DOWNWARD BIAS OF USING HIGH-LOW AVERAGES FOR LOSS DEVELOPMENT FACTORS

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Abstract

This paper extends previous research that studied the downward bias associated with high-low averages, which occurs when high-low averages are applied to data that exhibits a long-tailed property. The current study conducted a comprehensive review of insurance industry data when three-of-five averages are used to determine the age-to-age development factors in setting reserves. The downward bias was analyzed by line of business, premium size, development age, paid and incurred loss development methods, for one hundred and forty paid and incurred loss triangles from seventy insurance companies/groups compiled from the A.M. Best database. The study assumes that the age-to-age development factors are lognormally distributed. The three-of-five average was selected as the representative high-low average because it is commonly used by property/casualty actuaries. The results for this average can be generalized to other types of high-low averages. The results given in the paper are based on a bias formula for a large volume of data. Since the real-world loss development data is limited in volume, the study used large scale simulations to review the effect of limited volume data on the bias.

1. INTRODUCTION

1.A. Downward Bias of Using High-Low Averages for Age-to-Age Factors

Property/casualty actuaries often employ an averaging technique that excludes the same number of observations, split equally between the lowest and highest ranking observations. These averages will be called the high-low averages in this paper. One common application of the averages is the selection of loss development factors.

There are many types of high-low averages, for example, the middle three of the latest five years (three-of-five averages) and the middle six of the latest eight quarters (six-of-eight averages).

The purpose of using high-low averages is to exclude outliers and their disproportional influence on the results. Exclusion of observations requires a great deal of caution, however. According to Neter, Wasserman, and Kutner [8]:

> "… an outlying influential case should not be automatically discarded, because it may be entirely correct and simply represents an unlikely event. Discarding of such an outlying case could lead to the undesirable consequences of increased variances of some of the estimated regression coefficients."

In other words, systematic exclusion of high and low data points would lead to less statistically significant and, hence, less credible estimators.

Moreover, the distribution of insurance loss data exhibits unsymmetrical behavior of skewing toward the right (higher values). This is called the *long-tailed* property. Most typical insurance claims are small amount claims, probably less than a few thousand dollars. However, the remaining small number of claims can have very large losses. For example, automobile large loss claims will reach a few hundred thousand dollars, while medical malpractice or environmental claims can even be multimillion-dollar claims in today's legal climate. Therefore, long-tailed distributions such as lognormal, Pareto, and gamma distributions are better in describing the loss data than the symmetric normal distribution because they reflect the large loss probability. Exhibit 1 shows graphically a lognormal distribution and its long-tailed property of skewing to the right.

Applying high-low averages to loss development factors will result in a systematic downward bias when the loss development data exhibits a *long-tailed* property. This can be illustrated through the following example based on a lognormal assumption.

First, assume that:

- At development age i, the aggregate reported loss or paid loss is equal to L_i.
- From age i to $i+1$, a total loss of l_{i+1} is reported or paid.
- Since insurance losses have a long-tailed property, both L_i and l_{i+1} can be represented by lognormal distributions. If this is the case, then both $\ln(L_i)$ and $\ln(l_{i+1})$ are normally distributed. For the use of lognormal distributions to approximate insurance losses, please see Bowers, et al. [2], Finger [3], and Hogg and Klugman [5].

Based on these assumptions, the age-to-age development factor from age i to $i+1$ can be expressed as follows:

$$D_{i,i+1} = (L_i + l_{i+1})/L_i = 1 + l_{i+1}/L_i.$$

Since the multiplication or division result of two lognormally distributed variables also has a lognormal distribution, $1 + l_{i+1}/L_i$ and $D_{i,i+1}$ are lognormally distributed and should have a long-tailed property:

$$\ln(D_{i,i+1}) \sim N(\mu_i, \sigma_i^2),$$

where μ_i is the mean and σ_i^2 is the variance of the normal distribution for $\ln(D_{i,i+1})$.

One advantage of assuming lognormal distributions for the age-to-age development factors is that the age-to-ultimate factors and, consequently, the ultimate loss estimates are also lognormally distributed:

$$UD_i = D_{i,i+1} \times D_{i+1,i+2} \times D_{i+2,i+3} \times \cdots,$$

where

$$\ln(UD_i) = \ln(D_{i,i+1}) + \ln(D_{i+1,i+2}) + \ln(D_{i,i+1}) + \cdots$$

and

$$\ln(UD_i) \sim N(\mu_i + \mu_{i+1} + \mu_{i+2} + \cdots, \sigma_i^2 + \sigma_{i+1}^2 + \sigma_{i+2}^2 + \cdots).$$

The fact that age-to-age development factors may have a long tail has been noted previously. Hayne's study [4], in quantifying the variability of loss reserves, assumes that age-to-age development factors are lognormally distributed. Kelly [6] and McNichols [7] also conclude that a lognormal assumption is better in describing age-to-age development factors than a normal assumption, based on the fact that lognormal distributions can take only positive values and their long-tailed property reflects the distinct possibility of large development factors.

However, if $D_{i,i+1}$ is lognormally distributed, using high-low averages to estimate $D_{i,i+1}$ will result in a downward bias. Bias is defined as the percentage difference between the mean and the conditional mean, given that the data lie between a specified lower and upper pair of percentile points. The bias is expressed in the following formula whose detailed derivations can be found in the Appendix:

$$\begin{aligned}\text{Bias} &= \frac{\mathrm{E}(D_{i,i+1})'}{\mathrm{E}(D_{i,i+1})} - 1\\ &= \frac{1}{(1-2p)}[\Phi(\Phi^{-1}(1-p) - \sigma_i) - \Phi(\Phi^{-1}(p) - \sigma_i)] - 1,\end{aligned} \tag{1.1}$$

where:

$E(D_{i,i+1})$ is the expected value of $D_{i,i+1}$,

$E(D_{i,i+1})'$ is the expected value of $D_{i,i+1}$, given that $D_{i,i+1}$ lies between its upper and lower p percentile points

$$\left(\text{i.e.,}\quad \frac{1}{1-2p}\int_{d1}^{d2} t\times f(t)\,dt\right),$$

$f(d)$ is the probability distribution function for $D_{i,i+1}$,

$F(d)$ is the cumulative distribution function for $D_{i,i+1}$,

p represents percentile,

d_1 is the value of $D_{i,i+1}$ when $F(d) = p$,

d_2 is the value of $D_{i,i+1}$ when $F(d) = 1-p$,

and

$\Phi(X)$ is the standard normal distribution function,

$$\int_{-\infty}^{X} \frac{\exp(\frac{1}{2}t^2)}{\sqrt{2\pi}}\,dt.$$

Equation (1.1) indicates that the degree of bias depends only on p and σ_i, the percentage of data being excluded and the shape parameter, but not on μ_i, the location parameter. This suggests that the more data excluded or the more skewed and volatile the distribution, the higher the downward bias is. Exhibit 1 illustrates the downward bias graphically.

Note that we are not limited to only the lognormal assumption. For example, one other commonly used long-tailed distribution is the Pareto distribution. The bias formula similar to Equation (1.1) for the Pareto distribution is also derived in the Appendix. Further analysis indicates that for the age-to-age development factors reviewed in this study, there is no significant difference in the bias result between the lognormal distribution and the Pareto distribution.

1.B. Modified High-Low Averages for the Correction of Downward Bias

Results from Equation (1.1) can be extended to the high-low averages used by property/casualty actuaries. For example, a three-of-five average also excludes the upper and lower 20% of the data. The only difference is that the high-low average is based on a limited volume of data (five data points) and a sample distribution function, while Equation (1.1) is based on a very large volume of data and a cumulative distribution function.

Equation (1.1) provides a basis to correct the bias for the sample high-low average:

$$\text{Modified High-Low Average} = \text{Sample High-Low Average}/(1 + \text{Bias}), \qquad (1.2)$$

where the bias is given in Equation (1.1).

Exhibits 2 to 5 display how to correct the downward bias for the three-of-five averages based on Equations (1.1) and (1.2). This example uses product liability paid loss data for a sample company from the A. M. Best database [1].

Exhibit 2 shows two types of averages: five-year straight averages and three-of-five averages. These are factor averages, not volume-weighted averages. Because the data has 10 years of experience, the three-of-five averages can be applied to only the first five development ages. After the fifth development age, all-years averages are used.

The tail factor of 1.0261 selected in Exhibit 2 should be noted. This factor is the ratio of incurred loss to paid loss for the earliest year in the triangle. No further tail development is assumed. The choice of the tail factor will not affect the relative bias level because it is a constant that will be multiplied by the age-to-age development factors.

Results from Exhibit 2 clearly indicate that the five-year averages result in higher estimates than the three-of-five averages. This is consistent with the assumption that age-to-age loss development factors have a long-tailed property.

Fitting lognormal distributions to the age-to-age development factors in Exhibit 2 produces the parameter estimates in Exhibit 3. First, μ_i and σ_i^2 are estimated for each development period. All of the data in each development period are used to estimate these sample parameters, although only the latest five data points are used to select the age-to-age development factors. This approach is used to increase the credibility of the sample parameters. Then, the parameters for the age-to-ultimate development factors for a development age are the sum of all the parameters of the age-to-age factors from that age to ultimate.

Given these lognormal parameter estimates, the three-of-five averages in Exhibit 2 can be modified to correct the downward bias for the averages. The modified three-of-five factors are given in Exhibit 4. For example, the lognormal parameters for the 12-to-24 development factors are: $\mu_1 = 1.9221$, and $\sigma_1^2 = 0.3057$. With $p = 20\%$, a bias of -11.33% is indicated for the three-of-five average based on Equation (1.1).

Exhibit 4 shows the indicated bias for each development period and the modified three-of-five averages. Exhibit 5 compares the estimated ultimate losses and reserves between the five-year averages, the three-of-five averages, and the modified three-of-five averages. For example, the total reserve for the three-of-five averages is approximately 12.0% lower than the reserve for the five-year averages, and is 8.9% lower than the reserve for the modified three-of-five averages. Exhibit 5 does not show the results for the oldest five accident years since there is no difference among methods for these five accident years.

This specific example is for product liability paid loss data. The results of the comprehensive review, testing the biases with differing data volumes, differing lines of business, and paid and incurred loss data will be shown in later sections.

1.C. Limited Volume Data

As mentioned previously, the bias formula given in Equation (1.1) is based on a very large volume of data and a cumulative distribution function, while the real-world data is limited in volume.

Two issues in dealing with a limited volume of data should be noted. First, additional parameter variation is introduced because sample parameters are assumed in place of true parameters. Therefore, when Equation (1.1) is used to estimate the level of bias of real-world data, sample parameters, not the true parameters, are generally used. For example, in Exhibits 3 and 4, the lognormal parameters, $\mu_1 = 1.9221$ and $\sigma_1^2 = 0.3057$, for the 12-to-24 development factors, distribution are based on the nine sample data points in the 12-to-24 development period. We assumed these parameters were the true parameters when the -11.33% of downward bias was indicated by Equation (1.1).

Second, even if the true parameters are known, the indicated bias when sample size is small will not be the same as the indicated bias when sample size is large. For example, Equation (1.1) provides an accurate estimate of bias if 20% of high and low data are excluded from a data set of, for example, a million data points. However, when a three-of-five average is used to estimate the loss development factors, 20% of the high and low data are excluded from a data set of only five data points.

Resolving these limited volume data issues through statistical methods is very difficult, if not impossible, and is beyond the scope of this study. Instead, large scale simulations have been conducted and the simulation results will be presented in the later sections.

2. CURRENT STUDY

2.A. Purposes

The previous section illustrates the potential bias of using high-low averages for loss development factors, and more details can be found in Wu [9]. In light of these results, however, many outstanding questions remain to be answered:

- Do the real-world loss development factors really exhibit a long-tailed property?
- What is the level of the downward bias when the high-low averages are used in setting reserves?
- How does the downward bias vary by line of business, data volume, development age, and between paid and incurred loss development methods?
- What is the effect of limited volume data on the bias?

This study attempts to answer these questions through a comprehensive review of industry data and large scale simulations.

2.B. Data

Data from the A.M. Best database [1] were gathered for the following seven major liability lines:

- workers compensation;
- private passenger automobile liability;
- commercial automobile liability;
- medical malpractice, occurrence;
- medical malpractice, claims-made;
- product liability; and
- other liability.

For each line of business, paid loss and incurred loss triangles on an annual basis were compiled from ten randomly selected insurance companies/groups. In general, the same ten companies were not used for each line of business, but a few companies were repeatedly selected. A total of one hundred and forty triangles were collected. The loss triangles have ten years of experience and cover the period from 1986 to 1995.

The collected data were further broken down into two groups based on the volume of the data. One group, Group A, contains large multi-line and multi-state companies, while the other group, Group B, contains small local and regional companies. Exhibit 6 shows the range of the annual earned premium for the companies within each groups.

2.C. Review Approach

The loss development procedures used to review the A. M. Best data are the same as the procedures given in Exhibits 2 to 5. The following list summarizes the important assumptions in the approach:

- The three-of-five average was selected as the representative high-low average. The results for that average can be extended to other types of high-low averages.
- Due to the fact that the collected loss triangle data have only ten years of history, the three-of-five averages can be applied to only the first five development ages. For the development ages after 72 months, all-years averages were used.
- There is no tail development assumed for the incurred loss method. For the paid tail, the ratio of incurred to paid loss for the oldest accident year in the triangle was used.
- All data points in each development period were used to calculate the lognormal parameters. This was done to increase the credibility of the sample parameters. However, only the lat-

est five points were used to select the age-to-age development factors.

- Large scale simulations were conducted to study the effect of a limited of volume data on the bias when sample parameters are assumed as the true parameters. The simulations also measure the differences between the simulated bias and the bias based on Equation [1].

3. RESULTS AND DISCUSSION

3.A. Long-Tailed Property for Age-to-Age Development Factors

First, the reserve indications for the five-year averages and the three-of-five are compared. Exhibit 6 gives the comparison results by line of business, company size, and paid versus incurred methods.

Exhibit 6 indicates that approximately 70% of the data reviewed show lower reserve indications for the three-of-five averages. This is consistent with the assumption that the age-to-age development factors may have a long tail and the use of high-low averages will result in a downward bias.

Exhibit 6 further indicates that the long tail assumption is more valid for the more volatile lines such as medical malpractice and product liability. On the other hand, the assumption is equally valid for both large and small groups, and for both incurred and paid methods.

3.B. Results by Line of Business

Exhibits 7 to 13 give two types of downward bias by line of business: the bias for the age-to-age development factors and the bias for the reserve indications. The tests were conducted on both the total reserve and the incurred but not reported reserve (IBNR). In each exhibit, the downward bias is indicated by company size and paid versus incurred methods.

The indicated bias given in these exhibits is based on Equation (1.1). For example, Exhibit 11 shows that for the malpractice claims-made data of the large companies in Group A, the indicated minimum, maximum, and average downward biases associated with the three-of-five averages for the 12–24 paid factors are 0.86%, 2.88%, and 2.06%, respectively.

The bias for the reserve indications is the difference between indications based on the three-of-five averages and the modified three-of-five averages. For example, Exhibit 11 shows that for the malpractice claims-made data of the large companies in Group A, the indicated minimum, maximum, and average downward bias for the total reserves for the paid method are 0.61%, 2.86%, and 1.87%, respectively.

From Exhibits 7 to 13, the following observations can be made:

- The indicated bias for the age-to-age factors decreases as the loss data become mature. For workers compensation, private passenger automobile liability, and commercial automobile liability, the bias appears to be insignificant after 72 months of development. On the other hand, the bias is still noticeable after 72 months for medical malpractice, product liability, and other liability.
- The indicated bias for the reserve indications can be substantial, especially for the highly volatile lines such as medical malpractice, product liability, and other liability. The use of high-low averages can easily lead to a downward bias of over 10% for these lines of business.
- In general, the data of small companies shows higher downward bias than the data of large companies. This is because the age-to-age factors become more volatile as the volume of the data decreases.
- There is no systematic difference in the bias level between the paid and incurred factors. At a first glance, this result is some-

what surprising and counterintuitive, because paid loss development factors are larger and more leveraged than incurred loss development factors. However, most internal and external factors, such as claim processing, late reported claims, inflation, underwriting cycles, and economic cycles, affect both paid and incurred loss development factors. As indicated in Equation (1.1), the bias depends on the skewness and volatility of the data, as represented by σ_i, but not on the level or the magnitude of the data, as represented by μ_i. Further research indicates that the sample paid loss factors and incurred loss factors used in the study have similar degrees of skewness. For example, the averages of the sample σ for 12–24 paid and incurred factors for product liability data are not very different, 0.518 and 0.563, respectively.

3.C. Large Scale Simulations for the Limited Volume Data

As mentioned before, in theory, we need to have an infinitely large amount of loss development data in order to apply Equation (1.1) in calculating the downward bias of high-low averages. The real-world data is limited and, therefore, will deviate somewhat from the asymptotic assumptions underlying Equation (1.1). As a result, there are two issues when Equation (1.1) is used with a limited volume of data. First, true means and variances are usually unknown, and sample means and variances from the data need to be used. Second, Equation (1.1) calculates the bias when one assumes that the data volume is very large, while the three-of-five average, for example, uses only five data points.

In order to study the limited volume data effect, we designed a large scale simulation test. The simulation procedures and results are as follows:

1. A set of μ_i and σ_i are selected. The range for μ_i is between 0.1 and 2.0 and the range for σ_i is between 0.002 to 1.2. These ranges are based on the A. M. Best data reviewed in the study. See Exhibits 14 and 15 for the

selected combinations of μ_i and σ_i. These selected combinations of μ_i and σ_i represent the true parameters of the underlying distribution for the simulations.

2. 4,000 lognormal observations based on the selected μ_i and σ_i are generated. Each observation contains five random data points.

3. For each observation, the sample parameters from the five random data points are calculated. The bias using Equation (1.1) with the sample parameters is calculated. The bias result is compared to the bias based on the true parameters of μ_i and σ_i. Since the sample parameters are different from the true parameters of μ_i and σ_i, the bias based on the sample parameters may be higher or lower than the bias based on the true parameters. This is the effect of the use of the sample parameters. Exhibit 14 shows the comparison based on the overall 4,000 generated observations. The result indicates that the bias based on the sample parameters on average will be lower than the bias based on the true parameters. For example, when $\sigma_i = 1.2$ and $\mu_i = 1.0$, the bias on average will be understated by 8.5% for the sample parameters.

4. Finally, for each observation, the three-of-five average is calculated by excluding the lowest and highest data points. The three-of-five average is compared to the expected average of the lognormal distribution with the selected σ_i and μ_i to obtain the downward bias. The downward bias for the observation is compared to the expected downward bias based on Equation (1.1) with the selected σ_i, μ_i, and $p = 20\%$. This is the effect of the limited volume of data since the bias for each of the observations is based on only five data points, while the bias based on Equation (1.1) is based on a large volume of data. Exhibit 15 shows that the bias is tempered somewhat for the limited volume data. For example, when $\sigma_i = 1.2$ and

$\mu_i = 1.0$, the simulated bias for the three-of-five on average is approximately 67.5% of the bias calculated by Equation (1.1) for a large volume of data.

Exhibits 14 and 15 also show that the effects of the limited volume of data on the bias depend primarily on σ_i, not on μ_i. The effects diminish quickly as σ_i decreases.

Please note that the two effects in Exhibits 14 and 15 are separately studied because, in theory, the effect of sample parameters may not exist. This occurs when there is prior knowledge of the true values for μ and σ. With known μ and σ, there still exists the effect for limited sample size as given in Exhibit 15 when only five data points are used to calculate the three-of-five averages.

3.D. Summary of the Results

The current study presents strong evidence, through a comprehensive review of property and casualty insurance industry data, that downward bias will occur when high-low averages are used to determine age-to-age development factors. The review results show the level of the bias by line of business, development age, premium size, and paid versus incurred methods. The results indicate that the downward bias can be substantial, especially for small companies and highly volatile lines.

Equations (1.1) and (1.2) provide a basis to quantify and correct the bias. Equation (1.1) is based on a large volume of data, while only a limited volume of data is available for most real-world applications. The simulation results show that the bias for the limited volume of data, on average, is somewhat lower than what is indicated by Equation (1.1).

4. CONCLUSIONS

Many property and casualty actuaries are undoubtedly aware of the downward bias associated with the high-low averages. While this study focuses on the loss development application, the results and implications should go beyond that application, and can be extended to many other actuarial applications if the underlying data shows a long-tail property.

Also, the real-world data that actuaries deal with daily may have even higher levels of bias than indicated in this study for the following reasons:

- The bias will increase if less mature data or quarterly and semi-annual data are used.
- Due to the data limitation, the results given in this study only include the bias for the first five development periods and real-world data would allow a more thorough bias analysis beyond the fifth development age.
- The bias is demonstrated and quantified through the lognormal assumption in this study. The assumption may understate the thickness of the tail for insurance data (see Hogg and Klugman [5]). If the tail of the loss development factors distribution is more skewed than what is suggested by the lognormal distribution, the bias will be higher than indicated by Equation (1.1).

As usual, many assumptions used in the current study are ideal. Attempts to study the bias under more complicated assumptions are beyond the scope of the current study because they require advanced statistical knowledge. They can be topics for future research, however. For example, nonparametric methods may be used to explain the effects of limited volume. Another interesting topic would be to study the bias when loss development factors are highly correlated between development periods.

Finally, it should be noted that this paper does not attempt to suggest the high-low averaging approach be completely excluded from consideration by actuaries. The paper does attempt to indicate the potential bias if the approach is applied to insurance data on a comprehensive basis without an in-depth understanding of the data. The principle that no one arithmetic approach is superior to or inferior to all others will not and should not be altered by the results given in the paper. Perhaps, the key message delivered by the paper is the need for even more substantial professional judgment by actuaries in promulgating reserving and pricing estimates.

REFERENCES

[1] A. M. Best, *Best's Casualty Loss Reserve Development Series*., Best Database Services, 1996.

[2] Bowers, N. L., et al., *Actuarial Mathematics*, Society of Actuaries, 1986.

[3] Finger, R. J., "Estimating Pure Premiums by Layer—An Approach," *PCAS* LXIII, 1976, pp. 34–52.

[4] Hayne, R. M., "An Estimate of Statistical Variation in Development Factor Methods," *PCAS* LXXII, 1985, pp. 25–43.

[5] Hogg, R. V., and S. A. Klugman, *Loss Distributions*, John Wiley & Sons, Inc., 1984.

[6] Kelly, M. V., "Practical Loss Reserving Method with Stochastic Development Factors," *Insurer Financial Solvency*, Casualty Actuarial Society Discussion Paper Program, 1992, pp. 355–381.

[7] McNichols, J. P., "Simplified Confidence Boundaries Associated with Calendar Year Projections," *Insurer Financial Solvency*, Casualty Actuarial Society Discussion Paper Program, 1992, pp. 465–509.

[8] Neter, J., W. Wasserman, and M. H. Kutner, *Applied Liner Regression Models* (2nd edition), Richard D. Irwin, Inc., 1989.

[9] Wu, C. P., "Bias of Excluding High and Low Data for Long-Tailed Distributions," *Journal of Actuarial Practice* 4, 1996, pp. 143–158.

EXHIBIT 1

Shape of a Typical Lognormal Distribution with Downward Bias of High-Low Average for a Lognormal Distribution

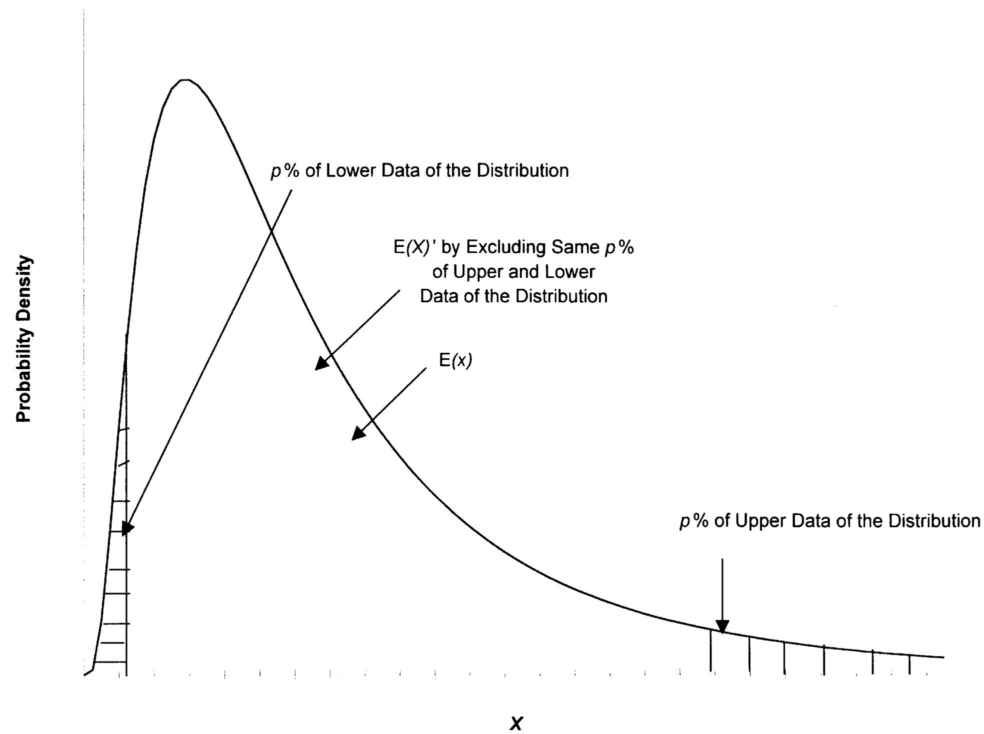

EXHIBIT 2

Product Liability Paid Loss and Loss Development Factor Triangles*

Paid Losses: (in Thousands)

Accident Year	Earned Premium	Development Period, Months 12	24	36	48	60	72	84	96	108	120
1986	$ 55,779	$ 446	$ 1,618	$ 4,685	$ 7,809	$13,722	$17,849	$18,240	$18,742	$19,076	$19,244
1987	$ 60,737	$ 61	$ 1,336	$ 3,341	$ 6,377	$ 9,596	$11,662	$12,876	$13,301	$13,909	
1988	$ 75,602	$ 302	$ 3,326	$ 6,804	$14,516	$16,254	$17,918	$21,169	$22,000		
1989	$ 82,764	$ 414	$ 3,228	$ 6,125	$12,994	$17,298	$23,505	$25,491			
1990	$103,688	$ 415	$ 3,111	$11,406	$18,249	$20,738	$23,537				
1991	$116,481	$1,747	$ 8,037	$23,063	$33,663	$41,467					
1992	$128,505	$2,956	$14,907	$30,199	$43,563						
1993	$137,629	$2,064	$12,249	$25,737							
1994	$153,565	$2,764	$12,746								
1995	$170,085	$3,232									

Age-to-Age Factors:

Accident Year	Earned Premium	Development Period, Months 12–24	24–36	36–48	48–60	60–72	72–84	84–96	96–108	108–120	
1986	$ 55,779	3.6250	2.8966	1.6667	1.7571	1.3008	1.0219	1.0275	1.0179	1.0088	
1987	$ 60,737	22.0000	2.5000	1.9091	1.5048	1.2152	1.1042	1.0330	1.0457		
1988	$ 75,602	11.0000	2.0455	2.1333	1.1198	1.1023	1.1814	1.0393			
1989	$ 82,764	7.8000	1.8974	2.1216	1.3312	1.3589	1.0845				
1990	$103,688	7.5000	3.6667	1.6000	1.1364	1.1350					
1991	$116,481	4.6000	2.8696	1.4596	1.2318						
1992	$128,505	5.0435	2.0259	1.4426							
1993	$137,629	5.9333	2.1011								
1994	$153,565	4.6111									
1995	$170,085										
Age-to-Age Development Factors:											Tail**
5-Year Average***		5.5376	2.5121	1.7514	1.2648	1.2224	1.0980	1.0333	1.0318	1.0088	1.0261
3-of-5 Average***		5.1960	2.3322	1.7271	1.2331	1.2170	1.0980	1.0333	1.0318	1.0088	1.0261
Age-to-Ultimate Development Factors:											
5-Year Average***		45.6427	8.2423	3.2810	1.8733	1.4811	1.2116	1.1035	1.0680	1.0351	1.0261
3-of-5 Average***		38.0553	7.3240	3.1404	1.8183	1.4746	1.2116	1.1035	1.0680	1.0351	1.0261

*This is product liability paid loss data from a sample company in the A.M. Best database [1].
**The tail factor of 1.0261 is the ratio of incurred to paid loss for 1986.
***For the last four development periods, straight averages are used in place of the specified averages.

EXHIBIT 3

LOGNORMAL PARAMETERS FOR LOSS DEVELOPMENT FACTORS

Natural Logarithm Transformation of the Age-to-Age Factors in Exhibit 2: Accident Year	Development Period, Months 12–24	24–36	36–48	48–60	60–72	72–84	84–96	96–108	108–120
1986	1.2879	1.0635	0.5108	0.5637	0.2630	0.0216	0.0272	0.0177	0.0087
1987	3.0910	0.9163	0.6466	0.4086	0.1949	0.0991	0.0325	0.0447	
1988	2.3979	0.7156	0.7577	0.1131	0.0974	0.1667	0.0385		
1989	2.0541	0.6405	0.7522	0.2861	0.3066	0.0811			
1990	2.0149	1.2993	0.4700	0.1278	0.1266				
1991	1.5261	1.0542	0.3782	0.2085					
1992	1.6181	0.7060	0.3664						
1993	1.7806	0.7425							
1994	1.5285								
1995									
Age-to-Age Development Factors:									
Estimated Mu	1.9221	0.8922	0.5546	0.2846	0.1977	0.0921	0.0327	0.0312	0.0087
Estimated Sigma Square	0.3057	0.0534	0.0274	0.0306	0.0078	0.0036	0.0000		
Age-to-Ultimate Development Factors:									
Estimated Mu	4.0160	2.0939	1.2017	0.6471	0.3625	0.1648	0.0726	0.0399	0.0087
Estimated Sigma Square	0.4284	0.1228	0.0694	0.0420	0.0114	0.0036	0.0000	0.0000	0.0000

EXHIBIT 4

MODIFIED HIGH-LOW AVERAGES FOR LOSS DEVELOPMENT FACTORS

Age-to-Age Factors in Exhibit 2: Accident Year	12–24	24–36	36–48	48–60	60–72	72–84	84–96	96–108	108–120	
	Development Period, Months									
1986	3.6250	2.8966	1.6667	1.7571	1.3008	1.0219	1.0275	1.0179	1.0088	
1987	22.0000	2.5000	1.9091	1.5048	1.2152	1.1042	1.0330	1.0457		
1988	11.0000	2.0455	2.1333	1.1198	1.1023	1.1814	1.0393			
1989	7.8000	1.8974	2.1216	1.3312	1.3589	1.0845				
1990	7.5000	3.6667	1.6000	1.1364	1.1350					
1991	4.6000	2.8696	1.4596	1.2318						
1992	5.0435	2.0259	1.4426							
1993	5.9333	2.1011								
1994	4.6111									
1995										
Age-to-Age Development Factors:										Tail
5-Year Average	5.5376	2.5121	1.7514	1.2648	1.2224	1.0980	1.0333	1.0318	1.0088	1.0261
Lognormal Parameters from Exhibit 3:										
Estimated Mu	1.9221	0.8922	0.5546	0.2846	0.1977	0.0921	0.0327	0.0312	0.0087	
Estimated Sigma Square	0.3057	0.0534	0.0274	0.0306	0.0078	0.0036	0.0000	0.0000	0.0000	
3-of-5 Average	5.1960	2.3322	1.7271	1.2331	1.2170	1.0980	1.0333	1.0318	1.0088	1.0261
% of High and Low Data Excluded	20.0%	20.0%	20.0%	20.0%	20.0%					
Indicated Downward Bias	−11.33%	−2.07%	−1.07%	−1.19%	−0.31%					
Modified 3-of-5 Average	5.8598	2.3816	1.7458	1.2480	1.2207	1.0980	1.0333	1.0318	1.0088	1.0261
Age-to-Ultimate Development Factors:										
5-Year Average	45.6427	8.2423	3.2810	1.8733	1.4811	1.2116	1.1035	1.0680	1.0351	1.0261
3-of-5 Average	38.0553	7.3240	3.1404	1.8183	1.4746	1.2116	1.1035	1.0680	1.0351	1.0261
Modified 3-of-5 Average	44.9738	7.6750	3.2226	1.8460	1.4791	1.2116	1.1035	1.0680	1.0351	1.0261

EXHIBIT 5

COMPARISON OF ULTIMATE LOSSES AND RESERVES ACROSS DIFFERENT AVERAGING TECHNIQUES

Age-to-Ult Loss Development Factors

Accident Year	Undeveloped Paid Losses	5-Year Average	3-of-5 Average	Modified 3-of-5 Average
1991	$ 41,467	1.4811	1.4746	1.4791
1992	$ 43,563	1.8733	1.8183	1.8460
1993	$ 25,737	3.2810	3.1404	3.2226
1994	$ 12,746	8.2423	7.3240	7.6750
1995	$ 3,232	45.6427	38.0553	44.9738
Total:	$126,745			

Ultimate Losses

Accident Year	5-Year Average	3-of-5 Average	Modified 3-of-5 Average	Difference 3-of-5 and 5-Year	Difference 3-of-5 and Mod 3-of-5
1991	$ 61,419	$ 61,146	$ 61,334	−0.4%	−0.3%
1992	$ 81,609	$ 79,213	$ 80,417	−2.9%	−1.5%
1993	$ 84,442	$ 80,823	$ 82,940	−4.3%	−2.6%
1994	$105,056	$ 93,351	$ 97,824	−11.1%	−4.6%
1995	$147,500	$122,980	$145,338	−16.6%	−15.4%
Total:	$480,026	$437,513	$467,853	−8.9%	−6.5%

Total Reserves

Accident Year	5-Year Average	3-of-5 Average	Modified 3-of-5 Average	Difference 3-of-5 and 5-Year	Difference 3-of-5 and Mod 3-of-5
1991	$ 19,952	$ 19,679	$ 19,867	−1.4%	−0.9%
1992	$ 38,046	$ 35,649	$ 36,854	−6.3%	−3.3%
1993	$ 58,706	$ 55,087	$ 57,203	−6.2%	−3.7%
1994	$ 92,310	$ 80,605	$ 85,078	−12.7%	−5.3%
1995	$144,268	$119,748	$142,106	−17.0%	−15.7%
Total:	$353,281	$310,768	$341,108	−12.0%	−8.9%

EXHIBIT 6

A.M. Best Data

		Annual Earned Premium from 1986 to 1995 (in Millions)			Data with Lower Reserve Indications for 3-of-5 Averages*	
	Number of Companies Sampled	Minimum	Maximum	Average	Paid Loss Method	Incurred Loss Method
Group A: Multistate, Multiline Insurance Companies/Groups:						
Workers Compensation	5	$426	$ 1,823	$1,029	3	3
Private Passenger Automobile Liability	5	$543	$14,126	$3,651	2	3
Commercial Automobile Liability	5	$151	$ 682	$ 354	2	2
Medical Malpractice-Occurrence	5	$ 14	$ 270	$ 71	5	5
Medical Malpractice-Claims-Made	5	$ 44	$ 700	$ 186	4	3
Product Liability	5	$ 43	$ 218	$ 115	5	5
Other Liability	5	$199	$ 1,221	$ 611	3	3
Total	35				24	24
Group B: Regional or Single State Insurance Companies:						
Workers Compensation	5	$ 14	$ 137	$ 60	2	3
Private Passenger Automobile Liability	5	$ 26	$ 122	$ 62	3	3
Commercial Automobile Liability	5	$ 19	$ 99	$ 47	3	2
Medical Malpractice-Occurrence	5	$ 2	$ 53	$ 17	5	5
Medical Malpractice-Claims-Made	5	$ 20	$ 64	$ 39	5	3
Product Liability	5	$ 5	$ 50	$ 29	5	5
Other Liability	5	$ 12	$ 98	$ 54	5	3
Total	35				28	24
Group A and Group B Combined:						
Workers Compensation	10	$ 14	$ 1,823	—	5	6
Private Passenger Automobile Liability	10	$ 26	$14,126	—	5	6
Commercial Automobile Liability	10	$ 19	$ 682	—	5	4
Medical Malpractice-Occurrence	10	$ 2	$ 270	—	10	10
Medical Malpractice-Claims-Made	10	$ 20	$ 700	—	9	6
Product Liability	10	$ 5	$ 218	—	10	10
Other Liability	10	$ 12	$ 1,221	—	8	6
Total	70				52	48

*Reserve indications were compared between 5-year averages and 3-of-5 averages. This is the data where 3-of-5 averages have a lower reserve indication.

EXHIBIT 7

Review Results of A.M. Best Workers Compensation Data

Indicated Downward Bias for 3-of-5 Age-to-Age Factors*:

Paid 3-of-5 Averages		12–24 Months	24–36 Months	36–48 Months	48–60 Months	60–72 Months
Group A-Large Companies	Miminum	−0.25%	0.00%	0.00%	0.00%	0.00%
	Maximum	−2.68%	−0.10%	−0.02%	−0.02%	0.00%
	Average	−0.80%	−0.05%	−0.01%	−0.01%	0.00%
Group B-Small to Medium Companies	Miminum	−0.03%	−0.02%	−0.01%	0.00%	0.00%
	Maximum	−0.72%	−0.22%	−0.20%	−0.09%	−0.09%
	Average	−0.25%	−0.07%	−0.05%	−0.02%	−0.02%

Incurred 3-of-5 Averages		12–24 Months	24–36 Months	36–48 Months	48–60 Months	60–72 Months
Group A-Large Companies	Miminum	−0.10%	−0.01%	−0.01%	−0.01%	−0.01%
	Maximum	−0.78%	−0.06%	−0.05%	−0.03%	−0.02%
	Average	−0.37%	−0.03%	−0.02%	−0.02%	−0.01%
Group B-Small to Medium Companies	Miminum	−0.07%	−0.02%	−0.01%	0.00%	0.00%
	Maximum	−1.07%	−0.16%	−0.13%	−0.05%	−0.02%
	Average	−0.57%	−0.10%	−0.05%	−0.02%	−0.01%

Indicated Downward Bias for 3-of-5 Reserve Indications:**

		Paid Loss Development Method		Incurred Loss Development Method	
		Total Reserves	IBNR Reserves	Total Reserves	IBNR Reserves
Group A-Large Companies	Miminum	−0.05%	−0.11%	−0.11%	−0.32%
	Maximum	−1.37%	−2.92%	−0.30%	−0.77%
	Average	−0.37%	−0.85%	−0.22%	−0.54%
Group B-Small to Medium Companies	Miminum	−0.06%	−0.15%	−0.09%	−0.32%
	Maximum	−1.37%	−3.63%	−0.73%	−1.73%
	Average	−0.38%	−0.96%	−0.45%	−1.04%

*The indicated downward bias for 3-of-5 factors is based on Equation (1.1).

**The indicated downward bias for reserves is the difference in reserve indications between 3-of-5 averages and modified 3-of-5 averages.

EXHIBIT 8

REVIEW RESULTS OF A.M. BEST PRIVATE PASSENGER AUTOMOBILE LIABILITY DATA

Indicated Downward Bias for 3-of-5 Age-to-Age Factors*:

Paid 3-of-5 Averages		12–24 Months	24–36 Months	36–48 Months	48–60 Months	60–72 Months
Group A-Large Companies	Miminum	−0.04%	0.00%	0.00%	0.00%	0.00%
	Maximum	−0.22%	−0.01%	−0.02%	0.00%	0.00%
	Average	−0.08%	−0.01%	−0.01%	0.00%	0.00%
Group B-Small to Medium Companies	Miminum	−0.09%	−0.01%	0.00%	0.00%	0.00%
	Maximum	−0.39%	−0.14%	−0.04%	−0.03%	−0.02%
	Average	−0.16%	−0.04%	−0.02%	−0.01%	0.00%

Incurred 3-of-5 Averages		12–24 Months	24–36 Months	36–48 Months	48–60 Months	60–72 Months
Group A-Large Companies	Miminum	−0.01%	0.00%	0.00%	0.00%	0.00%
	Maximum	−0.14%	−0.02%	−0.01%	0.00%	0.00%
	Average	−0.06%	−0.01%	0.00%	0.00%	0.00%
Group B-Small to Medium Companies	Miminum	−0.03%	−0.02%	−0.01%	0.00%	0.00%
	Maximum	−0.20%	−0.07%	−0.04%	−0.03%	−0.01%
	Average	−0.13%	−0.04%	−0.02%	−0.01%	0.00%

Indicated Downward Bias for 3-of-5 Reserve Indications:**

		Paid Loss Development Method		Incurred Loss Development Method	
		Total Reserves	IBNR Reserves	Total Reserves	IBNR Reserves
Group A-Large Companies	Miminum	−0.04%	−0.08%	−0.03%	−0.08%
	Maximum	−0.17%	−0.38%	−0.13%	−1.93%
	Average	−0.08%	−0.21%	−0.08%	−0.56%
Group B-Small to Medium Companies	Miminum	−0.11%	−0.59%	−0.08%	−0.23%
	Maximum	−0.59%	−1.55%	−2.31%	−7.39%
	Average	−0.27%	−0.98%	−0.60%	−2.14%

*The indicated downward bias for 3-of-5 factors is based on Equation (1.1).

**The indicated downward bias for reserves is the difference in reserve indications between 3-of-5 averages and modified 3-of-5 averages.

EXHIBIT 9

REVIEW RESULTS OF A.M. BEST COMMERCIAL AUTOMOBILE LIABILITY DATA

Indicated Downward Bias for 3-of-5 Age-to-Age Factors*:

Paid 3-of-5 Averages		12–24 Months	24–36 Months	36–48 Months	48–60 Months	60–72 Months
Group A–Large Companies	Miminum	−0.03%	−0.01%	0.00%	0.00%	0.00%
	Maximum	−1.77%	−0.16%	−0.03%	−0.04%	−0.03%
	Average	−0.56%	−0.07%	−0.02%	−0.02%	−0.01%
Group B-Small to Medium Companies	Miminum	−0.10%	−0.15%	−0.02%	−0.01%	0.00%
	Maximum	−1.21%	−0.43%	−0.13%	−0.07%	−0.07%
	Average	−0.46%	−0.22%	−0.07%	−0.03%	−0.02%

Incurred 3-of-5 Averages		12–24 Months	24–36 Months	36–48 Months	48–60 Months	60–72 Months
Group A-Large Companies	Miminum	−0.04%	−0.02%	−0.01%	0.00%	0.00%
	Maximum	−0.91%	−0.18%	−0.11%	−0.06%	−0.02%
	Average	−0.35%	−0.07%	−0.04%	−0.02%	−0.01%
Group B-Small to Medium Companies	Miminum	−0.18%	−0.04%	−0.01%	−0.01%	0.00%
	Maximum	−0.48%	−0.21%	−0.06%	−0.06%	−0.01%
	Average	−0.31%	−0.09%	−0.02%	−0.03%	−0.01%

Indicated Downward Bias for 3-of-5 Reserve Indications:**

		Paid Loss Development Method		Incurred Loss Development Method	
		Total Reserves	IBNR Reserves	Total Reserves	IBNR Reserves
Group A-Large Companies	Miminum	−0.06%	−0.09%	−0.07%	−0.12%
	Maximum	−1.31%	−2.17%	−0.92%	−1.85%
	Average	−0.50%	−0.98%	−0.29%	−0.66%
Group B-Small to Medium Companies	Miminum	−0.39%	−0.77%	−0.22%	−0.59%
	Maximum	−1.32%	−7.32%	−3.63%	−10.37%
	Average	−0.78%	−2.66%	−1.11%	−4.57%

*The indicated downward bias for 3-of-5 factors is based on Equation (1.1).

**The indicated downward bias for reserves is the difference in reserve indications between 3-of-5 averages and modified 3-of-5 averages.

EXHIBIT 10

REVIEW RESULTS OF A.M. BEST MEDICAL MALPRACTICE—OCCURRENCE DATA

Indicated Downward Bias for 3-of-5 Age-to-Age Factors*:

Paid 3-of-5 Averages		12–24 Months	24–36 Months	36–48 Months	48–60 Months	60–72 Months
Group A-Large Companies	Miminum	−3.84%	− 2.02%	−0.34%	− 0.12%	− 0.09%
	Maximum	−23.00%	−14.79%	−6.24%	− 2.01%	− 1.55%
	Average	−14.39%	− 5.46%	−1.96%	− 0.87%	− 0.52%
Group B-Small to Medium Companies	Miminum	−10.30%	− 1.22%	−0.51%	− 0.10%	− 0.14%
	Maximum	−22.99%	− 9.79%	−2.25%	− 2.73%	− 0.99%
	Average	−15.75%	− 5.75%	−1.31%	− 0.92%	− 0.37%

Incurred 3-of-5 Averages		12–24 Months	24–36 Months	36–48 Months	48–60 Months	60–72 Months
Group A-Large Companies	Miminum	−0.68%	− 0.57%	−0.28%	− 0.12%	− 0.07%
	Maximum	−30.02%	−21.60%	−2.70%	− 1.33%	− 0.69%
	Average	−12.84%	− 7.67%	−0.96%	− 0.52%	− 0.30%
Group B-Small to Medium Companies	Miminum	− 0.23%	− 0.32%	−0.27%	− 0.14%	− 0.07%
	Maximum	− 7.88%	− 6.08%	−1.33%	− 3.61%	− 4.88%
	Average	− 4.69%	− 2.09%	−0.93%	− 1.13%	− 1.52%

Indicated Downward Bias for 3-of-5 Reserve Indications:**

		Paid Loss Development Method		Incurred Loss Development Method	
		Total Reserves	IBNR Reserves	Total Reserves	IBNR Reserves
Group A-Large Companies	Miminum	−3.19%	−9.40%	−1.14%	−5.43%
	Maximum	−13.49%	−24.56%	−60.72%	−68.37%
	Average	−9.92%	−15.81%	−17.50%	−22.11%
Group B-Small to Medium Companies	Miminum	−4.19%	−8.20%	−0.76%	−1.35%
	Maximum	−18.89%	−39.64%	−43.22%	−283.92%
	Average	−13.58%	−23.67%	−16.65%	−91.94%

*The indicated downward bias for 3-of-5 factors is based on Equation (1.1).

**The indicated downward bias for reserves is the difference in reserve indications between 3-of-5 averages and modified 3-of-5 averages.

EXHIBIT 11

REVIEW RESULTS OF A.M. BEST MEDICAL MALPRACTICE—CLAIMS-MADE DATA

Indicated Downward Bias for 3-of-5 Age-to-Age Factors*:

Paid 3-of-5 Averages		12–24 Months	24–36 Months	36–48 Months	48–60 Months	60–72 Months
Group A-Large Companies	Miminum	−0.86%	−0.10%	−0.08%	−0.04%	0.00%
	Maximum	−2.88%	−0.44%	−0.63%	−0.60%	−0.22%
	Average	−2.06%	−0.28%	−0.26%	−0.21%	−0.12%
Group B-Small to Medium Companies	Miminum	−1.45%	−0.39%	−0.11%	−0.05%	−0.01%
	Maximum	−6.95%	−2.31%	−1.04%	−0.24%	−0.78%
	Average	−4.49%	−1.30%	−0.39%	−0.10%	−0.21%

Incurred 3-of-5 Averages		12–24 Months	24–36 Months	36–48 Months	48–60 Months	60–72 Months
Group A-Large Companies	Miminum	−0.27%	−0.19%	−0.12%	−0.03%	0.00%
	Maximum	−2.33%	−0.94%	−0.44%	−0.24%	−0.06%
	Average	−0.95%	−0.44%	−0.27%	−0.11%	−0.03%
Group B-Small to Medium Companies	Miminum	−0.49%	−0.07%	−0.07%	−0.04%	−0.03%
	Maximum	−1.45%	−0.36%	−0.32%	−0.26%	−0.54%
	Average	−0.98%	−0.26%	−0.17%	−0.12%	−0.16%

Indicated Downward Bias for 3-of-5 Reserve Indications:**

		Paid Loss Development Method		Incurred Loss Development Method	
		Total Reserves	IBNR Reserves	Total Reserves	IBNR Reserves
Group A-Large Companies	Miminum	−0.61%	−3.10%	−0.52%	−1.27%
	Maximum	−2.86%	−13.79%	−3.64%	−8.05%
	Average	−1.87%	−6.82%	−1.41%	−4.99%
Group B-Small to Medium Companies	Miminum	−3.05%	−3.90%	−1.41%	−0.94%
	Maximum	−4.28%	−68.72%	−7.36%	−51.12%
	Average	−3.89%	−20.40%	−3.01%	−13.71%

*The indicated downward bias for 3-of-5 factors is based on Equation (1.1).
**The indicated downward bias for reserves is the difference in reserve indications between 3-of-5 averages and modified 3-of-5 averages.

EXHIBIT 12

REVIEW RESULTS OF A.M. BEST PRODUCT LIABILITY DATA

Indicated Downward Bias for 3-of-5 Age-to-Age Factors*:

Paid 3-of-5 Averages		12–24 Months	24–36 Months	36–48 Months	48–60 Months	60–72 Months
Group A-Large Companies	Miminum	−2.44%	−1.45%	−1.02%	−0.30%	−0.16%
	Maximum	−42.19%	−35.08%	−10.36%	−2.04%	−7.65%
	Average	−17.40%	−9.39%	−2.93%	−1.00%	−1.73%
Group B-Small to Medium Companies	Miminum	−1.44%	−0.70%	−0.13%	−0.16%	−0.03%
	Maximum	−13.52%	−5.34%	−3.33%	−1.72%	−0.90%
	Average	−7.08%	−2.59%	−1.19%	−0.62%	−0.26%

Incurred 3-of-5 Averages		12–24 Months	24–36 Months	36–48 Months	48–60 Months	60–72 Months
Group A-Large Companies	Miminum	−1.42%	−1.00%	−0.17%	−0.18%	−0.09%
	Maximum	−27.35%	−17.13%	−2.49%	−3.51%	−4.15%
	Average	−18.17%	−7.00%	−1.34%	−1.02%	−1.15%
Group B-Small to Medium Companies	Miminum	−4.23%	−0.85%	−0.50%	−0.34%	−0.06%
	Maximum	−21.73%	−6.71%	−4.27%	−3.70%	−1.83%
	Average	−9.84%	−3.34%	−2.64%	−1.70%	−0.73%

Indicated Downward Bias for 3-of-5 Reserve Indications:**

		Paid Loss Development Method		Incurred Loss Development Method	
		Total Reserves	IBNR Reserves	Total Reserves	IBNR Reserves
Group A-Large Companies	Miminum	−3.04%	−6.11%	−1.94%	−4.63%
	Maximum	−68.50%	−77.61%	−39.88%	−45.01%
	Average	−22.20%	−27.14%	−22.00%	−28.70%
Group B-Small to Medium Companies	Miminum	−1.59%	10.47%	−1.55%	−5.61%
	Maximum	−5.82%	−15.54%	−12.89%	−35.88%
	Average	−3.26%	−1.52%	−5.48%	−22.19%

*The indicated downward bias for 3-of-5 factors is based on Equation (1.1).

**The indicated downward bias for reserves is the difference in reserve indications between 3-of-5 averages and modified 3-of-5 averages.

EXHIBIT 13

REVIEW RESULTS OF A.M. BEST OTHER LIABILITY DATA

Indicated Downward Bias for 3-of-5 Age-to-Age Factors*:

Paid 3-of-5 Averages		12–24 Months	24–36 Months	36–48 Months	48–60 Months	60–72 Months
Group A-Large Companies	Miminum	−0.30%	−0.12%	−0.04%	−0.05%	−0.02%
	Maximum	−21.90%	−2.04%	−0.41%	−0.21%	−0.23%
	Average	−7.16%	−0.63%	−0.17%	−0.12%	−0.09%
Group B-Small to Medium Companies	Miminum	−1.03%	−0.40%	−0.17%	−0.03%	−0.02%
	Maximum	−8.18%	−3.97%	−4.41%	−0.67%	−0.24%
	Average	−2.98%	−2.28%	−1.29%	−0.33%	−0.10%

Incurred 3-of-5 Averages		12–24 Months	24–36 Months	36–48 Months	48–60 Months	60–72 Months
Group A-Large Companies	Miminum	−0.12%	−0.09%	−0.03%	−0.02%	−0.01%
	Maximum	−3.31%	−0.59%	−0.16%	−0.11%	−0.10%
	Average	−1.23%	−0.29%	−0.09%	−0.07%	−0.05%
Group B-Small to Medium Companies	Miminum	−0.42%	−0.38%	−0.07%	−0.05%	−0.02%
	Maximum	−21.96%	−2.24%	−1.53%	−0.50%	−0.32%
	Average	−8.06%	−0.87%	−0.47%	−0.20%	−0.12%

Indicated Downward Bias for 3-of-5 Reserve Indications:**

		Paid Loss Development Method		Incurred Loss Development Method	
		Total Reserves	IBNR Reserves	Total Reserves	IBNR Reserves
Group A-Large Companies	Miminum	−0.70%	−0.91%	−0.46%	−0.80%
	Maximum	−11.64%	−27.59%	−1.99%	−2.85%
	Average	−3.90%	−8.33%	−1.01%	−1.76%
Group B-Small to Medium Companies	Miminum	−1.47%	−3.91%	−1.14%	−2.08%
	Maximum	−14.28%	−21.24%	−9.29%	−18.19%
	Average	−5.27%	−8.29%	−4.45%	−9.50%

*The indicated downward bias for 3-of-5 factors is based on Equation (1.1).

**The indicated downward bias for reserves is the difference in reserve indications between 3-of-5 averages and modified 3-of-5 averages.

EXHIBIT 14

EFFECT OF SAMPLE PARAMETERS
RATIO OF AVERAGE BIAS
BASED ON SIMULATED SAMPLE PARAMETERS VS. TRUE PARAMETERS

	μ			
σ	2.000	1.000	0.500	0.100
1.200	90.6%	91.5%	91.2%	91.8%
0.900	93.2%	93.2%	94.9%	94.1%
0.500	97.5%	97.7%	97.3%	97.9%
0.100	99.5%	99.9%	99.5%	99.6%
0.050	100.2%	98.8%	100.4%	100.9%
0.002	99.4%	100.6%	100.9%	97.9%

EXHIBIT 15

EFFECT OF LIMITED SAMPLE SIZE
RATIO OF SIMULATED BIAS TO BIAS BASED ON EQUATION (1.1) FOR THREE-OF-FIVE AVERAGES

	μ			
σ	2.000	1.000	0.500	0.100
1.200	68.3%	67.5%	67.4%	67.1%
0.900	80.7%	80.2%	80.6%	80.6%
0.500	93.1%	92.8%	93.6%	93.8%
0.100	99.8%	99.8%	99.9%	99.7%
0.050	99.9%	99.9%	99.9%	99.9%
0.002	100.0%	100.0%	100.0%	100.0%

APPENDIX

DOWNWARD BIAS FOR TWO LONG-TAILED DISTRIBUTIONS

This Appendix shows the derivations of the downward bias based on the cumulative distribution functions for two long-tailed distributions, lognormal and Pareto. Many of the details of these two distributions can be found in Hogg and Klugman [5] or other statistical texts.

First, the following list specifies the global notations for the two distributions:

$\mathrm{E}(X)$: expected value for random variable X;

$\mathrm{E}(X)'$: expected value of X when excluding the upper $p\%$ and lower $p\%$ of data;

$F(x)$: cumulative probability function;

$f(x)$: probability density function;

p: percentile;

x_1: value of X when $F(x) = p$;

x_2: value of X when $F(x) = 1 - p$;

Φ: standard normal distribution function $= \int_{-\infty}^{x} \dfrac{\exp(\frac{1}{2}x^2)}{\sqrt{2\pi}}\,dx$;

ϕ: standard normal density function $= \exp(\frac{1}{2}x^2)/\sqrt{2\pi}$.

A.1. Lognormal Distribution

a. Probability Density Function:

$$f(x) = \frac{\exp\left(\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2\right)}{x\sigma\sqrt{2\pi}}.$$

b. Cumulative Probability Function:

$$F(x) = \int_0^\infty \frac{\exp\left(\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2\right)}{x\sigma\sqrt{2\pi}} dx.$$

Let

$$x = e^{\sigma y + \mu}, \qquad \text{then} \quad y = \frac{\ln x - \mu}{\sigma}, \quad \text{and} \quad dx = e^{\sigma y + \mu}\sigma\, dy.$$

$$F(x) = \int_{-\infty}^{\ln x - \mu/\sigma} \frac{e^{-y^2/2} e^{\sigma y + \mu}\sigma}{e^{\sigma y + \mu}\sigma\sqrt{2\pi}} dy = \Phi\left(\frac{\ln x - \mu}{\sigma}\right).$$

$$F(x_1) = \Phi\left(\frac{\ln x_1 - \mu}{\sigma}\right) = p, \qquad x_1 = e^{(\Phi^{-1}(p)\sigma + \mu)}.$$

$$F(x_2) = \Phi\left(\frac{\ln x_2 - \mu}{\sigma}\right) = 1 - p, \qquad x_2 = e^{(\Phi^{-1}(1-p)\sigma + \mu)}.$$

c. Expected Value of X:

$$\mathrm{E}(X) = \int_0^\infty x \frac{e^{-1/2(\ln x - \mu/\sigma)^2}}{x\sigma\sqrt{2\pi}} dx = \int_0^\infty \frac{e^{-1/2(\ln x - \mu/\sigma)^2}}{\sigma\sqrt{2\pi}} dx.$$

Let

$$y = \frac{\ln x - \mu - \sigma^2}{\sigma}, \qquad \text{then} \quad x = e^{\sigma y + \mu + \sigma^2}, \quad \text{and}$$

$$dx = e^{\sigma y + \mu + \sigma^2}\sigma\, dy.$$

$$\mathrm{E}(X) = \int_0^\infty \frac{e^{-1/2(y+\sigma)^2} e^{\sigma y + \mu + \sigma^2}\sigma}{\sigma\sqrt{2\pi}} dx$$

$$= e^{(\mu + (1/2)\sigma^2)} \int_0^\infty \frac{e^{-(1/2)y^2}}{\sqrt{2\pi}} dx = e^{(\mu + (1/2)\sigma^2)}.$$

d. Expected Value of X when Excluding Upper $p\%$ and Lower $p\%$ of Data:

$$\mathrm{E}(X)' = \int_{x_1}^{x_2} x \frac{e^{-1/2(\ln x - \mu/\sigma)^2}}{(1-2p)x\sigma\sqrt{2\pi}} dx = \int_{x_1}^{x_2} \frac{e^{-1/2(\ln x - \mu/\sigma)^2}}{(1-2p)\sigma\sqrt{2\pi}} dx.$$

Let

$$y = \frac{\ln x - \mu - \sigma^2}{\sigma}, \qquad \text{then} \quad x = e^{\sigma y + \mu + \sigma^2}, \quad \text{and}$$

$$dx = e^{\sigma y + \mu + \sigma^2} \sigma \, dy.$$

$$\mathrm{E}(X)' = \frac{e^{(\mu + (1/2)\sigma^2)}}{(1-2p)} \int_{(\ln x_1 - \mu - \sigma^2)/\sigma}^{(\ln x_2 \mu - \sigma^2/\sigma} \frac{e^{-1/2y^2}}{\sqrt{2\pi}} dx$$

$$= \frac{e^{(\mu + (1/2)\sigma^2)}}{(1-2p)} \left(\Phi\left(\frac{\ln x_2 - \mu - \sigma^2)}{\sigma} \right) - \Phi\left(\frac{\ln x_1 - \mu - \sigma^2}{\sigma} \right) \right).$$

$$x_1 = e^{(\Phi^{-1}(p)\sigma + \mu)} \qquad \text{and} \qquad x_2 = e^{(\Phi^{-1}(1-p)\sigma + \mu)}, \quad \text{then}$$

$$\mathrm{E}(X)' = \frac{e^{(\mu + (1/2)\sigma^2)}}{(1-2p)} [\Phi(\Phi^{-1}(1-p) - \sigma)) - \Phi(\Phi^{-1}(p) - \sigma)].$$

e. Downward Bias for Excluding Upper $p\%$ and Lower $p\%$ of Data:

$$\text{Bias} = \frac{\mathrm{E}(x)'}{\mathrm{E}(x)} - 1$$

$$= \frac{1}{(1-2p)} [\Phi(\Phi^{-1}(1-p) - \sigma)) - \Phi(\Phi^{-1}(p) - \sigma)] - 1.$$

The above result indicates that the degree of bias depends on p, the percentage of data being excluded, and σ, the shape factor, only. The bias does not depend on μ, the location parameter.

A.2. Pareto Distribution

a. Probability Density Function:

$$f(x) = \alpha \lambda^{\alpha} (\lambda + x)^{-\alpha - 1}, \qquad x > 0.$$

b. Cumulative Probability Function:

$$F(x) = \int_0^x \alpha\lambda^\alpha(\lambda + x)^{-\alpha-1}\,dx = -\left(\frac{\lambda}{\lambda + x}\right)^\alpha\Bigg|_0^x = 1 - \left(\frac{\lambda}{\lambda + x}\right)^\alpha.$$

$$F(x_1) = p, \qquad \text{then} \quad x_1 = \lambda \times \left(\frac{1}{(1-p)^{1/\alpha}} - 1\right).$$

$$F(x_2) = 1 - p, \qquad \text{then} \quad x_2 = \lambda \times \left(\frac{1}{p^{1/\alpha}} - 1\right).$$

c. Expected Value of X:

$$\begin{aligned}
\mathrm{E}(X) &= \int_0^\infty x\alpha\lambda^\alpha(\lambda + x)^{-\alpha-1}\,dx = -\left(\frac{\lambda}{\lambda + x}\right)^\alpha x\Bigg|_0^\infty \\
&\quad + \int_0^\infty \lambda^\alpha(\lambda + x)^{-\alpha}\,dx \\
&= \int_0^\infty \lambda^\alpha(\lambda + x)^{-\alpha}\,dx = -\frac{\lambda}{\alpha - 1}\left(\frac{\lambda}{\lambda + x}\right)^{-(\alpha-1)}\Bigg|_0^\infty \\
&= \frac{\lambda}{\alpha - 1}.
\end{aligned}$$

d. Expected Value of X when Excluding Upper $p\%$ and Lower $p\%$ of Data:

$$\begin{aligned}
\mathrm{E}(X)' &= \int_{x_1}^{x_2} x\frac{\alpha\lambda^\alpha(\lambda + x)^{-\alpha-1}}{1 - 2p}dx = -x\,\frac{\left(\frac{\lambda}{\lambda + x}\right)^\alpha}{1 - 2p}\Bigg|_{x_1}^{x_2} \\
&\quad + \int_{x_1}^{x_2} \frac{\lambda^\alpha(\lambda + x)^{-\alpha}}{1 - 2p}dx \\
&= -x\,\frac{\left(\frac{\lambda}{\lambda + x}\right)^\alpha}{1 - 2p}\Bigg|_{x_1}^{x_2} - \frac{\lambda\left(\frac{\lambda}{\lambda + x}\right)^{(\alpha-1)}}{(\alpha - 1)(1 - 2p)}\Bigg|_{x_1}^{x_2}.
\end{aligned}$$

Since

$$\frac{\lambda}{\lambda + x_1} = \frac{\lambda}{\lambda + \lambda\left(\frac{1}{(1-p)^{1/\alpha}} - 1\right)} = (1-p)^{1/\alpha}, \qquad \text{and}$$

$$\frac{\lambda}{\lambda + x_2} = \frac{\lambda}{\lambda + \lambda\left(\frac{1}{p^{1/\alpha}} - 1\right)} = p^{1/\alpha},$$

then,

$$\begin{aligned}\mathrm{E}(X)' &= \frac{\lambda}{1-2p}\Bigg[-p^{\alpha-1/\alpha}(1-p^{1/\alpha}) + (1-p)^{\alpha-1/\alpha}(1-(1-p)^{1/\alpha}) \\ &\qquad - \frac{p^{\alpha-1/\alpha}}{\alpha-1} + \frac{(1-p)^{\alpha-1/\alpha}}{\alpha-1}\Bigg] \\ &= \frac{\lambda}{(\alpha-1)(1-2p)}[\alpha(-p^{\alpha-1/\alpha} + (1-p)^{\alpha-1/\alpha}) - (\alpha-1)(1-2p)].\end{aligned}$$

e. Downward Bias for Excluding Upper p% and Lower p% of Data:

$$\begin{aligned}\text{Bias} &= \frac{\mathrm{E}(X)'}{\mathrm{E}(X)} - 1 \\ &= \frac{\alpha}{(1-2p)}[-p^{\alpha-1/\alpha} + (1-p)^{\alpha-1/\alpha} - (1-2p)].\end{aligned}$$

Again, the degree of bias for Pareto distribution depends on p and α only, the percentage of excluded data and the shape factor, but not on λ, the location parameter.