Grouping Loss Distributions by Tail Behavior Part III: Ordering Distributions

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Abstract: *This three part paper addresses the task of modelling the right hand tail of a severity distribution. In Part I the excess ratio function is used to define a discrete sequence of loss distributions with related moments and similar tail behavior. Part II extends this to continuous one-parameter families and provides some examples. Part III provides the main result: that under some reasonable conditions, each such family has a limiting distribution which is exponential. The paper then exploits this to 1) group loss distributions based on tail behavior and 2) promote the choice of (mixed) exponentials to model tail behavior.*

Remark 85 *This is the final part of a three part paper. We assume familiarity with Parts I and II and continue our numbering from those earlier parts.*

6 Orbits and Tail Behavior

We have seen that the orbit $\widetilde{F}^{[\mathbb{R}]}$ of an SLDFn F says something about the existence of moments and the $\sim$invariant τ_F. In this section we investigate the structural possibilities for the orbits $\widetilde{F}^{[\mathbb{R}]}$ and relate it to analytic behavior naturally associated with tail behavior. We make the following:

Definition 86 *A C^{∞} function $T\colon [0,\infty) \to \mathbb{R}$ is* ***monotone of degree n*** *provided*

$$(-1)^k \frac{d^kT}{dx^k}(x) \geq 0 \text{ for } k = 0, 1, 2, ..., n \text{ and for every } x \in (0,\infty).$$

T is ***completely monotone*** *provided T is monotone of degree n for every $n \in \mathbb{N}$.*

Note that while the concept of monotone of degree n is peculiar to this paper, this is the standard definition of completely monotone (sometimes called totally monotone). As an immediate consequence of this definition we have:

Proposition 87 *For any SLDFn F:*

$$\widetilde{F}^{[-n]} \text{ exists for } n \in \mathbb{N} \Leftrightarrow S_F \text{ is monotone of degree } n \Leftrightarrow S_{\widetilde{F}} \text{ is monotone of degree } n{+}1$$

$$\widetilde{F}^{[-n]} \text{ exists for every } n \in \mathbb{N} \quad \Leftrightarrow \quad S_F \text{ is completely monotone.}$$

Proof. Clear from the definition of the backward coderived LDFn.

Example 88 *The survival function $S_F(x) = \sum_{i=1}^{m} w_i e^{-\frac{x}{\mu_i}}$ for a mixture of exponentials is completely monotone.*

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Proposition 89 *If $T(x) = \int_0^\infty e^{-xt} g(t)dt$ for some integrable function $g\colon (0,\infty) \to [0,\infty)$, then T is completely monotone.*

Proof. This follows from differentiation under the integral

$$\frac{d^n T}{dx^n}(x) = \frac{d^n \int_0^\infty e^{-xt} g(t)dt}{dx^n} = \int_0^\infty \frac{d^n g(t)e^{-xt}}{dx^n} dt$$

$$= \int_0^\infty g(t) \frac{d^n e^{-xt}}{dx^n} dt = (-1)^n \int_0^\infty t^n g(t) e^{-xt} dt$$

$$\Rightarrow (-1)^n \frac{d^n T}{dx^n}(x) = \int_0^\infty t^n g(t) e^{-xt} dt \geq 0$$

completing the proof. ■

Remark 90 *A theorem of Bernstein establishes the converse; and we will soon make use of that theorem.*

Example 91 *Consider the survival function $S_F(x) = e^{-\sqrt{x}}$. In this case we have ([1], #29.3.83, p. 1026)*

$$S_F(x) = e^{-\sqrt{x}} = \int_0^\infty e^{-xt} g(t)dt \text{ where } g(t) = \frac{e^{-\frac{1}{4t}}}{2\sqrt{\pi t^3}}$$

and so S_F is completely monotone. Observe that we also have

$$f_F(x) = -\frac{dS_F}{dx}(x) = \frac{e^{-\sqrt{x}}}{2\sqrt{x}}$$

$$\lambda_F(x) = \frac{1}{2\sqrt{x}}$$

$$\Rightarrow \tau_F = 0.$$

Note too that the SLDFn F has all finite moments by Proposition 40, since

$$\begin{aligned}
\mu_F^{(n)} &= n\int_0^\infty x^{n-1}S_F(x)dx = n\int_0^\infty x^{n-1}e^{-\sqrt{x}}dx \\
&= n\int_0^\infty u^{2n-2}e^{-u}2udu \text{ where } u=\sqrt{x}, x=u^2, dx=2udu \\
&= 2n\int_0^\infty u^{2n-1}e^{-u}du = 2n\,(2n-1)! < \infty.
\end{aligned}$$

The following generalizes an earlier observation on mixed exponentials:

Proposition 92 *For any SLDFn F, if the survival function $S_F(x)$ has the form*

$$S_F(x) = \int_0^\infty e^{-xt}g(t)dt$$

for some integrable function $g\colon (0,\infty) \to [0,\infty)$, then $CV_F \geq 1$.

Proof. Note first that

$$1 = S_F(0) = \int_0^\infty g(t)dt.$$

and so by Schwartz

$$\begin{aligned}
\left(\int_0^\infty \frac{g(t)}{t}dt\right)^2 &= \left(\int_0^\infty \sqrt{g(t)}\frac{\sqrt{g(t)}}{t}dt\right)^2 \\
&\leq \int_0^\infty \left(\sqrt{g(t)}\right)^2 dt \int_0^\infty \left(\frac{\sqrt{g(t)}}{t}\right)^2 dt \\
&= \int_0^\infty g(t)dt \int_0^\infty \frac{g(t)}{t^2}dt = \int_0^\infty \frac{g(t)}{t^2}dt.
\end{aligned}$$

Observe next that for any fixed $t > 0$, from what has been observed for the exponential distribution of parameter $\theta = \frac{1}{t}$ (example 76)

$$\begin{gathered}
1 = \int_0^\infty \frac{e^{-\frac{x}{\theta}}}{\theta}dx = t\int_0^\infty e^{-xt}dx \\
\Rightarrow \int_0^\infty e^{-xt}dx = \frac{1}{t} \\
\theta = \int_0^\infty x\frac{e^{-\frac{x}{\theta}}}{\theta}dx = t\int_0^\infty xe^{-xt}dx \\
\Rightarrow \int_0^\infty xe^{-xt}dx = \frac{\theta}{t} = \frac{1}{t^2}.
\end{gathered}$$

72

Now we compute, using Fubini

$$\begin{aligned}\mu_F &= \int_0^\infty S_F(x)dx = \int_0^\infty \int_0^\infty e^{-xt}g(t)dtdx \\ &= \int_0^\infty \int_0^\infty e^{-xt}g(t)dxdt = \int_0^\infty g(t)\int_0^\infty e^{-xt}dxdt \\ &= \int_0^\infty \frac{g(t)}{t}dt.\end{aligned}$$

Similarly, from Proposition 27

$$\begin{aligned}\mu_F^{(2)} &= 2\int_0^\infty xS_F(x)dx = 2\int_0^\infty x\int_0^\infty e^{-xt}g(t)dtdx \\ &= 2\int_0^\infty g(t)\int_0^\infty xe^{-xt}dxdt = 2\int_0^\infty \frac{g(t)}{t^2}dt.\end{aligned}$$

Now it follows that

$$\begin{aligned}&\sigma_F^2+\mu_F^2=\mu_F^{(2)} \\ &\Rightarrow CV_F^2+1=\frac{\sigma_F^2}{\mu_F^2}+1=\frac{\mu_F^{(2)}}{\mu_F^2} \\ &= \frac{2\int_0^\infty \frac{g(t)}{t^2}dt}{\left(\int_0^\infty \frac{g(t)}{t}dt\right)^2} \geq 2 \\ &\Rightarrow CV_F^2 \geq 1 \Rightarrow CV_F \geq 1\end{aligned}$$

as required. ■

We noted in the examples that the fixed points under the coderived loss construction are exactly the exponential densities. In fact, by a theorem of Serge Bernstein, we have the following characterization of the exponential survival curve that we will find useful and that may even be of some independent interest:

Proposition 93 *For any C^∞ function $T\colon (0,\infty)\to\mathbb{R}$:*

$$\left\{\begin{array}{c} T \text{ is completely monotone} \\ \\ 1=\int_0^\infty T(x)dx \\ \\ \text{There exists some } m\in\mathbb{N} \text{ such that} \\ (-1)^m \frac{d^mT}{dx^m}(x)=T(x) \text{ for every } x\in(0,\infty)\end{array}\right\} \Leftrightarrow T(x)=e^{-x}.$$

Proof. $\Leftarrow$) Clear.

$\Rightarrow$) We have already noted that the case $m = 1$ holds, so assume $m > 1$. Letting $f = -\frac{dT}{dx}(x)$ we clearly have

$$1 = \int_0^\infty T(x)dx \Rightarrow \lim_{x\to\infty} T(x) = 0 \text{ and so } f(x) \geq 0 \text{ with } 1 = \int_0^\infty f(x)dx$$

and $f = f_F$ is the PDF for an SLDFn F for which $T = S_F$. As per Proposition 87, since $T = S_F$ is completely monotone we have the series of backward coderived loss variables

$$\begin{aligned}
S_{\widetilde{F}^{[-1]}}(x) &= \frac{f(x)}{f(0)} = \frac{-\frac{dT}{dx}(x)}{-\frac{dT}{dx}(0)} = \frac{\frac{dT}{dx}(x)}{\frac{dT}{dx}(0)} \\
S_{\widetilde{F}^{[-2]}}(x) &= \frac{-\frac{d}{dx}\left(S_{\widetilde{F}^{[-1]}}\right)(x)}{-\frac{d}{dx}\left(S_{\widetilde{F}^{[-1]}}\right)(0)} = \frac{\frac{d^2T}{dx^2}(x)}{\frac{d^2T}{dx^2}(0)} \\
&\vdots \\
S_{\widetilde{F}^{[-k]}}(x) &= \frac{-\frac{d}{dx}\left(S_{\widetilde{F}^{[-k+1]}}\right)(x)}{-\frac{d}{dx}\left(S_{\widetilde{F}^{[-k+1]}}\right)(0)} = \frac{\frac{d^kT}{dx^k}(x)}{\frac{d^kT}{dx^k}(0)}
\end{aligned}$$

Let $G = \widetilde{F}^{[-m]}$. We have

$$S_G = S_{\widetilde{F}^{[-m]}} = \frac{\frac{d^mT}{dx^m}}{\frac{d^mT}{dx^m}(0)} = \frac{T}{T(0)} = S_F$$

$$\Rightarrow G = F$$

$$\Rightarrow \widetilde{G}^{[k]} = \left(\widetilde{\widetilde{F}^{[-m]}}\right)^{[k]} = \widetilde{F}^{[k-m]}, \text{ for every } k \in \mathbb{Z}.$$

Now Bernstein's theorem implies that since T, $\frac{dT}{dx}$, $\frac{d^2T}{dx^2}$,...are all completely monotone, we can represent each of the $S_{\widetilde{F}^{[k]}}$ as a Laplace transform, as in Proposition 92 from which we conclude from Proposition 92 that $CV_{\widetilde{F}^{[k]}} \geq 1$ for every $k \in \mathbb{Z}$. But then by Proposition 46

$$\mu_F \leq \mu_{\widetilde{F}} \leq \mu_{\widetilde{F}^{[2]}} \leq ... \leq \mu_{\widetilde{F}^{[m]}} = \mu_F$$

$$\Rightarrow \mu_{\widetilde{F}^{[k]}} = \mu_F = \int_0^\infty T(x)dx = 1, \text{ for every } k \in \mathbb{Z}.$$

We claim that $\mu_F^{(k)} = k!$ for every $k \in \mathbb{N}$. We verify this by induction. Indeed we just observed the case $n = 1$ and by Proposition 40 and the induction hypothesis

$$1 = \mu_{\widetilde{F}^{[k]}} = \frac{\mu_F^{(k+1)}}{(k+1)\,\mu_F^{(k)}} = \frac{\mu_F^{(k+1)}}{(k+1)\,k!} = \frac{\mu_F^{(k+1)}}{(k+1)!}$$

$$\Rightarrow \mu_F^{(k+1)} = (k+1)!.$$

74

It only remains to observe that

$$\mu_F^{(k)} = k! \text{ for every } k \in \mathbb{N} \cup \{0\} \Rightarrow L_F(t) = \frac{1}{1+t} = L_{\daleth(1,\langle 1\rangle,\langle 1\rangle)}(t)$$
$$\Rightarrow F = \daleth(1, \langle 1\rangle, \langle 1\rangle)$$
$$\Rightarrow T = S_F = S_{\daleth(1,\langle 1\rangle,\langle 1\rangle)} = e^{-x}$$

completing the proof. ■

Lemma 94 *If $c > 0$ is a fixed irrational number and $g\colon [0,\infty) \to \mathbb{R}$ is a continuous function satisfying*

$$g(n) \le g(n+1) \text{ for every } n \in \mathbb{N} \text{ and } g(x) = g(x+c) \text{ for every } x \in [0,\infty),$$

then g is constant, i.e., $g(x) = g(0)$ for every $x \in [0,\infty)$.

Proof. Consider the equivalence relation $\equiv$ on $[0,\infty)$ defined by

$$x \equiv y \Leftrightarrow \frac{x-y}{c} \in \mathbb{Z}.$$

Note that because $g(x) = g(x+c)$ for every $x \in [0,\infty)$, the function g is a continuous function well-defined on the equivalence classes of $[0,\infty)$. Note that

$$x \equiv x_1 \text{ and } y \equiv y_1$$
$$\Rightarrow \frac{x-x_1}{c} = z \in \mathbb{Z} \text{ and } \frac{y-y_1}{c} = w \in \mathbb{Z}$$
$$\text{but then } \frac{(x+y)-(x_1+y_1)}{c} = z+w \in \mathbb{Z}$$
$$\Rightarrow x+y \equiv x_1+y_1$$
$$\text{and } \frac{(x-y)-(x_1-y_1)}{c} = z-w \in \mathbb{Z}$$
$$\Rightarrow x-y \equiv x_1-y_1.$$

We claim that the sequence $A = \{\overline{n} | n \in \mathbb{N}, \overline{n} \in [0,c)\}$ of equivalence class representatives is dense in $[0,c)$. Assume given $d \in [0,c)$ and $0 < \epsilon_1 < c-d$. We have

$$\text{for every } n,m \in \mathbb{N},\quad n \equiv m \ne n \Rightarrow 0 \ne \frac{m-n}{c} = z \in \mathbb{Z} \Rightarrow c = \frac{m-n}{z} \in \mathbb{Q}, \text{ a contradiction} \Rightarrow\Leftarrow$$
$$\Rightarrow \text{ sequence } A \text{ has distinct numbers in compact set } [0,c]$$
$$\Rightarrow A \text{ has a cluster point.}$$

Since there is a cluster point and the elements of A are distinct, it follows that

$$\text{there exist } m,n \in \mathbb{N} \text{ such that } m > n,\ \overline{m}, \overline{n} \in [0,c) \text{ and } |\overline{m} - \overline{n}| < \frac{\epsilon_1}{4}$$

$$\Rightarrow \text{there exists } l \in \mathbb{N} \text{ such that } \sum_{k=1}^{l} (\overline{m} - \overline{n}) \in (d - \epsilon_1, d + \epsilon_1) \subset [0, c)$$

$$\text{but } \sum_{k=1}^{l} (\overline{m} - \overline{n}) = \overline{\sum_{k=1}^{l} (m - n)} = \overline{l\,(m - n)}$$

$$l\,(m-n) \in \mathbb{N} \Rightarrow \phi \neq A \cap (d - \epsilon_1, d + \epsilon_1) \cap [0, c)$$

and so A is dense in $[0, c)$ as claimed. Now since g is continuous

$$A \text{ dense in } [0, c)$$

$$\Rightarrow g(A) \text{ dense in } \{g(x)\,|x \in [0, c)\} = \{g(x)\,|x \in [0, \infty)\} = \mathrm{Im}(g).$$

Note that since g is continuous on the compact set $[0, c]$, we know that g has a maximum. But then by our assumptions, the sequence $\{g(n)|n \in \mathbb{N}\}$ is nondecreasing and bounded above. So we can set

$$\lim_{n \to \infty} g(n) = \alpha < \infty.$$

Now assume given any $\epsilon_2 > 0$, it follows that

$$\text{there exists } M \in \mathbb{N} \text{ such that } g(n) \in (\alpha - \epsilon_2, \alpha] \text{ for every } n > M$$

$$g \text{ periodic} \Rightarrow g(n) \in (\alpha - \epsilon_2, \alpha] \text{ for every } n \in \mathbb{N}$$

$$\{g(n)|n \in \mathbb{N}\} \text{ dense in } \mathrm{Im}(g) \Rightarrow \mathrm{Im}(g) \subseteq (\alpha - \epsilon_2, \alpha].$$

But since $\epsilon_2 > 0$, was arbitrary, we have

$$\{g(n)|n \in \mathbb{N}\} \subseteq \bigcap_{n \in \mathbb{N}} (\alpha - \frac{1}{n}, \alpha] = [\alpha, \alpha]$$

$$\{g(n)|n \in \mathbb{N}\} \text{ dense in } \mathrm{Im}(g) \Rightarrow \mathrm{Im}(g) \subseteq \overline{\{g(n)|n \in \mathbb{N}\}} \subseteq \overline{[\alpha, \alpha]} = [\alpha, \alpha]$$

$$\mathrm{Im}(g) \subseteq [\alpha, \alpha] \Longrightarrow g(x) = \alpha = g(0) \text{ for every } x \in [0, \infty)$$

and the proof is complete. ■

Theorem 95 *For any SLDFn F with finite mean, the following are equivalent:*

1. there exists $r \in \mathbb{R}, r > 0$ such that $F = \widetilde{F}^{[r]}$
2. $S_F(x) = e^{-\frac{x}{\mu}}$
3. $\widetilde{F}^{[\mathbb{R}]} = \{F\}$

Proof. Suppose $F = \widetilde{F}^{[r]}$ for some $r > 0$ We first claim that $\omega_F = \infty$. Indeed, if $\omega_F = b < \infty$. then clearly $\mu_F^{(k)} < \infty$ for every $k \in \mathbb{N}$ and by the intermediate value theorem

$$\begin{aligned}
\text{there exist } a_k &\in (0,b] \text{ such that } a_1 = \mu_F = \mu_F^{(1)} \\
\text{and } \mu_F^{(k+1)} &= \int_0^b x^{k+1} f_F(x)dx = \int_0^b x\left(x^k f_F(x)\right)dx \\
&= a_k \int_0^b x^k f_F(x)dx = a_k \mu_F^{(k)} \\
&\Rightarrow \quad \mu_{\widetilde{F}^{[k]}} = \frac{\mu_F^{(k+1)}}{(k+1)\,\mu_F^{(k)}} \\
&= \frac{a_k \mu_F^{(k)}}{(k+1)\,\mu_F^{(k)}} \\
&= \frac{a_k}{(k+1)} \le \frac{b}{(k+1)} \\
&\Rightarrow \quad \lim_{k\to\infty} \mu_{\widetilde{F}^{[k]}} = 0
\end{aligned}$$

But $F = \widetilde{F}^{[r]} \Rightarrow \mu_{\widetilde{F}^{[kr]}} = \mu_F > 0$ for all $k \in \mathbb{N}$. Since the function $m(c) = \mu_{\widetilde{F}^{[c]}}$ is evidently continuous, we have

$$\begin{aligned}
0 &= \lim_{k\to\infty} \mu_{\widetilde{F}^{[k]}} = \lim_{c\to\infty} \mu_{\widetilde{F}^{[c]}} \\
&= \lim_{k\to\infty} \mu_{\widetilde{F}^{[kr]}} = \lim_{k\to\infty} \mu_F = \mu_F > 0 \quad \Rightarrow\Leftarrow
\end{aligned}$$

This contradiction implies that $\omega_F = \infty$ and $S_F\colon (0,\infty) \to \mathbb{R}$ is a C^∞ function. We now prove that $S_F(x) = e^{-\frac{x}{\mu}}$. Consider first the case $r = m \in \mathbb{N}$. Assume that $F = \widetilde{F}^{[\mathbb{R}]}$ and let $G = F_\mu$. Then by Proposition 48 and the fact that

$$\begin{aligned}
\widetilde{G}^{[m]} &= \widetilde{(F_\mu)}^{[m]} = \left(\widetilde{F}^{[m]}\right)_\mu = F_\mu = G \\
&\Rightarrow \quad \widetilde{G}^{[-m]} = G \\
&\Rightarrow \quad S_G = S_{\widetilde{G}^{[-m]}}.
\end{aligned}$$

Since S_G is clearly completely monotone, it follows from the same CV argument as in the proof of Proposition 93 that

$$\begin{aligned}
1 &= \mu_G = \mu_{\widetilde{G}^{[k]}} \quad \text{for every } k \in \mathbb{Z} \\
&\Rightarrow \quad f_{\widetilde{G}^{[k]}}(0) = 1 \quad \text{for every } k \in \mathbb{Z} \\
&\Rightarrow \quad S_G = S_{\widetilde{G}^{[-m]}} = \frac{(-1)^m \frac{d^m S_G}{dx^m}}{f_{\widetilde{G}^{[-m+1]}}(0)} = (-1)^m \frac{d^m S_G}{dx^m}.
\end{aligned}$$

and so the characterization of that lemma implies that

$$\begin{aligned} S_G(y) &= e^{-y} \\ F &= G_{\frac{1}{\mu}} \\ &\Rightarrow S_F(x) = S_{G_{\frac{1}{\mu}}}(x) = S_G\left(\frac{x}{\mu}\right) = e^{-\frac{y}{\mu}}. \end{aligned}$$

This completes the proof for $m \in \mathbb{N}$. Consider next the case $r \in \mathbb{Q}$ let $r = \frac{n}{m}$, with $m, n \in \mathbb{N}$.and $F = \widetilde{F}^{[r]}$ We claim that

$$F = \widetilde{F}^{\left[\frac{kn}{m}\right]} \text{ for every } k \in \mathbb{N}.$$

This is a simple verification by induction, for $k = 1$ this reduces to $F = \widetilde{F}^{\left[\frac{n}{m}\right]}$, which is true, and then

$$\begin{aligned} F &= \widetilde{F}^{\left[\frac{kn}{m}\right]} \\ &\Rightarrow \widetilde{F}^{\left[\frac{(k+1)n}{m}\right]} = \widetilde{F}^{\left[\frac{(k+1)n}{m}\right]} = \widetilde{F}^{\left[\frac{kn}{m}+\frac{n}{m}\right]} = \widetilde{F}^{\left[\frac{kn}{m}+r\right]} \\ &= \widetilde{\widetilde{F}^{\left[\frac{kn}{m}\right]}}^{[r]} = \widetilde{F}^{[r]} = F \end{aligned}$$

completing the induction. But then it follows that

$$F = \widetilde{F}^{\left[\frac{mn}{m}\right]} = \widetilde{F}^{[n]} \text{ for } n \in \mathbb{N} \Rightarrow S_F(x) = e^{-\frac{x}{\mu}}$$

by the case $r = m \in \mathbb{N}$, completing the proof in the rational case. Finally, consider next the case $r \in \mathbb{R}$ with r irrational. As above, the assumptions imply that we can represent each of the $S_{\widetilde{F}^{[c]}}$ as a Laplace transform, as in Proposition 92 from which we conclude that $CV_{\widetilde{F}^{[c]}} \geq 1$ for every $c \in \mathbb{R}$ We clearly have $\mu_{\widetilde{F}^{[c]}} < \infty$ for every $c \geq 0$ so we define

$$g(c) = \mu_{\widetilde{F}^{[c]}} \text{ for } c \geq 0.$$

Then g is continuous on $[0, \infty)$ and by Proposition 46

$$\begin{aligned} g(c) &= \mu_{\widetilde{F}^{[c]}} \leq \mu_{\widetilde{\widetilde{F}^{[c]}}} = \mu_{\widetilde{F}^{[c+1]}} = g(c+1) \\ g(c+r) &= \mu_{\widetilde{F}^{[r+c]}} = \mu_{\widetilde{\widetilde{F}^{[r]}}^{[c]}} = \mu_{\widetilde{F}^{[c]}} = g(c). \end{aligned}$$

And so the lemma implies that

$$\mu_{\widetilde{F}^{[c]}} = g(c) = g(0) = \mu_{\widetilde{F}^{[0]}} = \mu_F = \mu \text{ for every } c \geq 0.$$

We claim that $\mu_F^{(k)} = k!\mu^k$ for every $k \in \mathbb{N}$. We verify this by induction. Indeed the case $n = 1$ being apparent. By Proposition 40 and the induction hypothesis

$$\mu = \mu_{\widetilde{F}^{[k]}} = \frac{\mu_F^{(k+1)}}{(k+1)\,\mu_F^{(k)}} = \frac{\mu_F^{(k+1)}}{(k+1)\,k!\mu^k} = \frac{\mu_F^{(k+1)}}{(k+1)!\mu^k}$$

78

$$\Rightarrow (k+1)!\mu^{k+1} = \mu_F^{(k+1)}$$

completing the induction. It only remains to observe that

$$\mu_F^{(k)} = k!\mu^k \text{ for every } k \in \mathbb{N} \cup \{0\} \Rightarrow L_F(t) = L_{\daleth(1,\langle 1\rangle,\langle \mu\rangle)}(t)$$
$$\Rightarrow F = \daleth(1, \langle 1\rangle, \langle \mu\rangle)$$
$$\Rightarrow T(x) = S_F(x) = S_{\daleth(1,\langle 1\rangle,\langle \mu\rangle)}(x) = e^{-\frac{x}{\mu}}.$$

We have shown

$$\left(\text{there exists some } r \in \mathbb{R}, r > 0 \text{ such that } F = \widetilde{F}^{[r]}\right) \Rightarrow \left(S_F(x) = e^{-\frac{x}{\mu}}\right)$$

but clearly

$$\left(S_F(x) = e^{-\frac{x}{\mu}}\right) \Rightarrow \left(\widetilde{F}^{[\mathbb{R}]} = \{F\}\right)$$
$$\text{and} \quad \left(\widetilde{F}^{[\mathbb{R}]} = \{F\}\right) \Rightarrow \left(\text{there exists } r \in \mathbb{R}, r > 0 \text{ such that } F = \widetilde{F}^{[r]}\right)$$

and the result follows. ■

We observe that, except when $\widetilde{F}^{[\mathbb{R}]} = \{F\}$ is the singleton orbit of an exponential, once we have selected an SLDFn $G \in \widetilde{F}^{[\mathbb{R}]}$ the elements $H \in \widetilde{F}^{[\mathbb{R}]}$ are uniquely expressible in the form $H = \widetilde{G}^{[c]}$ in the sense $c = c(H)$ is uniquely determined. It is most natural to just take $G = F$. This enables us to describe the possibilities for the structure of the orbit $\widetilde{F}^{[\mathbb{R}]}$ as related to a subset of $\mathbb{R}$, an interval actually, via the bijection

$$\Omega \colon \widetilde{F}^{[\mathbb{R}]} \to \Omega\left(\widetilde{F}^{[\mathbb{R}]}\right) \subseteq \mathbb{R} \qquad \text{where } \Omega\left(\widetilde{F}^{[c]}\right) = c \in \mathbb{R}.$$

In effect, this is a canonical 1-dimensional continuous parametrization of the orbits $\widetilde{F}^{[\mathbb{R}]}$ of non-exponential SLDFns F. We summarize this observation in:

Proposition 96 *For any SLDFn $F \neq \daleth(1, \langle 1\rangle, \langle \mu_F\rangle)$ with finite mean, $[0, 1] \subseteq \Omega\left(\widetilde{F}^{[\mathbb{R}]}\right)$ and the possibilities for $\Omega\left(\widetilde{F}^{[\mathbb{R}]}\right)$ and τ_F are:*

1. there exist $c, d \in \mathbb{R}$ with $\Omega\left(\widetilde{F}^{[\mathbb{R}]}\right) \in \{[c, , d], [c, d), (c, d], (c, d)\}, \tau_F = 0$

   $\Leftrightarrow$there exists $n \in \mathbb{N}$ such that $\mu_F^{(n)} = \infty$ and S_F is not completely monotone.

2. there exists $c \in \mathbb{R}$ with $\Omega\left(\widetilde{F}^{[\mathbb{R}]}\right) \in \{[c, \infty), (c, \infty)\}, c \leq 0, \tau_F \in [0, \infty]$

   $\Leftrightarrow \mu_F^{(n)} < \infty$ for every $n \in \mathbb{N}$ and S_F is not completely monotone.

3. there exists $d \in \mathbb{R}$ with $\Omega\left(\widetilde{F}^{[\mathbb{R}]}\right) \in \{(-\infty, d], (-\infty, d)\}, d > 0, \tau_F = 0$

   $\Leftrightarrow$there exists $n \in \mathbb{N}$ such that $\mu_F^{(n)} = \infty$ and S_F is completely monotone.

79

4. $\Omega\left(\widetilde{F}^{[\mathbb{R}]}\right) = (-\infty, \infty), \tau_F \in (0, \infty)$
   $\Leftrightarrow \mu_F^{(n)} < \infty$ for every $n \in \mathbb{N}$ and S_F is completely monotone.

With all possibilities actually occurring. Indeed we have:

1. Let $F(x) = \frac{x^2}{1+x^2}$. We have

$$
\begin{aligned}
S_F(x) &= \frac{1}{1+x^2} \\
\mu_F &= \frac{\pi}{2} < \infty \\
f_F(x) &= \frac{2x}{\left(1+x^2\right)^2} \\
\mu_F^{(a)} &= 2\int_0^\infty \frac{x^{a+1}}{x^4+2x^2+1} dx < \infty \\
&\Rightarrow a+1 < 3 \Rightarrow a < 2 \\
&\Rightarrow d \leq 2 \\
\frac{df_F}{dx} &= 2\frac{1-3x^2}{\left(1+x^2\right)^3} \Rightarrow \text{there exists a mode at } \frac{1}{\sqrt{3}} > 0 \\
&\Rightarrow c > -1.
\end{aligned}
$$

2. Lognormal or any loss distribution with finite support and mode > 0.
3. Pareto.
4. Mixed Exponential.

The following is also clear from the above:

Proposition 97 $\Omega\left(\widetilde{F}^{[\mathbb{R}]}\right) = (\infty, \infty) \Leftrightarrow$ *there exists LDFn G with $F = \mathcal{L}^*(G)$ and $\mu_G^{(n)} < \infty$ for every $n \in \mathbb{N}$.*

7 Ordering Loss Distributions

In this section we introduce a way to order SLDFns based on differences between hazard rate functions. We then relate this with the orbit structure of the previous section.

Proposition 98 *For any SLDFns F and G with $\omega_F = \omega_G$:*

$$
\lim_{x\to\omega_F} \frac{f_F(x)}{f_G(x)} = \lim_{x\to\omega_F} \frac{S_F(x)}{S_G(x)} = e^{\int_0^{\omega_F}(\lambda_G-\lambda_F)(t)dt}
$$

80

Proof. All but the last equality is clear from l'Hôpital, but then

$$\begin{aligned}
\lim_{x\to\omega_F}\frac{S_F(x)}{S_G(x)} &= \lim_{x\to\omega_F}\frac{e^{-\int_0^x\lambda_F(t)dt}}{e^{-\int_0^x\lambda_G(t)dt}} \\
&= \lim_{x\to\omega_F} e^{-\int_0^x\lambda_F(t)dt+\int_0^x\lambda_G(t)dt} \\
&= \lim_{x\to\omega_F} e^{\int_0^x(\lambda_G-\lambda_F)(t)dt} \\
&= e^{\lim\limits_{x\to\omega_F}\int_0^x(\lambda_G-\lambda_F)(t)dt} \\
&= e^{\int_0^{\omega_F}(\lambda_G-\lambda_F)(t)dt}
\end{aligned}$$

as required. ■

Definition 99 *For two SLDFns F and G set*

$$\upsilon(F,G)=e^{\int_0^{Min(\omega_F,\omega_G)}(\lambda_G-\lambda_F)(t)dt}.$$

Provided $\upsilon(F,G)$ exists, define the relations ***thicker than*** *and* ***strictly thicker than*** *by*

$$\begin{aligned}
F &\succeq G\Leftrightarrow\omega_F\geq\omega_G \text{ or } (\omega_F=\omega_G \text{ and } \upsilon(F,G)\geq 1) \\
F &\succ G\Leftrightarrow\omega_F>\omega_G \text{ or } (\omega_F=\omega_G \text{ and } \upsilon(F,G)>1).
\end{aligned}$$

Remark 100 *Note that*

$$\begin{aligned}
\omega_F &= \omega_G=\infty \text{ and } \tau_F<\tau_G\Rightarrow\int_0^{\omega_F}(\lambda_G-\lambda_F)(t)dt=\infty\Rightarrow\upsilon(F,G)=e^{\infty}=\infty>1 \\
&\Rightarrow F\succ G.
\end{aligned}$$

Example 101 *Let $F(x)=1-(x+1)e^{-x}$. We have*

$$S_F(x)=(x+1)e^{-x}$$

$$f_F(x)=-\frac{dS_F}{dx}(x)=-\left((x+1)e^{-x}(-1)+e^{-x}\right)=xe^{-x}$$

$$\lambda_F(x)=\frac{f_F(x)}{S_F(x)}=\frac{xe^{-x}}{(x+1)e^{-x}}=\frac{x}{x+1}\Rightarrow\tau_F=1$$

$$\mu_F=\int_0^{\infty}xf_F(x)dx=\int_0^{\infty}x^2e^{-x}dx=2$$

$$f_{\widetilde{F}}(x)=\frac{S_F(x)}{\mu_F}=\frac{(x+1)e^{-x}}{2}$$

$$\upsilon(F,\widetilde{F})=\lim_{x\to\infty}\frac{f_F(x)}{f_{\widetilde{F}}(x)}=\lim_{x\to\infty}\frac{xe^{-x}}{\frac{(x+1)e^{-x}}{2}}=2\lim_{x\to\infty}\frac{x}{x+1}=2>1$$

$$\Rightarrow F\succ\widetilde{F}.$$

81

Example 102 *Let* $F(x) = 1 - \frac{e^{-x}}{x+1}$*. We have*

$$S_F(x) = \frac{e^{-x}}{x+1}$$

$$f_F(x) = -\frac{dS_F}{dx}(x) = -\left(\frac{(x+1)\,e^{-x}(-1) - e^{-x}}{(x+1)^2}\right) = \frac{(x+2)\,e^{-x}}{(x+1)^2}$$

$$\lambda_F(x) = \frac{f_F(x)}{S_F(x)} = \frac{\frac{(x+2)e^{-x}}{(x+1)^2}}{\frac{e^{-x}}{x+1}} = \frac{x+2}{x+1} \Rightarrow \tau_F = 1$$

$$\begin{aligned}\mu_F &= \int_0^\infty S_F(x)dx = \int_0^\infty \frac{e^{-x}}{x+1}dx = \int_1^\infty \frac{e^{-u+1}}{u}du \\ &= e\int_1^\infty \frac{e^{-u}}{u}du < e\int_1^\infty e^{-u}du = e\left[-e^{-u}\right]_1^\infty = \frac{e}{e} = 1\end{aligned}$$

$$f_{\widetilde{F}}(x) = \frac{S_F(x)}{\mu_F} = \frac{\frac{e^{-x}}{x+1}}{\mu_F}$$

$$\upsilon(F,\widetilde{F}) = \lim_{x\to\infty}\frac{f_F(x)}{f_{\widetilde{F}}(x)} = \lim_{x\to\infty}\frac{\frac{(x+2)e^{-x}}{(x+1)^2}}{\frac{\frac{e^{-x}}{x+1}}{\mu_F}} = \mu_F\lim_{x\to\infty}\frac{x+2}{x+1} = \mu_F < 1$$

$$\Rightarrow F \prec \widetilde{F}.$$

Proposition 103 *Given SLDFns* F *and* G *with* $\omega_F = \omega_G$ *and constants* $a, b > 0$ *such that the limit* $\rho_F(\frac{a}{b}) = \lim_{x\to\omega_F}\frac{S_F(\frac{a}{b}x)}{S_F(x)}$ *exists. Then*

$$\upsilon(F_a, G_b) = \rho_F(\frac{a}{b})\upsilon(F,G)$$

Proof. We have

$$\begin{aligned}\upsilon(F_a, G_b) &= \lim_{x\to\omega_F}\frac{S_{F_a}(x)}{S_{G_b}(x)} = \lim_{x\to\omega_F}\frac{S_F(ax)}{S_G(bx)} \\ &= \lim_{x\to\omega_F}\frac{S_F\left(a(\frac{x}{b})\right)}{S_G\left(b(\frac{x}{b})\right)} = \lim_{x\to\omega_F}\frac{S_F\left((\frac{a}{b})x\right)}{S_G(x)} \\ &= \lim_{x\to\omega_F}\frac{S_F((\frac{a}{b})x)}{S_F(x)}\frac{S_F(x)}{S_G(x)} \\ &= \lim_{x\to\omega_F}\frac{S_F((\frac{a}{b})x)}{S_F(x)}\lim_{x\to\omega_F}\frac{S_F(x)}{S_G(x)} \\ &= \rho_F(\frac{a}{b})\upsilon(F,G)\end{aligned}$$

as required. ■

Recall the following from set theory:

Definition 104 *Given a set A with relation $\geqslant$, A is* ***partially ordered*** *under $\geqslant$ provided for every $a, b, c \in A$*

1. *(reflexive)* $a \geqslant a$
2. *(antisymmetric)* $(a \geqslant b$ *and* $b \geqslant a) \Rightarrow a = b$
3. *(transitive)* $(a \geqslant b$ *and* $b \geqslant c) \Rightarrow a \geqslant c$

It is straightforward to verify that $\succeq$ defines a partial order relation on the equivalence classes of SLDFns modulo the equivalence relation

$$F \approx G \Leftrightarrow \omega_F = \omega_G \text{ and } \upsilon(F,G) = 1.$$

As usual, the case in which the hazard function is either increasing or decreasing is especially easy:

Proposition 105 *If F is any SLDFn with finite mean and $\tau_F > 0$, then for all $m < n \in \mathbb{N}$:*

$$\begin{aligned} \lambda_F \textit{ increasing} &\quad\Rightarrow\quad \widetilde{F}^{[m]} \succ \widetilde{F}^{[n]} \\ \lambda_F \textit{ decreasing} &\quad\Rightarrow\quad \widetilde{F}^{[n]} \succ \widetilde{F}^{[m]}. \end{aligned}$$

Proof. Clear from Propositions 33 and 32. Observe that λ_F increasing or decreasing implies that $\lambda_F > 0$ on (α_F, ω_F). Now, all moments are finite, so $\widetilde{F}^{[n]}$ exists and so the assertion at least makes sense. We have

$$\begin{aligned} &\lambda_F \text{ increasing } \Rightarrow \lambda_{\widetilde{F}^{[m]}} \text{ increasing} \\ &\Rightarrow \lambda_{\widetilde{F}^{[n]}} = \lambda_{\widetilde{\widetilde{F}^{[n-1]}}} > \lambda_{\widetilde{F}^{[n-1]}} \geq \lambda_{\widetilde{F}^{[m]}} \text{ on } (\alpha_F, \omega_F) \\ &\Rightarrow \int_{\alpha_F}^{\omega_F} \left(\lambda_{\widetilde{F}^{[n]}} - \lambda_{\widetilde{F}^{[m]}}\right)(t)dt > 0 \\ &\Rightarrow \upsilon(\widetilde{F}^{[m]}, \widetilde{F}^{[n]}) = e^{\int_{\alpha_F}^{\omega_F}\left(\lambda_{\widetilde{F}^{[n]}} - \lambda_{\widetilde{F}^{[m]}}\right)(t)dt} > e^0 = 1 \\ &\Rightarrow \widetilde{F}^{[m]} \succ \widetilde{F}^{[n]} \end{aligned}$$

as asserted. The result for λ_F decreasing follows similarly, reversing inequalities. ■

Proposition 106 *For any SLDFns F and G with $\omega_F = \omega_G = \infty$*

$$\begin{aligned} \upsilon &= \upsilon(F,G) < \infty \\ &\Rightarrow \textit{ for every } \epsilon > 0 \textit{ there exists an } M \textit{ such that } |S_F(x) - \upsilon S_G(x)| < \epsilon \textit{ for every } x > M. \end{aligned}$$

Proof. Clear. Given $\epsilon > 0$

$$\omega_G = \infty \Rightarrow S_G(x) > 0 \text{ for every } x \geq 0$$

$$0 = \lim_{x\to\infty} S_G(x)$$

$$\Rightarrow \text{there exists } M_1 \text{ such that } 0 < |S_G(x)| < \sqrt{\epsilon} \text{ for every } x > M_1$$

$$\upsilon = \upsilon(F,G) = \lim_{x\to\infty} \frac{S_F(x)}{S_G(x)} < \infty$$

$$\Rightarrow \text{there exists } M_2 \text{ such that } 0 \leq \left|\frac{S_F(x)}{S_G(x)} - \upsilon\right| < \sqrt{\epsilon} \text{ for every } x > M_2.$$

Then setting $M = \max(M_1, M_2)$ we have

$$x > M$$

$$\Rightarrow |S_F(x) - \upsilon S_G(x)| = |S_G(x)| \left|\frac{S_F(x)}{S_G(x)} - \upsilon\right| < \sqrt{\epsilon}\sqrt{\epsilon} = \epsilon$$

as required. ∎

Proposition 107 *For any SLDFns F and G with $\omega_F = \omega_G = \infty$ and for which* $0 \leq \tau_F, \tau_G \leq \infty$:

$$0 < \upsilon(F,G) < \infty \Rightarrow \tau_F = \tau_G.$$

Proof. Set $\lim_{x\to\infty} \frac{f_F(x)}{f_G(x)} = \lim_{x\to\infty} \frac{S_F(x)}{S_G(x)} = \upsilon$

$$\begin{aligned} 1 &= \upsilon\frac{1}{\upsilon} = \lim_{x\to\infty} \frac{f_F(x)}{f_G(x)} \lim_{x\to\infty} \frac{S_G(x)}{S_F(x)} = \lim_{x\to\infty} \frac{f_F(x)}{f_G(x)} \frac{S_G(x)}{S_F(x)} \\ &= \lim_{x\to\infty} \frac{f_F(x)}{S_F(x)} \frac{S_G(x)}{f_G(x)} = \lim_{x\to\infty} \lambda_F(x) \frac{1}{\lambda_G(x)}. \end{aligned}$$

Consider first the case $0 < \tau_F, \tau_G < \infty$

$$1 = \lim_{x\to\infty} \lambda_F(x) \frac{1}{\lim_{x\to\infty} \lambda_G(x)} = \tau_F \frac{1}{\tau_G} \Rightarrow \tau_F = \tau_G.$$

We have

$$1 = \lim_{x\to\infty} \lambda_F(x) \frac{1}{\lim_{x\to\infty} \lambda_G(x)} \text{ and so}$$

$$0 = \tau_F = \lim_{x\to\infty} \lambda_F(x) \Rightarrow 0 = \lim_{x\to\infty} \lambda_G(x) = \tau_G.$$

and by the same token

$$1 = \lim_{x\to\infty} \lambda_F(x) \frac{1}{\lim_{x\to\infty} \lambda_G(x)} \text{ and so}$$

$$\infty = \tau_F = \lim_{x\to\infty} \lambda_F(x) \Rightarrow \infty = \lim_{x\to\infty} \lambda_G(x) = \tau_G$$

and the result follows. ∎

84

Example 108 *The converse is false:*

$$\lambda_F(x) = 1, \lambda_G(x) = 1 + \frac{1}{x}$$
$$\Rightarrow \int_0^\infty (\lambda_G - \lambda_F)(t)dt = \int_0^\infty \frac{dt}{t} = \infty$$
$$\Rightarrow \tau_F = \tau_G = 1 \text{ with } \upsilon(F,G) = \infty.$$

Proposition 109 *For any SLDFns F and G with $\omega_F = \omega_G = \infty$ and for which $F \succeq G$:*

$$c > 0 \text{ such that } \mu_F^{(c)} < \infty \Rightarrow \mu_G^{(c)} < \infty.$$

Proof. We have

$$F \succeq G$$
$$\Rightarrow \upsilon = \upsilon(F,G) = \lim_{x\to\infty} \frac{S_F(x)}{S_G(x)} \geq 1.$$

Consider first the case $\upsilon > 1$

$$\text{there exists } M_1 \text{ such that } S_F(x) > S_G(x), \text{ for every } x > M_1$$
$$\Rightarrow \mu_G^{(c)} = c\int_0^\infty x^{c-1} S_G(x)dx$$
$$= c\int_0^{M_1} x^{c-1} S_G(x)dx + c\int_{M_1}^\infty x^{c-1} S_G(x)dx$$
$$\leq cM_1^{c-1}M_1 + c\int_{M_1}^\infty x^{c-1} S_F(x)dx$$
$$\leq cM_1^c + c\mu_F^{(c)} < \infty.$$

So now consider the case $\upsilon = 1$

$$\lim_{x\to\infty} \frac{S_G(x)}{S_F(x)} = \frac{1}{\upsilon} = 1$$

$$\Rightarrow \text{there exists } M_2 \text{ such that } S_F(x) > 0 \text{ and } \left|\frac{S_G(x)}{S_F(x)} - 1\right| < \frac{1}{2}, \text{ for every } x > M_2$$
$$\Rightarrow \frac{S_G(x)}{S_F(x)} < \frac{3}{2}, \text{ for every } x > M_2$$
$$\Rightarrow S_G(x) < \frac{3}{2} S_F(x), \text{ for every } x > M_2$$
$$\Rightarrow \mu_G^{(c)} = c\int_0^\infty x^{c-1} S_G(x)dx$$
$$= c\int_0^{M_2} x^{c-1} S_G(x)dx + c\int_{M_2}^\infty x^{c-1} S_G(x)dx$$

$$\leq cM_2^{c-1}M_2 + c\int_{M_2}^{\infty} x^{c-1}\frac{3}{2}S_F(x)dx$$
$$\leq cM_2^c + \frac{3c}{2}\int_0^{\infty} x^{c-1}S_F(x)dx \leq cM_2^c + \frac{3}{2}\mu_F^{(c)} < \infty$$

and the proof is complete. ■

Proposition 110 *For any SLDFns F and G with $\omega_F = \omega_G = \infty$:*

$$0 \leq \tau_F < \tau_G \Rightarrow F \succ G.$$

Proof. This is straightforward:

$$\tau_F < \tau_G$$
$$\Rightarrow \text{there exist } M, \epsilon > 0 \text{ such that } \lambda_F(x) < \lambda_G(x) - \epsilon \text{ for every } x > M$$
$$\Rightarrow \int_M^{\infty} (\lambda_G - \lambda_F)(t)dt \geq \int_M^{\infty} \epsilon dt = \infty.$$

But then

$$\begin{aligned}\int_0^{\infty} (\lambda_G - \lambda_F)(t)dt &= \int_0^{M} (\lambda_G - \lambda_F)(t)dt + \int_M^{\infty} (\lambda_G - \lambda_F)(t)dt \\ &= \int_0^{M} (\lambda_G - \lambda_F)(t)dt + \infty = \infty\end{aligned}$$
$$\Rightarrow \lim_{x\to\infty}\frac{S_F(x)}{S_G(x)} = e^{\int_0^{\infty}(\lambda_G - \lambda_F)(t)dt} = e^{\infty} = \infty > 1$$
$$\Rightarrow F \succ G$$

as asserted. ■

Given any two SLDFns F and G and assuming τ_F and τ_G are known, the comparative thickness reduces to evaluating the limit $\upsilon(F,G)$ when $\tau_F = \tau_G$. But we know that the set of SLDFns F for which τ_F is a specified constant is acted on by the additive group $\mathbb{R}$ via taking the coderived distributions (when they exist) and is thus decomposed into orbits $\widetilde{F}^{[\mathbb{R}]}$under that action. The structure of those orbits was described in the previous section and we can orient ourselves within an orbit as to the "more or less tail-like" the distribution is in the "analytic" sense that $\widetilde{F}^{[c]}$ is more tail-like than $\widetilde{F}^{[d]}$exactly when $c > d$ (here "more tail like" means higher degree of monotonality. And we have seen that one may sacrifice the existence of moments to achieve that). The next result finally draws together the two perspectives of the paper and shows how the structure of those orbits relates with "thickness":

Proposition 111 *If F and G are SLDFns with $\omega_F = \omega_G$ and $0 < \tau_F = \tau_G < \infty$, then:*

$$\text{for every } m, n \in \mathbb{N},\ \upsilon\left(\widetilde{F}^{[m]}, \widetilde{G}^{[n]}\right) = \upsilon(F,G)\frac{m!\tau_F^{n-m}\mu_G^{(n)}}{n!\mu_F^{(m)}}.$$

Proof. From the Corollary 43

$$\lim_{x\to\infty}\frac{f_{\widetilde{F}^{[n]}}(x)}{f_{\widetilde{F}^{[m]}}(x)}=\frac{\tau_F^{m-n}n!\mu_F^{(m)}}{m!\mu_F^{(n)}}$$

whence

$$\begin{aligned}\upsilon\left(\widetilde{F}^{[m]},\widetilde{G}^{[n]}\right)&=\lim_{x\to\omega_F}\frac{f_{\widetilde{F}^{[m]}}(x)}{f_{\widetilde{G}^{[n]}}(x)}=\lim_{x\to\omega_F}\frac{f_{\widetilde{F}^{[m]}}(x)}{f_F(x)}\frac{f_F(x)}{f_G(x)}\frac{f_G(x)}{f_{\widetilde{G}^{[n]}}(x)}\\&=\lim_{x\to\omega_F}\frac{f_{\widetilde{F}^{[m]}}(x)}{f_F(x)}\lim_{x\to\omega_F}\frac{f_F(x)}{f_G(x)}\lim_{x\to\omega_G}\frac{f_{G^{[0]}}(x)}{f_{\widetilde{G}^{[n]}}(x)}\\&=\left(\lim_{x\to\omega_F}\frac{f_{F^{[0]}}(x)}{f_{\widetilde{F}^{[m]}}(x)}\right)^{-1}\upsilon(F,G)\frac{\tau_G^{n-0}(0!)\mu_G^{(n)}}{n!\mu_G^{(0)}}\\&=\left(\frac{\tau_F^m\mu_F^{(m)}}{m!}\right)^{-1}\upsilon(F,G)\frac{\tau_G^n\mu_G^{(n)}}{n!}\\&=\upsilon(F,G)\frac{m!}{\tau_F^m\mu_F^{(m)}}\frac{\tau_G^n\mu_G^{(n)}}{n!}=\upsilon(F,G)\frac{m!}{n!}\frac{\tau_F^{n-m}\mu_G^{(n)}}{\mu_F^{(m)}}\end{aligned}$$

as required. ∎

Corollary 112 *For any SLDFns F and G with $\omega_F=\omega_G$ and $0<\tau_F=\tau_G<\infty$:*

$$\widetilde{F}^{[m]}\succeq\widetilde{G}^{[n]}\Leftrightarrow\upsilon(F,G)\frac{m!\tau_F^{n-m}\mu_G^{(n)}}{n!\mu_F^{(m)}}\geq 1.$$

Corollary 113 *For any SLDFn F with $0<\tau_F<\infty$:*

$$\widetilde{F}^{[m]}\succeq\widetilde{F}^{[n]}\Leftrightarrow\frac{m!\tau_F^{n-m}\mu_F^{(n)}}{n!\mu_F^{(m)}}\geq 1.$$

Corollary 114 *For any SLDFn F with $0<\tau_F<\infty$:*

$$\widetilde{F}^{[m]}\succeq F\Leftrightarrow m!\geq\tau_F^m\mu_F^{(m)}.$$

Corollary 115 *For any SLDFn F with $0<\tau_F<\infty$:*

$$\text{there exists } k\geq 0 \text{ such that } \widetilde{F}^{[m]}\succeq F\quad\text{for every } m\geq k.$$

Proof. Observe that for all $x,t>0$:

$$0<e^{-tx}<1$$

and so the integral

$$L(t)=\int_0^\infty e^{-tx}f(x)dx=\int_0^\infty\left|e^{-tx}f(x)\right|dx\leq\int_0^\infty|f(x)|\,dx=1$$

is absolutely convergent. We have for any $x > 0$:

$$e^{\tau_F x} = \sum_{k=0}^{\infty} \frac{(\tau_F x)^k}{k!} = \sum_{k=0}^{\infty} \left| \frac{(-1)^k \tau_F^k x^k}{k!} \right|$$

and the power series expansion

$$e^{-\tau_F x} = \sum_{k=0}^{\infty} \frac{(-\tau_F x)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k \tau_F^k x^k}{k!}$$

is absolutely convergent and so can be integrated term by term:

$$\begin{aligned} L_F(\tau_F) &= \int_0^{\infty} e^{-\tau_F x} f(x)dx = \int_0^{\infty} \sum_{k=0}^{\infty} \left(\frac{(-1)^k \tau_F^k x^k f(x)}{k!} \right) dx \\ &= \sum_{k=0}^{\infty} \left(\left(\frac{(-1)^k \tau_F^k}{k!} \right) \int_0^{\infty} x^k f(x)dx \right) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k \mu_F^{(k)} \tau_F^k}{k!}. \end{aligned}$$

Since the terms of any convergent series must converge to 0:

$$\text{there exists } k \geq 0 \text{ such that } \frac{\mu_F^{(m)} \tau_F^m}{m!} = \left| \frac{(-1)^m \mu_F^{(m)} \tau_F^m}{m!} \right| < 1 \quad \text{for every } m \geq k.$$

And by the previous corollary:

$$\widetilde{F}^{[m]} \succeq F \quad \text{for every } m \geq k$$

as required. ■

Proposition 116 *If F is an SLDFn for which the orbit $\widetilde{F}^{[\mathbb{R}]}$ has a last element then:*

$$\widetilde{F}^{[m]} \succeq \widetilde{F}^{[n]} \Leftrightarrow m \geq n.$$

Proof. Let $\widetilde{F}^{[l]}$ be the last element of $\widetilde{F}^{[\mathbb{R}]}, \Omega\left(\widetilde{F}^{[\mathbb{R}]}\right) = (-\infty, l]$. Suppose first that $\widetilde{F}^{[m]} \succeq \widetilde{F}^{[n]}$, in this case, we have

$$l \geq m, l \geq n$$

$$\Rightarrow l - m = \text{highest finite moment of } \widetilde{F}^{[m]}$$

$$\Rightarrow l - n = \text{highest finite moment of } \widetilde{F}^{[n]}$$

but then from Proposition 109

$$\widetilde{F}^{[m]} \succeq \widetilde{F}^{[n]} \text{ and } \mu_{\widetilde{F}^{[m]}}^{(l-m)} < \infty$$

$$\Rightarrow \mu_{\widetilde{F}^{[n]}}^{(l-m)} < \infty$$
$$\Rightarrow l - m \leq l - n$$
$$\Rightarrow -m \leq -n$$
$$\Rightarrow m \geq n$$

establishing one direction. For the converse, suppose that $m \geq n$, and by way of contradiction that $\widetilde{F}^{[m]} \succeq \widetilde{F}^{[n]}$ is false. Then since by Proposition 42 the limit $\upsilon\left(\widetilde{F}^{[m]}, \widetilde{F}^{[n]}\right)$ exists, we must have

$$\upsilon\left(\widetilde{F}^{[m]}, \widetilde{F}^{[n]}\right) < 1$$
$$\Rightarrow \upsilon\left(\widetilde{F}^{[n]}, \widetilde{F}^{[m]}\right) = \frac{1}{\upsilon\left(\widetilde{F}^{[m]}, \widetilde{F}^{[n]}\right)} > 1$$
$$\Rightarrow \widetilde{F}^{[n]} \succ \widetilde{F}^{[m]}$$
$$\text{above direction} \Rightarrow n \geq m \quad \text{and } n \neq m$$
$$\Rightarrow n > m \quad \Rightarrow\Leftarrow$$

and this contradiction completes the proof. ■

Proposition 117 *If F is the SLDFn of a mixed exponential density, then for all $k, n \in \mathbb{N}$:*

$$\widetilde{F}^{[k]} \succeq \widetilde{F}^{[n]} \quad \Leftrightarrow \quad k \geq n.$$

Proof. Let F be a mixture of exponential densities. More precisely, for some m, $1 \leq m \leq \infty$, weights $w_i > 0$ with $1 = \sum_{i=1}^{m} w_i$ and parameters $\mu_i > 0$ ordered so that $\mu_i < \mu_{i+1}$ and with $\sum_{i=1}^{m} w_i\mu_i < \infty$ set

$$F = \beth(m, \langle w_i \rangle, \langle \mu_i \rangle)$$

with survival function

$$S_F(x) = \sum_{i=1}^{m} w_i e^{-\frac{x}{\mu_i}}.$$

Then we have, by Proposition 49

$$\mu_F = \sum_{i=1}^{m} w_i\mu_i < \infty$$

$$S_{\widetilde{F}}(x) = \frac{\sum_{i=1}^{m} w_i\mu_i e^{-\frac{x}{\mu_i}}}{\mu_F} = \sum_{i=1}^{m} u_i e^{-\frac{x}{\mu_i}} \text{ where } u_i = \frac{w_i\mu_i}{\mu_F}.$$

And it follows that

$$\frac{S_F(x)}{S_{\widetilde{F}}(x)} = \frac{\sum_{i=1}^{m} w_i e^{-\frac{x}{\mu_i}}}{\sum_{i=1}^{m} u_i e^{-\frac{x}{\mu_i}}} = \frac{e^{\frac{x}{\mu_m}} \sum_{i=1}^{m} w_i e^{-\frac{x}{\mu_i}}}{e^{\frac{x}{\mu_m}} \sum_{i=1}^{m} u_i e^{-\frac{x}{\mu_i}}}$$

89

$$= \frac{\sum_{i=1}^{m} w_i e^{\frac{x}{\mu_m} - \frac{x}{\mu_i}}}{\sum_{i=1}^{m} u_i e^{\frac{x}{\mu_m} - \frac{x}{\mu_i}}} = \frac{w_m + \sum_{i=1}^{m-1} w_i e^{\left(\frac{\mu_i - \mu_m}{\mu_i \mu_m}\right)x}}{u_m + \sum_{i=1}^{m-1} u_i e^{\left(\frac{\mu_i - \mu_m}{\mu_i \mu_m}\right)x}}$$

$$\mu_i - \mu_m < 0, 1 \leq i \leq m-1$$

$$\Rightarrow \lim_{x\to\infty} \frac{S_F(x)}{S_{\widetilde{F}}(x)} = \lim_{x\to\infty} \frac{w_m + \sum_{i=1}^{m-1} w_i e^{\left(\frac{\mu_i - \mu_m}{\mu_i \mu_m}\right)x}}{u_m + \sum_{i=1}^{m-1} u_i e^{\left(\frac{\mu_i - \mu_m}{\mu_i \mu_m}\right)x}}$$

$$= \frac{w_m + \lim_{x\to\infty} \sum_{i=1}^{m-1} w_i e^{\left(\frac{\mu_i - \mu_m}{\mu_i \mu_m}\right)x}}{u_m + \lim_{x\to\infty} \sum_{i=1}^{m-1} u_i e^{\left(\frac{\mu_i - \mu_m}{\mu_i \mu_m}\right)x}} = \frac{w_m + 0}{u_m + 0} = \frac{w_m}{u_m}$$

$$= \frac{w_m}{\frac{w_m \mu_m}{\mu_F}} = \frac{\mu_F}{\mu_m} < 1$$

$$\Rightarrow \widetilde{F} \succeq F$$

and then by transitivity.$\widetilde{F}^{[k-n]} \succeq F$, and the result follows by replacing F with .$\widetilde{F}^{[n]}$, which also has a mixed exponential density.

As one would expect, there are continuous analogues for many of these "momentous" observations: ■

Proposition 118 *For any SLDFns F and G with $0 < \tau_F = \tau_G < \infty$ and positive $c, d \in \mathbb{R}$:*

$$\upsilon\left(\widetilde{F}^{[d]}, \widetilde{G}^{[c]}\right) = \upsilon(F, G) \frac{\Gamma(d+1)\tau_F^{c-d}\mu_G^{(c)}}{\Gamma(c+1)\mu_F^{(d)}}.$$

Proof. This is clear from Proposition 71

$$\upsilon\left(\widetilde{F}^{[d]}, \widetilde{G}^{[c]}\right) = \lim_{x\to\infty} \frac{f_{\widetilde{F}^{[d]}}(x)}{f_{\widetilde{G}^{[c]}}(x)} = \lim_{x\to\infty} \frac{f_{\widetilde{F}^{[d]}}(x)}{f_F(x)} \frac{f_F(x)}{f_G(x)} \frac{f_G(x)}{f_{\widetilde{G}^{[c]}}(x)}$$

$$= \lim_{x\to\infty} \frac{f_{\widetilde{F}^{[d]}}(x)}{f_F(x)} \lim_{x\to\infty} \frac{f_F(x)}{f_G(x)} \lim_{x\to\infty} \frac{f_G(x)}{f_{\widetilde{G}^{[c]}}(x)}$$

$$= \left(\lim_{x\to\infty} \frac{f_F(x)}{f_{\widetilde{F}^{[d]}}(x)}\right)^{-1} \upsilon(F, G) \frac{\tau_G^c \mu_G^{(c)}}{\Gamma(c+1)} = \left(\frac{\tau_F^d \mu_F^{(d)}}{\Gamma(d+1)}\right)^{-1} \upsilon(F, G) \frac{\tau_Y^c \mu_G^{(c)}}{\Gamma(c+1)}$$

$$= \upsilon(F, G) \frac{\Gamma(d+1)}{\tau_F^d \mu_F^{(d)}} \frac{\tau_G^c \mu_G^{(c)}}{\Gamma(c+1)} = \upsilon(F, G) \frac{\Gamma(d+1)}{\Gamma(c+1)} \frac{\tau_F^{c-d} \mu_G^{(c)}}{\mu_F^{(d)}}$$

as required. ■

Corollary 119 *For any SLDFns F and G with $0 < \tau_F = \tau_G < \infty$ and positive $c, d \in \mathbb{R}$:*

$$\widetilde{F}^{[d]} \succeq \widetilde{G}^{[c]} \Leftrightarrow \upsilon(F, G) \frac{\Gamma(d+1)\tau_F^{c-d}\mu_G^{(c)}}{\Gamma(c+1)\mu_F^{(d)}} \geq 1.$$

90

Corollary 120 *For any SLDFn F with $0 < \tau_F < \infty$ and positive $c, d \in \mathbb{R}$:*

$$\widetilde{F}^{[d]} \succeq \widetilde{F}^{[c]} \Leftrightarrow \frac{\Gamma(d+1)\tau_F^{c-d}\mu_F^{(c)}}{\Gamma(c+1)\mu_F^{(d)}} \geq 1.$$

Corollary 121 *For any SLDFn F with $0 < \tau_F < \infty$ and positive $d \in \mathbb{R}$:*

$$\widetilde{F}^{[d]} \succeq F \Leftrightarrow \Gamma(d+1) \geq \tau_F^d \mu_F^{(d)}.$$

We conclude this section with a couple more examples.

Example 122 *Consider the Pareto density $F = \Pi(\alpha, \theta)$:*

$$S_F(x) = \left(\frac{\theta}{x+\theta}\right)^{\alpha}$$

$$\frac{\partial S}{\partial \alpha} = \left(\frac{\theta}{x+\theta}\right)^{\alpha} \ln\left(\frac{\theta}{x+\theta}\right) \leq 0$$

$$\frac{\partial S}{\partial \theta} = \alpha\theta\left(\frac{\theta}{x+\theta}\right)^{\alpha-1}\left(\frac{1}{x+\theta}\right)^{2} = \alpha\theta^{\alpha}\left(x+\theta\right)^{-\alpha-1} > 0$$

which suggests that, all else equal, F gets "thicker" as θ increases or α decreases. Now let $F = \Pi(\alpha, \theta)$ $G = \Pi(\beta, \vartheta)$ be two Pareto densities, $\alpha, \beta, \theta, \vartheta \in (0, \infty)$. Then $\omega_F = \omega_G = \infty$ and $\tau_F = \tau_G = 0$ with

$$\upsilon(F, G) = e^{\int_0^\infty (\lambda_G - \lambda_F)(t)dt}$$

$$\int_0^\infty (\lambda_G - \lambda_F)(t)dt = \int_0^\infty \left(\frac{\beta}{t+\vartheta} - \frac{\alpha}{t+\theta}\right)dt$$

$$= \left[\beta \ln(t+\vartheta) - \alpha \ln(t+\theta)\right]_{t=0}^{t=\infty} = \left[\ln\left((t+\vartheta)^{\beta}\right) - \ln(t+\theta)^{\alpha}\right]_{t=0}^{t=\infty}$$

$$= \left[\ln\left(\frac{(t+\vartheta)^{\beta}}{(t+\theta)^{\alpha}}\right)\right]_{t=0}^{t=\infty} = \lim_{t\to\infty}\left[\ln\left(\frac{(t+\vartheta)^{\beta}}{(t+\theta)^{\alpha}}\right)\right] - \ln\left(\frac{\vartheta^{\beta}}{\theta^{\alpha}}\right)$$

$$= \ln\left(\lim_{t\to\infty}\left(\frac{(t+\vartheta)^{\beta}}{(t+\theta)^{\alpha}}\right)\right) + \ln\left(\frac{\theta^{\alpha}}{\vartheta^{\beta}}\right)$$

$$= \ln\left(\lim_{t\to\infty}\left(\frac{t+\vartheta}{t+\theta}\right)^{\alpha}(t+\vartheta)^{\beta-\alpha}\right) + \ln\left(\frac{\theta^{\alpha}}{\vartheta^{\beta}}\right)$$

$$= \ln\left(\lim_{t\to\infty} 1^{\alpha}(t+\vartheta)^{\beta-\alpha}\right) + \ln\left(\frac{\theta^{\alpha}}{\vartheta^{\beta}}\right)$$

$$= \ln\left(\lim_{t\to\infty}(t+\vartheta)^{\beta-\alpha}\right) + \ln\left(\frac{\theta^{\alpha}}{\vartheta^{\beta}}\right)$$

$$= \left\{ \begin{array}{cc} +\infty & \beta > \alpha \\ \ln\left(\left(\frac{\theta}{\vartheta}\right)^{\alpha}\right) & \beta = \alpha \\ -\infty & \beta < \alpha \end{array} \right\}.$$

We see that

$$\upsilon(F,G) = e^{\int_0^{\infty}(\lambda_G - \lambda_F)(t)dt} = \left\{ \begin{array}{cc} +\infty & \beta > \alpha \\ \left(\frac{\theta}{\vartheta}\right)^{\alpha} & \beta = \alpha \\ 0 & \beta < \alpha \end{array} \right\}$$

and that

$$\Pi(\alpha,\theta) \succeq \Pi(\beta,\vartheta) \Leftrightarrow \upsilon(F,G) \geq 1$$
$$\Leftrightarrow \beta > \alpha \text{ or } (\beta = \alpha \text{ and } \theta \geq \vartheta)$$

and

$$\Pi(\alpha,\theta) \succ \Pi(\beta,\vartheta) \Leftrightarrow \upsilon(F,G) > 1$$
$$\Leftrightarrow \beta > \alpha \text{ or } (\beta = \alpha \text{ and } \theta > \vartheta)$$

which conforms to what was suggested before.

Example 123 *Let* $F = \Pi(\alpha,\theta)$ *and* $G(x) = \frac{x^2}{1+x^2}$ *:*

$$\lambda_F(x) = \frac{\alpha}{\theta + x} \Rightarrow \tau_F = 0$$

$$\lambda_G(x) = \frac{f_G(x)}{S_G(x)} = \frac{\frac{2x}{(1+x^2)^2}}{\frac{1}{1+x^2}} = \frac{2x}{1+x^2} \Rightarrow \tau_G = 0$$

$$\int_0^{\infty} (\lambda_G - \lambda_F)(t)dt = \int_0^{\infty} \left(\frac{2t}{1+t^2} - \frac{\alpha}{\theta + t}\right) dt$$

$$= \left[\ln\left(1+t^2\right) - \alpha \ln(\theta + t)\right]_{t=0}^{t=\infty} = \lim_{t\to\infty} \left[\ln\left(1+t^2\right) - \ln(\theta+t)^{\alpha}\right] + \alpha \ln(\theta)$$

$$= \lim_{t\to\infty} \left[\ln\left(\frac{1+t^2}{(\theta+t)^{\alpha}}\right)\right] + \ln(\theta^{\alpha}) = \ln\left[\lim_{t\to\infty}\left(\frac{1+t^2}{(\theta+t)^{\alpha}}\right)\right] + \ln(\theta^{\alpha})$$

$$= \ln\left[\lim_{t\to\infty}\left(\frac{2t}{\alpha(\theta+t)^{\alpha-1}}\right)\right] + \ln(\theta^{\alpha})$$

$$= \left\{ \begin{array}{cc} +\infty & \alpha \leq 1 \\ \ln\left[\lim_{t\to\infty}\left(\frac{2}{\alpha(\alpha-1)(\theta+t)^{\alpha-2}}\right)\right] + \ln(\theta^{\alpha}) & \alpha > 1 \end{array} \right\}$$

$$= \left\{ \begin{array}{cc} +\infty & \alpha \leq 1 \\ 1 + \ln(\theta^{\alpha}) & \alpha = 2 \\ \ln\left[\frac{2}{\alpha(\alpha-1)} \lim_{t\to\infty}\left((\theta+t)^{2-\alpha}\right)\right] - \ln(\theta^{\alpha}) & \alpha > 1, \alpha \neq 2 \end{array} \right\}$$

92

$$= \left\{ \begin{array}{cc} +\infty & \alpha \leq 1 \\ \ln(1) + \ln\left(\theta^2\right) & \alpha = 2 \\ \ln\left[\frac{2}{\alpha(\alpha-1)}\left(+\infty\right)\right] + \ln\left(\theta^\alpha\right) = +\infty & 1 < \alpha < 2 \\ \ln\left[\frac{2}{\alpha(\alpha-1)}\left(0\right)\right] + \ln\left(\theta^\alpha\right) = -\infty & \alpha > 2 \end{array} \right\}$$

$$= \left\{ \begin{array}{cc} +\infty & \alpha < 2 \\ \ln\left(\theta^2\right) & \alpha = 2 \\ -\infty & \alpha > 2 \end{array} \right\}$$

$$\upsilon(F,G) = e^{\int_0^\infty (\lambda_G - \lambda_F)(t)dt} = \left\{ \begin{array}{cc} +\infty & \alpha < 2 \\ \theta^2 & \alpha = 2 \\ 0 & \alpha > 2 \end{array} \right\}$$

$$\begin{aligned} \alpha &< 2 \Rightarrow \upsilon(F,G) = \infty > 1 \Rightarrow F \succ G \\ \alpha &> 2 \Rightarrow \upsilon(F,G) = 0 < 1 \Rightarrow G \succ F \end{aligned}$$

$$\begin{aligned} \alpha &= 2, \theta > 1 \Rightarrow \upsilon(F,G) > 1 \Rightarrow F \succ G \\ \alpha &= 2, \theta = 1 \Rightarrow \upsilon(F,G) = 1 \Rightarrow F \approx G \\ \alpha &= 2, \theta < 1 \Rightarrow \upsilon(F,G) < 1 \Rightarrow G \succ F. \end{aligned}$$

8 Conclusion

For a given continuous loss distribution F with finite mean, we have seen that the ratio of losses in excess of a given loss limit x to total losses defines a function $R(x)$ that formally resembles a survival function. The loss distribution defined by that survival function was defined to be the "coderived" distribution $\widetilde{F}$. This coderived distribution was shown to exhibit (right hand) tail behavior and moments that are very closely related to those of the original loss distribution (Propositions 27 and 28). Moreover, this coderived distribution has a simpler, more "monotone", structure than the original (Proposition 87). We observed that this coderived distribution completely determines the original distribution (Proposition 26). Repeating this process yields a discrete sequence of loss distributions $F, \widetilde{F}, \widetilde{\widetilde{F}}, \ldots$within a continuous, one-parameter collection of loss distributions (Remark 58). Such collections all have tails with the same ultimate settlement rate $\tau_F = \tau_{\widetilde{F}} = \tau_{\widetilde{\widetilde{F}}}$ (Proposition 28). We described a simple approach to ordering loss distributions according to the "thickness" of their tails (Definition 99) and related thickness with monotonality and ultimate settlement rate (Proposition 111). A key finding is that the asymptotic behavior of the hazard rate as captured by the ultimate settlement rate $\tau_F = \lim\limits_{x \to \omega_F} \lambda_F(x)$, provides a natural bridge between these two perspectives. We observed that if the hazard rate function is increasing or decreasing, then the sequence of coderived distributions converges to an exponential loss distribution (Proposition 78). We conclude that when modeling loss severity (where the hazard rate function is reasonably well-behaved, e.g. with only finitely many turning points, and where there is no cap), there is a uniquely determined exponential distribution with canonical properties that favor it as a choice to splice onto to the model as the right hand tail. If it is impractical to go far enough out into the tail to make the tail close to monotone (near constant hazard rate), one should consider fitting a mixed exponential. The reader is invited to consult [2] for both a discussion of tail-splicing and as a case study of this approach.

References

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