS_G(x+y)}{dy} = -\frac{dS_G(x)S_G(y)}{dy}$$

$$= S_G(x)f_G(y) + S_G(y) \cdot 0 = S_G(x)f_G(y) \quad \textit{for every } x, y \geq 0$$

$$\Rightarrow f_G(x) = S_G(x)f_G(0)$$

$$\Rightarrow 1 = \int_0^\infty f_G(x)dx = f_G(0)\int_0^\infty S_G(x)dx = f_G(0)\mu_G$$

$$f_G(0) = S_G(x)f_G(0) = \frac{1}{\mu_G}$$

$$\Rightarrow f_G(x) = S_G(x)f_G(0) = \frac{S_G(x)}{\mu_G} = f_{\widetilde{G}}(x)$$

$$\Rightarrow G = \widetilde{G} \Rightarrow G = 1 - e^{-\frac{x}{\mu_G}}.$$

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Finally, we have as noted above

$$\begin{aligned}
\Gamma_F(c) &= \lim_{x\to\infty} \frac{\int_x^{\infty}(y-x)^{c-1}\frac{e^{-\frac{y}{\mu}}}{\mu}dy}{e^{-\frac{x}{\mu}}} \\
&= \lim_{x\to\infty}\int_x^{\infty}\mu^{1-c}\left(\frac{y-x}{\mu}\right)^{c-1}e^{-\left(\frac{y-x}{\mu}\right)}\frac{dy}{\mu} \\
&= \lim_{x\to\infty}\mu^{1-c}\int_0^{\infty}z^{c-1}e^{-z}dz \\
&= \lim_{x\to\infty}\mu^{1-c}\Gamma(c) = \mu^{1-c}\Gamma(c).
\end{aligned}$$

As a consequence of this example:

Proposition 77 *Suppose F is an SLDFn with $(\alpha_F, \omega_F) = (0, \infty)$, $\mu_F^{(k)} < M < \infty$ for every $k \in \mathbb{N}$, and for which $\widetilde{F}^{[\infty]} = \lim_{n\to\infty}\widetilde{F}^{[n]}$ exists as a pointwise limit function and where the convergence is uniform on $(0,\infty)$. Then for all $x > 0$:*

$$\widetilde{F}^{[\infty]}(x) = 1 - e^{-\tau_F x}.$$

Proof. Let $G = \lim_{n\to\infty}\widetilde{F}^{[n]}$, uniform convergence implies that G is an SLDFn with finite mean $\mu_G \le M < \infty$. We have

$$\tau_G = \lim_{n\to\infty}\tau_{\widetilde{F}^{[n]}} = \lim_{n\to\infty}\tau_F = \tau_F$$

$$\text{and } \widetilde{G} = \widetilde{\lim_{n\to\infty}\widetilde{F}^{[n]}} = \lim_{n\to\infty}\widetilde{\widetilde{F}^{[n]}} = \lim_{n\to\infty}\widetilde{F}^{[n+1]} = \lim_{n\to\infty}\widetilde{F}^{[n]} = G.$$

And so by the exponential example

$$\begin{aligned}
G(x) &= 1 - e^{-\frac{x}{\mu_G}} \\
&\Rightarrow \tau_F = \tau_G = \frac{1}{\mu_G} \\
&\Rightarrow G(x) = 1 - e^{-\tau_F x}
\end{aligned}$$

as required. ■

In practice, one would expect that far enough into the tail of a distribution the hazard function λ_F would be bounded and stabilized at least to being either nonincreasing or nondecreasing. And in that event, the hazard functions of the higher coderived distributions $\widetilde{F}^{[n]}$ are squeezed to the constant τ_F. Accordingly, when $\tau_F > 0$, as n increases we would expect the $\widetilde{F}^{[n]}$ to converge to the exponential density of mean $\mu = \frac{1}{\tau_F}$. This points toward a special role for the exponential density when fitting the tail of a loss distribution. More precisely, we have:

Proposition 78 *Suppose F is an SLDFn with λ_F either nonincreasing or nondecreasing and with $0 < \tau_F < \infty$. Then for any $x > 0$:*

$$\widetilde{F}^{[\infty]}(x) = \lim_{n\to\infty}\widetilde{F}^{[n]}(x) = 1 - e^{-\tau_F x}.$$

Proof. By Proposition 33, the assumption $0 < \tau_F < \infty$ assures all moments are finite. Since λ_F is either nondecreasing or nonincreasing, Propositions 31 and 32 imply that either $\lambda \leq \widetilde{\lambda} \leq \widetilde{\widetilde{\lambda}} \leq$ ...or $\lambda \geq \widetilde{\lambda} \geq \widetilde{\widetilde{\lambda}} \geq$ .... In either event, we have pointwise convergence on $(0, \omega_F) = (0, \infty)$ of the hazard function sequence $\lambda, \widetilde{\lambda}, \widetilde{\widetilde{\lambda}}, ...$ to a $\sim$invariant hazard function. As in the proof of the previous Proposition, this entails uniform convergence of $\lambda, \widetilde{\lambda}, \widetilde{\widetilde{\lambda}}, ...$ to the constant τ_F on the interval $[x, \infty)$, which then entails that $\widetilde{F}^{[n]}(y)$ converge to $1 - e^{-\tau_F y}$ as $n \to \infty$ for any $y \geq x$, and the result follows. ∎

Similarly, recall from Proposition 37 that $\lim_{x,c \to \omega_F} \frac{S_{\widetilde{F^{>c}}}(x)}{S_{F^{>c}}(x)} = 1$ where F is any SLDFn with finite mean and $0 < \tau_F < \infty$. Then the idea is again that the "far tail" of F is captured as $G = F^{>c}$ for c large and where we have

$$\lim_{x \to \omega_G} \frac{S_{\widetilde{G}}(x)}{S_G(x)} = 1$$

$$\Rightarrow G \approx \widetilde{G} \approx 1 - e^{-\tau_G x} = 1 - e^{-\tau_{(F^{>c})} x} = 1 - e^{-\tau_F x}.$$

This suggests that quite generally, when $0 < \tau_F, \mu_F < \infty$, the exponential density of mean $\frac{1}{\tau_F}$ appears as a natural way to model the structure of the far tail of the distribution. We will see in the next section that analytic properties of exponentials, and more generally mixed exponentials, again make them a natural choice for modeling tail behavior. This strengthens the theoretical justification for the methodology used to fit tails when calculating ELFs in [2] which also derives some general formulas for splicing tails on loss distributions.

Example 79 *Pareto Density: Let $F = \Pi(\alpha, \theta)$ have the Pareto density with parameters α and θ :*

$$F(x) = \Pi(\alpha, \theta; x) = 1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha} \qquad \omega_F = \infty$$

$$S_F(x) = \left(\frac{\theta}{x+\theta}\right)^{\alpha}$$

$$f_F(x) = \frac{\alpha\theta^{\alpha}}{(x+\theta)^{\alpha+1}} \quad \lambda_F(x) = \frac{\frac{\alpha\theta^{\alpha}}{(x+\theta)^{\alpha+1}}}{\left(\frac{\theta}{x+\theta}\right)^{\alpha}} = \frac{\alpha}{x+\theta} \quad \tau_F = 0$$

$$k \in \mathbb{N} \text{ and } k < \alpha \Rightarrow \mu_F^{(k)} = \frac{\theta^k k!}{(\alpha-1)\cdots(\alpha-k)}$$

$$\begin{aligned} F_a(x) &= F(ax) = 1 - \left(\frac{\theta}{ax+\theta}\right)^{\alpha} \\ &= 1 - \left(\frac{\frac{\theta}{a}}{x+\frac{\theta}{a}}\right)^{\alpha} = \Pi(\alpha, \frac{\theta}{a}; x) \end{aligned}$$

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$$\begin{aligned} F^{>c}(x) &= 1-\frac{S_F(x+c)}{S_F(c)} = 1-\frac{\left(\frac{\theta}{x+c+\theta}\right)^\alpha}{\left(\frac{\theta}{c+\theta}\right)^\alpha} \\ &= 1-\left(\frac{\theta+c}{x+(\theta+c)}\right)^\alpha = \Pi(\alpha,\theta+c;x) \end{aligned}$$

$$\Rightarrow \Pi(\alpha,\theta)^{>c} = \Pi(\alpha,\theta+c)$$

$$f_{\widetilde{F}}(x) = \frac{\left(\frac{\theta}{x+\theta}\right)^\alpha}{\frac{\theta}{(\alpha-1)}} = \frac{(\alpha-1)\,\theta^{\alpha-1}}{(x+\theta)^{(\alpha-1)+1}} = f_{\Pi(\alpha-1,\theta)}(x)$$

$$\Rightarrow \widetilde{\Pi}(\alpha,\theta) = \Pi(\alpha-1,\theta)$$

$$\Rightarrow \text{ for every } k\in\mathbb{Z},\ \widetilde{\Pi}^{[k]}(\alpha,\theta) = \left\{\begin{array}{cc} \Pi(\alpha-k,\theta) & k<\alpha \\ \nexists & k\geq\alpha \end{array}\right\}.$$

More generally we have for $c > 0$

$$\begin{aligned} S_{\widetilde{F}^{[c]}}(x) &= \frac{\int_x^\infty (y-x)^c f_F(y)dy}{\int_0^\infty y^c f_F(y)dy} \\ &= \frac{\int_0^\infty z^c f_f(z+x)dz}{\int_0^\infty y^c f_F(y)dy} \\ &= \frac{\int_0^\infty z^c \frac{\alpha\theta^\alpha}{(z+x+\theta)^{\alpha+1}}dz}{\int_0^\infty y^c \frac{\alpha\theta^\alpha}{(y+\theta)^{\alpha+1}}dy} \end{aligned}$$

$$\begin{aligned} &= \frac{\left(\frac{\theta^\alpha}{(x+\theta)^\alpha}\right)\int_0^\infty z^c \frac{\alpha(x+\theta)^\alpha}{(z+x+\theta)^{\alpha+1}}dz}{\frac{\theta^c\Gamma(c+1)\Gamma(\alpha-c)}{\Gamma(\alpha)}} \\ &= \frac{\left(\frac{\theta}{x+\theta}\right)^\alpha \frac{(x+\theta)^c\Gamma(c+1)\Gamma(\alpha-c)}{\Gamma(\alpha)}}{\frac{\theta^c\Gamma(c+1)\Gamma(\alpha-c)}{\Gamma(\alpha)}} = \left(\frac{\theta}{x+\theta}\right)^{\alpha-c} \end{aligned}$$

$$\Rightarrow \text{ for every } c\in\mathbb{R},\ \widetilde{\Pi}^{(c)}(\alpha,\theta) = \left\{\begin{array}{cc} \Pi(\alpha-c,\theta) & c<\alpha \\ \textit{Does not exist} & c\geq\alpha \end{array}\right\}$$

and we see that in this case the natural parametrization of the orbit $\widetilde{F}^{[\mathbb{R}]}$ *relates linearly with the* α *parameter of the usual arithmetic formula and with an orbit corresponding to a fixed value of the* θ *parameter*

$$\begin{aligned} \widetilde{F}^{[\mathbb{R}]} &= \left\{\widetilde{F}^{[r]} | r\in\mathbb{R} \text{ such that } \mu_F^{(r)}<\infty\right\} \\ &= \{\Pi(\alpha-r,\theta)|r\in(0,\alpha)\} \\ &= \{\Pi(s,\theta)|s>0\}. \end{aligned}$$

Note too that for $a > 0$

$$F(x) = \Pi(\alpha, \theta; x)$$

$$\begin{aligned}\Rightarrow \quad F_a(x) = F(ax) &= 1 - \left(\frac{\theta}{ax+\theta}\right)^{\alpha} \\ &= 1 - \left(\frac{\frac{\theta}{a}}{x+\frac{\theta}{a}}\right)^{\alpha} = \Pi\left(\alpha, \frac{\theta}{a}; x\right)\end{aligned}$$

$$\Rightarrow \Pi(\alpha,\theta)_a = \Pi\left(\alpha, \frac{\theta}{a}\right)$$

and we see that the "half plane" $(0,\infty)\widetilde{F}^{[\mathbb{R}]}$ ~invariant subset here more resembles a "quadrant" and corresponds to the Pareto density "family" of distributions

$$(0,\infty)\widetilde{\Pi(\alpha,\theta)}^{[\mathbb{R}]} = (0,\infty)\widetilde{F}^{[\mathbb{R}]} = \{\Pi(s,t) | s > 0, t > 0\}.$$

Example 80 *Lognormal Density: Let $F = \Lambda(\mu,\sigma)$ have the Lognormal density with $F(x) = \Phi\left(\frac{\ln x - \mu}{\mu}\right)$. In this case:*

$$\omega_F = \infty, \tau_F = 0 \text{ and } \mu_F^{(n)} = e^{n\mu + \frac{n^2\sigma^2}{2}} < \infty \text{ for every } n \in \mathbb{N}$$

$$\Rightarrow \widetilde{F}^{[n]} \text{ exists for every } n \in \mathbb{N}.$$

We see from Proposition 40 that

$$\mu_{\widetilde{F}^{[n]}} = \frac{\mu_F^{(n+1)}}{(n+1)\,\mu_F^{(n)}} = \frac{e^{(n+1)\mu + \frac{(n+1)^2\sigma^2}{2}}}{(n+1)\,e^{n\mu+\frac{n^2\sigma^2}{2}}} = \frac{e^{\mu + \frac{(2n+1)\sigma^2}{2}}}{n+1}$$

$$\Rightarrow \lim_{n\to\infty} \mu_{\widetilde{F}^{[n]}} = 0.$$

Also, the mode of F is $e^{\mu-\sigma^2} > 0 \Rightarrow \nexists\ \widetilde{F}^{[-1]}$.

Perhaps the most useful example for the practical application of these ideas is:

Example 81 *Mixed Exponential Distribution: Let F be a mixture of exponential densities. More precisely, for some m, $1 \le m \le \infty$, and for any real weights $w_i > 0$ with $1 = \sum_{i=1}^m w_i$ and parameters $\mu_i > 0$ ordered so that $\mu_i < \mu_{i+1}$ and with $\sum_{i=1}^m w_i\mu_i < \infty$. Then consider the weighted mixture SLDFn variable $F = \daleth(m, \langle w_i\rangle, \langle \mu_i\rangle)$*

$$\begin{aligned}\daleth(m, \langle w_i\rangle, \langle \mu_i\rangle; x) &= F(x) = 1 - \sum_{i=1}^m w_i e^{-\frac{x}{\mu_i}}. \\ S_F(x) &= \sum_{i=1}^m w_i e^{-\frac{x}{\mu_i}}.\end{aligned}$$

Then we have, by Proposition 49

$$\begin{aligned} \mu_F &= \sum_{i=1}^{m} w_i \mu_i < \infty \\ S_{\widetilde{F}}(x) &= \frac{\sum_{i=1}^{m} w_i \mu_i e^{-\frac{x}{\mu_i}}}{\mu_F} \end{aligned}$$

and generally

$$S_{\widetilde{F}^{[n]}}(x) = \frac{\sum_{i=1}^{m} w_i \mu_i^n e^{-\frac{x}{\mu_i}}}{\sum_{i=1}^{m} w_i \mu_i^n} \text{ for every } n \in \mathbb{Z} \text{ such that } \sum_{i=1}^{m} w_i \mu_i^n < \infty.$$

Proposition 9 can be used to verify that λ_F *is decreasing provided* $m > 1$. *Indeed*

$$m > 1 \Rightarrow$$

$$\frac{1}{\mu_1} = \frac{1}{\mu_1}\frac{S_F(x)}{S_F(x)} = \frac{1}{\mu_1}\frac{\sum_{i=1}^{m} w_i e^{-\frac{x}{\mu_i}}}{S_F(x)}$$

$$= \frac{\sum_{i=1}^{m} \frac{w_i}{\mu_1} e^{-\frac{x}{\mu_i}}}{S_F(x)} > \frac{\sum_{i=1}^{m} \frac{w_i}{\mu_i} e^{-\frac{x}{\mu_i}}}{S_F(x)} = \frac{f_F(x)}{S_F(x)} = \lambda_F(x)$$

$$> \frac{\sum_{i=1}^{m} \frac{w_i}{\mu_m} e^{-\frac{x}{\mu_i}}}{S_F(x)} = \frac{1}{\mu_m}\frac{\sum_{i=1}^{m} w_i e^{-\frac{x}{\mu_i}}}{S_F(x)} = \frac{1}{\mu_m}\frac{S_F(x)}{S_F(x)} = \frac{1}{\mu_m}$$

$$\Rightarrow \frac{1}{\mu_1} > \lambda_F(x) > \frac{1}{\mu_m}$$

and similarly we find that

$$\frac{-\frac{df}{dx}}{S(x)} = \frac{\sum_{i=1}^{m} \frac{w_i}{\mu_i^2} e^{-\frac{x}{\mu_i}}}{S(x)} > \frac{\frac{1}{\mu_1}\sum_{i=1}^{m} \frac{w_i}{\mu_i} e^{-\frac{x}{\mu_i}}}{S(x)} = \frac{1}{\mu_1}\frac{f(x)}{S(x)} = \frac{1}{\mu_1}\lambda(x) > \lambda(x)^2$$

$$\Rightarrow 0 > \lambda(x)^2 + \frac{\frac{df}{dx}}{S(x)} = \frac{d\lambda}{dx}$$

$$\Rightarrow \lambda_F \text{ is decreasing}$$

as asserted. Note also that Corollary 51 implies that the $CV_F \geq 1$ *for any mixed exponential. In fact*

$$m > 1 \Rightarrow CV_F > 1 \Rightarrow \mu_{\widetilde{F}^{[n]}} < \mu_{\widetilde{F}^{[n+1]}} \text{ for every } n \in \mathbb{Z}.$$

When $1 < m < \infty$ *we clearly have*

$$\sum_{i=1}^{m} w_i \mu_i^n < \infty \text{ for every } n \in \mathbb{Z}$$

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and it follows that $\widetilde{F}^{[\mathbb{R}]} \cong \mathbb{R}$ *as ordered sets, with no first or last element. From Proposition 49 we see that*

$$\lim_{n\to\infty} \mu_{\widetilde{F}^{[n]}} = \mu_m \text{ and } \lim_{n\to\infty} \mu_{\widetilde{F}^{[-n]}} = \mu_1$$

and we see that for mixed exponentials there are readily identified limiting distributions equal to exponential distributions

$$\begin{aligned} \gimel(m, \widetilde{\langle w_i \rangle}, \langle \mu_i \rangle)^{[\infty]} &= \lim_{n\to\infty} \gimel(m, \widetilde{\langle w_i \rangle}, \langle \mu_i \rangle)^{[n]} = \gimel(1, 1, \mu_m) \\ \gimel(m, \widetilde{\langle w_i \rangle}, \langle \mu_i \rangle)^{[-\infty]} &= \lim_{n\to\infty} \gimel(m, \widetilde{\langle w_i \rangle}, \langle \mu_i \rangle)^{[-n]} = \gimel(1, 1, \mu_1) \end{aligned}$$

This also illustrates what was just established more generally for the case of decreasing hazard functions.

The next example generalizes the mixed exponential and illustrates a construction that is "dual" to that of the coderived distribution:

Example 82 *Let* $G = G(w)$ *be a (not necessarily continuous) LDFn with PDF*

$$g(w) = \frac{dG}{dw}$$

and finite mean $\mu_G < \infty$. *As above, there is the related LDFn* $\widehat{G}$ *with PDF*

$$\widehat{g}(w) = \frac{wg(w)}{\mu_G}$$

$$\widehat{G}(w) = \int_0^w \widehat{g}(z)dz = \frac{\int_0^w zg(z)dz}{\mu_G}$$

which conforms with our earlier notation and as before we set

$$\widehat{G} = \widehat{G}^{[1]}$$

$$\widehat{G}^{[k]} = \widehat{\widehat{G}^{[k-1]}} \text{ for } k \in \mathbb{N} \text{ and } \mu_{\widehat{G}^{[k-1]}} < \infty$$

$$\mu_{\widehat{G}^{[k]}} = \frac{\mu_G^{(k+1)}}{\mu_G^{(k)}}.$$

This relates with the mixed exponential coderived distribution via a Laplace-like transformation. Define

$$L_G(x) = \int_0^\infty e^{-xw} dG = \int_0^\infty e^{-xw} g(w)dw$$

$$\Rightarrow L_G(0) = \int_0^\infty g(w)dw = 1$$

and the function L_G resembles a survival function of a mixed exponential distribution. We set

$$\begin{aligned}\mathcal{L}^*(G) &= 1 - L_G \\ \mathcal{L}^*(G)(x) &= 1 - \int_0^\infty e^{-xw} g(w)dw\end{aligned}$$

which associates with the LDFn G another LDFn $F = \mathcal{L}^*(G)$. Now observe that, differentiating under the integral

$$\begin{aligned}f_F(x) &= \frac{d\mathcal{L}^*(G)}{dx} = -\int_0^\infty \frac{d}{dx}\left(e^{-xw}\right) g(w)dw \\ &= -\frac{\mu_G}{\mu_G}\int_0^\infty \left(e^{-xw}\right)(-w)\, g(w)dw \\ &= \mu_G \int_0^\infty e^{-xw}\widehat{g}(w)dw \\ &= \mu_G\left(1 - \mathcal{L}^*(\widehat{G})(x)\right).\end{aligned}$$

Let $H = \mathcal{L}^*(\widehat{G})$, we have for the PDF of $\widetilde{H} = \widetilde{\mathcal{L}^*(\widehat{G})}$

$$\begin{aligned}f_{\widetilde{H}} &= \frac{S_H}{\mu_H} = \frac{1-H}{\mu_H} \\ &= \frac{\mu_G(1-H)}{\mu_G\mu_H} \\ &= \frac{\mu_G\left(1-\mathcal{L}^*(\widehat{G})\right)}{\mu_G\mu_H} \\ &= \frac{f_F}{\mu_G\mu_H}.\end{aligned}$$

Now since both $f_{\widetilde{H}}$ and f_F are PDFs of the LDFns $\widetilde{F}$ and $\mathcal{L}(\widehat{G})$, respectively, it follows that

$$\begin{aligned}1 &= \int_0^\infty f_{\widetilde{H}}(x)dx = \int_0^\infty \frac{f_F(x)}{\mu_G\mu_H}dx = \frac{\int_0^\infty f_F(x)dx}{\mu_G\mu_H} = \frac{1}{\mu_G\mu_H} \\ &\Rightarrow \mu_G = \frac{1}{\mu_H} = \frac{1}{\mu_{\mathcal{L}^*(\widehat{G})}} \Rightarrow \mu_{\mathcal{L}^*(\widehat{G})} = \frac{1}{\mu_G} \\ &\Rightarrow f_{\widetilde{H}} = f_F \\ &\Rightarrow \mathcal{L}^*(G) = F = \widetilde{H} = \widetilde{\mathcal{L}^*(\widehat{G})}.\end{aligned}$$

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This may be summarized in the commutative diagram:

$$\begin{array}{ccc} G & \xrightarrow{\mathcal{L}^*} & F \\ \hat{}\downarrow & & \uparrow\sim \\ \widehat{G} & \xrightarrow{\mathcal{L}^*} & H \end{array}$$

which illustrates that the rather trivially "derived" construction $G \to \widehat{G}$ of the time-biased distribution is "dual" under $\mathcal{L}^*$ to determining a coderived or equilibrium distribution or equivalently to determining the excess ratio curve. We have

$$\mathcal{L}^*(\widehat{G}) = \mathcal{L}^*\widetilde{\left(\widehat{\widehat{G}}\right)} = \mathcal{L}^*\widetilde{\left(\widehat{G}^{[2]}\right)}$$

$$\Rightarrow \mathcal{L}^*(G) = \widetilde{\mathcal{L}^*(\widehat{G})} = \left(\mathcal{L}^*\widetilde{\widetilde{\left(\widehat{G}^{[2]}\right)}}\right) = \mathcal{L}^*\widetilde{\left(\widehat{G}^{[2]}\right)}^{[2]}$$

and by induction

$$\mathcal{L}^*(G) = \widetilde{\mathcal{L}^*(\widehat{G}^{[1]})}^{[1]} = \mathcal{L}^*\widetilde{\left(\widehat{G}^{[2]}\right)}^{[2]} = ... = \mathcal{L}^*\widetilde{\left(\widehat{G}^{[n]}\right)}^{[n]} \quad \textit{for every } n \in \mathbb{N}$$

$$\Rightarrow \widetilde{\mathcal{L}^*(G)}^{[-n]} = \mathcal{L}^*\left(\widehat{G}^{[n]}\right) \quad \textit{for every } n \in \mathbb{Z}.$$

Example 83 *Gamma Density: Let $F = \Gamma(\alpha, \theta)$ have the Gamma density:*

$$f_F(x) = \frac{\left(\frac{x}{\theta}\right)^{\alpha} e^{-\frac{x}{\theta}}}{x\Gamma(\alpha)}$$

$$\mu_F^{(k)} = \theta^k (\alpha + k - 1) \cdots \alpha \textit{ for } -\alpha < k \in \mathbb{Z}.$$

We see from Proposition 40 that

$$\mu_{\widetilde{F}^{[n]}} = \frac{\mu_F^{(n+1)}}{(n+1)\,\mu_F^{(n)}} = \frac{\theta^{n+1}(\alpha+n)\cdots\alpha}{(n+1)\,\theta^n(\alpha+n-1)\cdots\alpha} = \frac{\theta(\alpha+n)}{n+1}$$

$$\Rightarrow \lim_{n\to\infty} \mu_{\widetilde{F}^{[n]}} = \theta.$$

Letting

$$G = \widetilde{F}^{[\infty]} = \lim_{n\to\infty} \widetilde{F}^{[n]}$$

we have

$$\widetilde{G} = G \quad \textit{and } \mu_G = \theta \Rightarrow G = \gimel(1, \langle 1\rangle, \langle\theta\rangle)$$

and the limiting distribution is independent of α and is recognized as exponential of mean θ. Finally, observe that

$$\mathcal{L}^*(F)(x) = 1 - \int_0^\infty e^{-xw} f_F(w)dw = 1 - \int_0^\infty e^{-xw}\frac{\left(\frac{w}{\theta}\right)^{\alpha} e^{-\frac{w}{\theta}}}{w\Gamma(\alpha)}dw$$

$$= 1 - \int_0^\infty \frac{w^{\alpha-1} e^{-\frac{w}{\theta} - xw}}{\theta^\alpha \Gamma(\alpha)} dw = 1 - \int_0^\infty \frac{w^{\alpha-1} e^{-w\left(x+\frac{1}{\theta}\right)}}{\theta^\alpha \Gamma(\alpha)} dw$$

$$\begin{aligned} &= 1 - \int_0^\infty \frac{\left(\frac{u}{x+\frac{1}{\theta}}\right)^{\alpha-1} e^{-u}}{\left(x+\frac{1}{\theta}\right)\theta^\alpha \Gamma(\alpha)} du \quad \textit{where} \quad u = w\left(x + \frac{1}{\theta}\right) \\ &= 1 - \frac{\left(x+\frac{1}{\theta}\right)^{-\alpha}}{\theta^\alpha} \int_0^\infty \frac{u^{\alpha-1} e^{-u}}{\Gamma(\alpha)} du \end{aligned}$$

$$= 1 - \frac{1}{\left(x+\frac{1}{\theta}\right)^\alpha \theta^\alpha} = 1 - \left(\frac{1}{\theta x + 1}\right)^\alpha = 1 - \left(\frac{\frac{1}{\theta}}{x + \frac{1}{\theta}}\right)^\alpha$$

$$= \Pi(\alpha, \frac{1}{\theta}; x)$$

and we have that $\mathcal{L}^*(F) = \Pi(\alpha, \frac{1}{\theta})$. *And so for* $\alpha > 1$

$$f_{\widehat{F}}(x) = \frac{x f_F(x)}{\mu_F} = \frac{\left(\frac{x}{\theta}\right)^\alpha e^{-\frac{x}{\theta}}}{\theta \alpha \Gamma(\alpha)} = \frac{\left(\frac{x}{\theta}\right)^{\alpha+1} e^{-\frac{x}{\theta}}}{x \Gamma(\alpha+1)} = f_{\Gamma(\alpha+1,\theta)}(x)$$

$$\Rightarrow \widehat{F} = \Gamma(\alpha+1, \theta)$$

$$\Pi(\alpha, \frac{1}{\theta}) = \mathcal{L}^*(F) = \widetilde{\mathcal{L}^*(\widehat{F})} = \mathcal{L}^*(\widetilde{\Gamma(\alpha+1}, \theta)) = \Pi(\widetilde{\alpha+1}, \frac{1}{\theta})$$

as had already been observed in Example 79 above.

Example 84 *Weibull: Let* $F = W(\tau, \theta)$ *have the Weibull density:*

$$F(x) = W(\tau, \theta; x) = 1 - e^{-\left(\frac{x}{\theta}\right)^\tau} \quad \omega_F = \infty$$

$$f_F(x) = \frac{\tau \left(\frac{x}{\theta}\right)^\tau e^{-\left(\frac{x}{\theta}\right)^\tau}}{x}$$

$$\begin{aligned} \lambda_F(x) &= \frac{f_F(x)}{S_F(x)} = \frac{\tau \left(\frac{x}{\theta}\right)^\tau e^{-\left(\frac{x}{\theta}\right)^\tau}}{x e^{-\left(\frac{x}{\theta}\right)^\tau}} \\ &= \frac{\tau}{x}\left(\frac{x}{\theta}\right)^\tau = \frac{\tau x^{\tau-1}}{\theta^\tau} \end{aligned}$$

$$\tau_F = \left\{ \begin{array}{cc} 0 & \tau < 1 \\ \frac{1}{\theta} & \tau = 1 \\ \infty & \tau > 1 \end{array} \right\}$$

$$\mu_F^{(k)} = \theta^k \Gamma\left(1 + \frac{k}{\tau}\right), \; k > -\tau.$$

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Recall the definition of the incomplete gamma function:

$$\Gamma(\alpha; x) = \int_0^x \frac{t^{\alpha-1}e^{-t}}{\Gamma(\alpha)} dt$$

and define

$$G(x) = 1 - \Gamma\left(1 + \frac{1}{\tau}; \left(\frac{x}{\theta}\right)^{\tau}\right) + \frac{xe^{-\left(\frac{x}{\theta}\right)^{\tau}}}{\theta\Gamma\left(1+\frac{1}{\tau}\right)}$$

then

$$\begin{aligned}
\frac{dG}{dx} &= -\frac{-\tau\left(\frac{x}{\theta}\right)^{\tau\left(1+\frac{1}{\tau}\right)} e^{-\left(\frac{x}{\theta}\right)^{\tau}}}{x\Gamma\left(1+\frac{1}{\tau}\right)} + \frac{-x\frac{\tau\left(\frac{x}{\theta}\right)^{\tau} e^{-\left(\frac{x}{\theta}\right)^{\tau}}}{x} + e^{-\left(\frac{x}{\theta}\right)^{\tau}}}{\theta\Gamma\left(1+\frac{1}{\tau}\right)} \\
&= \frac{\tau\left(\frac{x}{\theta}\right)^{\tau+1} e^{-\left(\frac{x}{\theta}\right)^{\tau}}}{x\Gamma\left(1+\frac{1}{\tau}\right)} + \frac{-\tau\left(\frac{x}{\theta}\right)^{\tau} e^{-\left(\frac{x}{\theta}\right)^{\tau}} + e^{-\left(\frac{x}{\theta}\right)^{\tau}}}{\theta\Gamma\left(1+\frac{1}{\tau}\right)} \\
&= \frac{\tau\left(\frac{x}{\theta}\right)^{\tau} e^{-\left(\frac{x}{\theta}\right)^{\tau}}}{\theta\Gamma\left(1+\frac{1}{\tau}\right)} + \frac{-\tau\left(\frac{x}{\theta}\right)^{\tau} e^{-\left(\frac{x}{\theta}\right)^{\tau}} + e^{-\left(\frac{x}{\theta}\right)^{\tau}}}{\theta\Gamma\left(1+\frac{1}{\tau}\right)} \\
&= \frac{\tau\left(\frac{x}{\theta}\right)^{\tau} e^{-\left(\frac{x}{\theta}\right)^{\tau}} - \tau\left(\frac{x}{\theta}\right)^{\tau} e^{-\left(\frac{x}{\theta}\right)^{\tau}} + e^{-\left(\frac{x}{\theta}\right)^{\tau}}}{\theta\Gamma\left(1+\frac{1}{\tau}\right)} \\
&= \frac{e^{-\left(\frac{x}{\theta}\right)^{\tau}}}{\theta\Gamma\left(1+\frac{1}{\tau}\right)} = \frac{S_F(x)}{\mu_F} = f_{\widetilde{F}}(x) \\
&\Rightarrow G = \widetilde{F} = \widetilde{W(\tau,\theta)}.
\end{aligned}$$

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