Appendix A: User's Guide to the Financial Scenario Model

The purpose of the Financial Scenario Model is to develop projections for future economic conditions and related financial data on an integrated basis, explicitly accounting for relationships between key variables. The model is intended to be a useful tool, general enough to pertain to a variety of actuarial applications including, but not limited to: dynamic financial analysis, cash flow testing, solvency testing, stress testing, reserving, and pricing.

This user guide documents the use of the Financial Scenario Model and includes a description of (1) the layout of the model, (2) the steps required to perform a simulation, and (3) a sensitivity analysis of the parameters.

PLATFORM: EXCEL AND @RISK

The Financial Scenario Model is an Excel spreadsheet that benefits from the use of a simulation software package called @RISK available through Palisade Corporation (www.palisade.com). @RISK leverages the simplicity of spreadsheets and integrates powerful analysis tools that help to randomly select future scenarios and examine risk in a stochastic financial environment. @RISK allows users to define variables as a distribution, then randomly select from these assumptions. Sampling from the distribution allows a modeler to understand the impact of risk on key financial results.

There are several benefits that @RISK provides over a stand alone spreadsheet. First, @RISK allows input variables to have explicit, user-defined distributions. The creator of an @RISK simulation project can choose from a variety of distributional assumptions, from common alternatives such as the normal and uniform distribution, to more complex distributions, which are difficult to simulate using only standard Excel functions, such as the beta, exponential, and Weibull distributions. @RISK also allows the modeler to define his own distribution and even select the distribution to best fit available empirical data.

A second benefit of @RISK is that the software can more easily capture the correlation of dependent variables. While correlating multi-normal distributions can be easily accomplished through direct means, programming correlations of other distributions can be more difficult. @RISK includes an explicit correlation matrix that allows the user to input key relationships, regardless of the distribution of the dependent variables. After the user defines a correlation matrix for use in a simulation, @RISK also checks to be sure that joint correlations are internally consistent.

Finally, a significant benefit of @RISK is the ability to capture, study, and report simulation results. @RISK allows users to use familiar spreadsheet models and makes it easy to track key output cells. However, @RISK's power is in the ability to analyze output and create a variety of reports for effective communication of any risk analysis.

NAVIGATING THE MODEL

As mentioned in the main report, the following variables are included in the Financial Scenario Model and projected forward for 50 years:

- Inflation
- Real interest rates
- Nominal interest rates, which are implied from the processes for inflation and real interest rates
- Large and small stock returns
- Equity dividends
- Returns from real estate investments
- Rate of unemployment

In projecting these series, the Financial Scenario Model is set up in eight worksheets. Each of the worksheets is described here.

(1) *@RISK Correlations* – The correlation matrix on this sheet determines the joint dependency of the underlying modeled variables. See the discussion in the sensitivity analysis section below to see the effects of changing the default correlation values.

(2) *Scenarios* – Users of the financial scenario model can define specific scenarios in several of the variables including interest rates, inflation, and equity returns. Creating user-defined scenarios begins on the *Scenarios* worksheet. Specific guidelines on inputting scenarios are described later in this user guide.

(3) *ModelInput* – This sheet is the main area for user input. Users must also enter information about the current environment (interest rates and inflation) and make any desired changes to the default parameters of the model before beginning the simulation.

(4) *StochProcs* – This sheet provides the details behind the month-by-month projections of each of the financial variables included in the model. The parameters from the *ModelInput* directly impact the time series shown on *StochProcs*. These details provide the basis for the model's output shown on the output sheets (see (5) and (6) below).

(5) *OutputIntRates* – Given the central importance of interest rates in the Financial Scenario Model, output for interest rates are shown on a separate sheet. Nominal interest rates are determined by the combination of inflation and real interest rates; all three of these variables are shown on *OutputIntRates*. Each row represents a projection date and shows the resulting term structure (e.g., nominal interest rate by maturity).

(6) *OtherOutput* – In addition to projections of interest rates as shown on *OutputIntRates*, returns on stocks (small and large), dividend yields, real estate returns, and unemployment are shown on the *OtherOutput* sheet (by projection date).

(7) and (8) *IntRateChart* and *InitTermStructure* – After the user initializes the Financial Scenario Model on the *ModelInput* page, these two sheets show the

implied starting term structure graphically (*IntRateChart)* and in tabular form (*InitTermStructure).* From this initial (implied) term structure, the model begins to project future scenarios.

PERFORMING PROJECTIONS

To simulate outcomes under the Financial Scenario Generator, there are four basic steps:

1. Initializing the model, adjusting any parameters, and specifying any scenarios of relevance to the modeler,
2. Indicating the important variables that form the basis for analysis (choosing output variables),
3. Selecting the simulation settings and performing the simulation, and
4. Viewing the output reports.

Each of these steps is described below.

Step 1: Parameter Selection

First, on the *ModelInput* worksheet, the user must initialize the model to the current interest rate environment. Users must include a recent measure of inflation, as well as current yield information on default free securities (Treasury yields). Some sources of this information include the Wall Street Journal, Yahoo! Finance (bonds.yahoo.com/rates.html), or the CNN/Money Magazine web site (money.cnn.com/markets/bondcenter/). This initialization determines the starting values when projecting nominal interest rates, inflation, and real interest rates.

The default parameter values included on the rest of the *ModelInput* sheet were selected based on an analysis of historical data. Where possible, the use of public information accessible through the World Wide Web was chosen to parameterize these processes (see Section 5 of the report for a description of this data, their sources, and the methodologies used to isolate the default parameters). If the user wishes to alter any of these assumptions, he can put the new parameter value in the *ModelInput* worksheet. To better understand the relationship between individual parameters and the resulting output, see the sensitivity analysis section below.

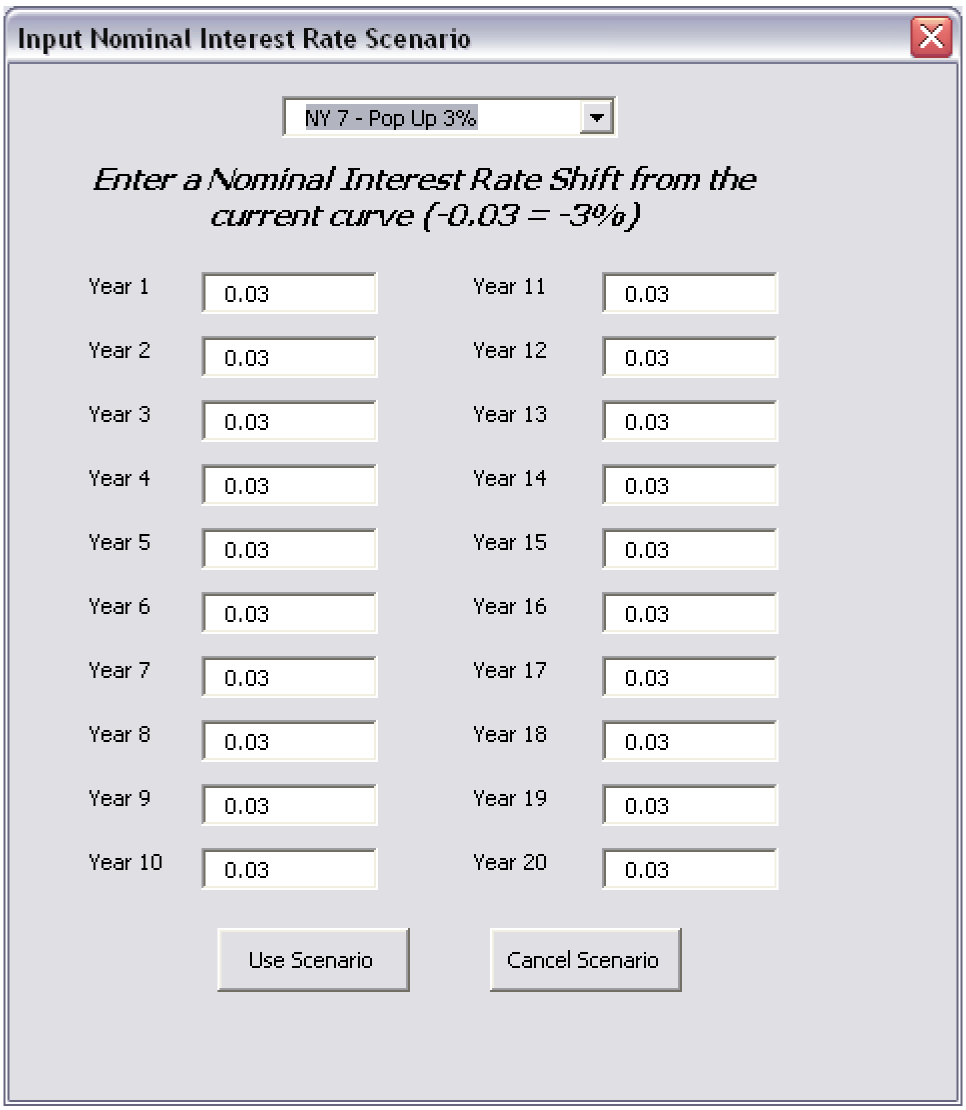
In addition to changing the parameters of the individual processes (if desired), the "Model Input" sheet includes check boxes for the following options:

- *Placing lower bounds on the level of inflation and real interest rates*. When placing lower bonds on inflation or real interest rates, the model projects these processes normally, and simply chooses the maximum of the lower bound and the resulting simulated value.
- *Eliminating the potential for negative nominal interest rates.* Recall that the realized nominal interest rate is the combination of inflation and real interest

rates. If negative nominal interest rates are not allowed, the process for inflation is not affected, but the model effectively puts a lower bound on resulting real interest rates so the resulting nominal interest rate is not negative.

In addition to the assumptions of lower bounds, users may be interested in isolating the effects of specified patterns of financial variables. For example, the user may be interested in specifying a path for interest rates over the next 30 years while allowing all other variables to remain stochastic. The *Scenarios* sheet provides the opportunity for these situations.

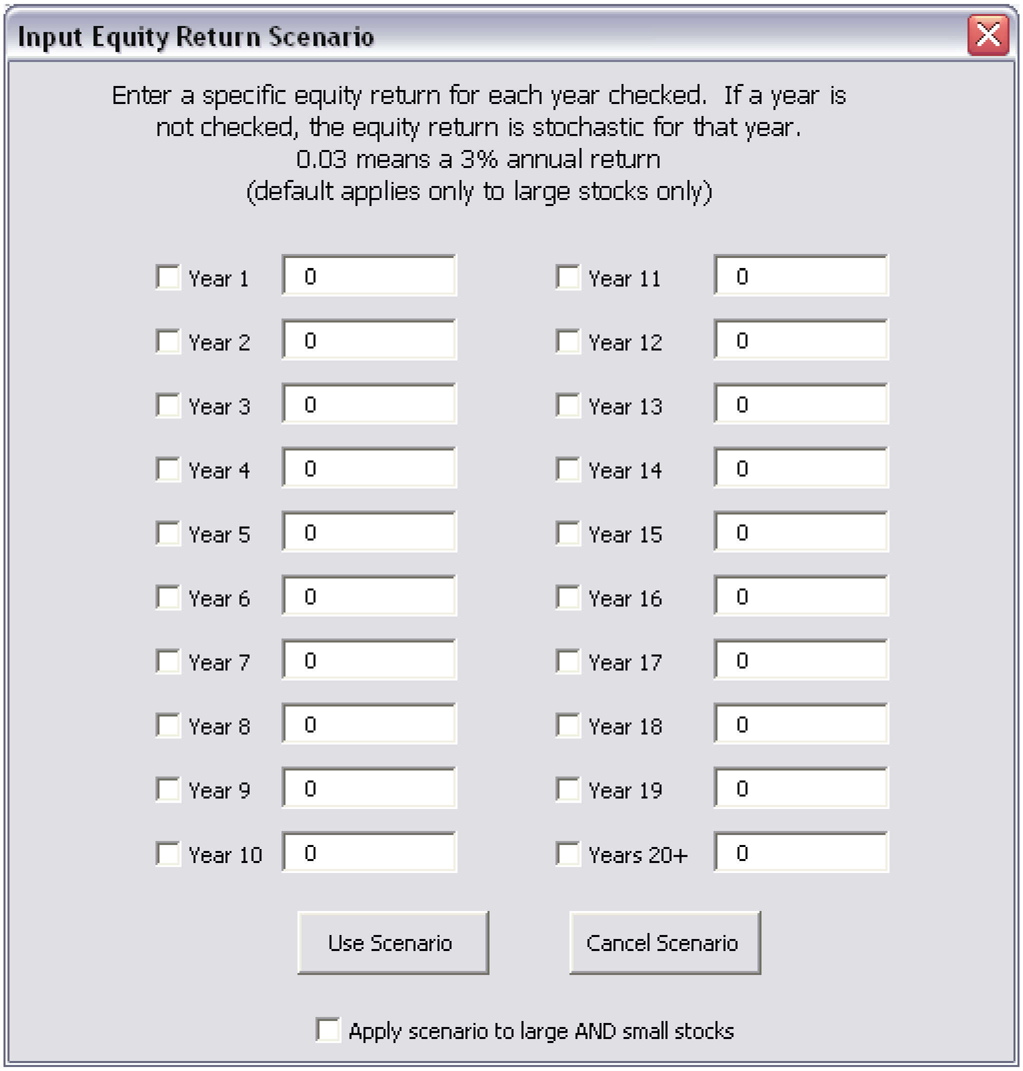
When the interest rate scenario box is checked, the following popup form allows the user to specify the path based on changes from the existing term structure. (NOTE: The interest rate scenario popup form specifies the percentage point change, in each of the subsequent simulated years, relative to the *initial* term structure implied by the model. If the user wishes to view the initial term structure, see the *IntRateChart* sheet.)



Note that the drop down box at the top of the form includes seven scenarios related to the New York Regulation 126. In low interest rate environments, the model automatically adjusts NY 126's decreasing interest rate scenarios by incorporating lower bounds. "Custom Scenarios" are any other interest rate movements desired by the user.

When specifying interest rate scenarios, the inflation process remains stochastic (similar to the case where negative nominal interest rates are prohibited). The model then backs into the real interest rate, based on the user-defined nominal interest rate less the modeled value for inflation.

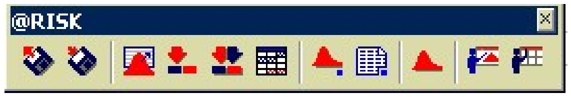
For equity return scenarios, users select the years for which they would like to specify equity returns (see the popup form below). The check box in front of each year indicates a user-supplied equity return. For those years that are not checked, equity returns are stochastic based on the model's parameters. As an example, one may be interested in the impact of a drop in the stock market of 50% in the tenth projection year. In this case, the user will check the box in front of "Year 10", and put -0.50 in the accompanying text box. Note that, by default, a user-selected return scenario applies only to large stocks; should the user be interested in applying the scenario to the entire stock portfolio (large *and* small stocks), s/he can check the box at the bottom of the popup form.

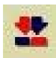


Inflation scenarios specify a rate of inflation over the next 50 years. A word of caution when using inflation scenarios: the user should be sure to understand the relationship between inflation scenarios and the assumed mean reversion level of inflation on *ModelInput*. Section 5 of the report discussed the link between long-term interest rates and long-term expectations of inflation. The latter is based on a term structure of inflation which is derived from the model's assumption of the long-term rate of inflation. The inflation scenario specifies the path of the *short-term* inflation rate, but long-term inflation expectations are a function of the current (short-term) inflation rate and the mean reversion level. The user is cautioned about potential inconsistencies between scenario-specific inflation rates and the assumed level of mean reversion.

Step 2: Choosing the Output Variables and @RISK Settings

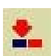
There are several @RISK settings that need to be adjusted before running a simulation. The @RISK Toolbar shown below contains shortcuts to all of the important functions performed in a simulation.



*List input and output*

Input cells are spreadsheet cells that contain specialized @RISK functions such as distribution definitions. In the financial scenario model, there are thousands of defined distributions; given that the model does projections for seven distinct financial series, each month for over 50 years, there are over 4,200 input cells in the model (12 months x 50 years x 7 variables). Output cells are user defined cells that become the focus for risk analysis. Any cell that the user wishes to study is an output cell. See the instructions immediately below to define output cells.

The input and output listing allows the user to view the input and output cells that will be tracked during a simulation. Users may also define names for individual input or output cells, or groups of cells, to make reports generated by @RISK easier to read. For example, in the 5th projection year, the nominal term structure from the model (1-month, 3-month, 1-year, 3-year, 5-year, 10-year, and 20-year spot rates) may be named “5yr projected nominal”.

 *Adding an output cell*

Adding an extra output cell is simply a matter of locating the cell (or group of cells) of interest in the model’s spreadsheet and adding it to the output list by clicking the “Add Output” button (illustrated above).

Users should not feel constrained to the cells already shown in the worksheet. Additional variables may be defined on the spreadsheet using standard Excel functions (or @RISK functions) of variables that are already tracked by the Financial Scenario Model. For example, the slope of the yield curve can be defined in any number of ways (e.g., the difference between the 1- and 10-year bond yields or the difference between the 1-month and 5-year yields). Users can define the variable of interest using the formula in Excel and then add the defined variable to the output list. In addition, if users incorporate the Financial Scenario Model as an engine for other models (such as a dynamic financial analysis model), one may define the output cell as the scenario-specific insurer surplus.

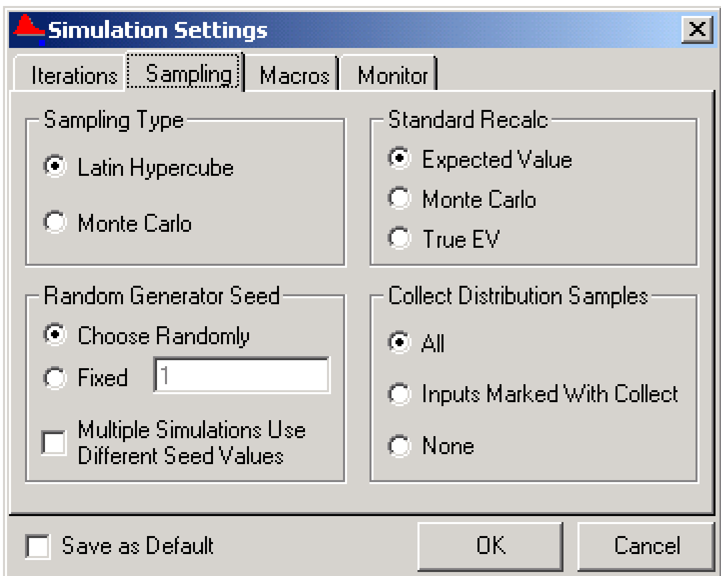
NOTE: When working with any user-defined scenarios described above (including lower bounds), @RISK may delete previously captured output variables. After the defining or canceling scenarios, it is best to list the input and output to be sure the relevant variables are still in the output list. Users may need to redefine the output cells if @RISK has eliminated it from the list. In fact, before running any simulation, it is good practice to review the output list to be sure the simulation captures all desired output cells. Nothing is more frustrating than running an hour long simulation, only to discover that @RISK has inadvertently deleted one of the variables in which you were interested!

Step 3: Simulation Settings and Running a Simulation

Simulation settings

Simulation settings provide information about how projections will be performed. After selecting the simulation setting button, the number of iterations (sample paths) must be selected. If the number of iterations selected is 1,000, the model projects 1,000 different paths for each of the financial variables defined. Depending on the speed of the computer and the size of the model, the user may be interested in more or fewer paths.

The screenshot below shows the Sampling tab of the Simulation Settings window in @RISK.



Sampling Type

For more efficient sampling of the input cells, the @RISK sampling type should be set to Latin Hypercube. Monte Carlo sampling often requires a larger number of iterations for the resulting sample distribution to converge to the true distributional assumption. The poorer performance of Monte Carlo sampling is especially pronounced if the distribution is slightly skewed or there are outcomes with very low probability. Latin Hypercube sampling, a type of stratified sampling technology, more accurately recreates true distributions with fewer iterations. The importance of sampling approaches for insurers selecting interest rate scenarios is fully discussed in Chueh (2002).

Random Generator Seed

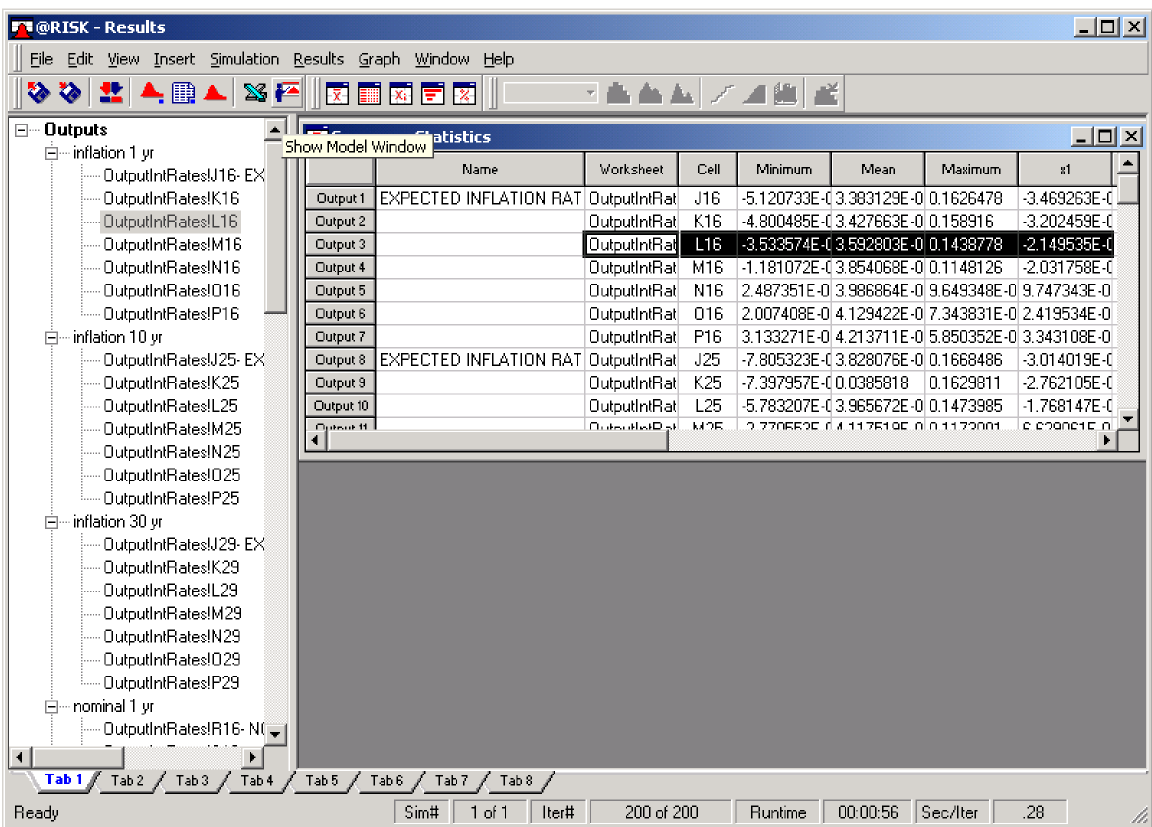
The seed value for the random number generator is useful when looking at the sensitivity of projections to initial values and model parameters. The seed assures the same set of random numbers will be used for each simulation tested in order to isolate the effects of alternative initial conditions. If the user wants a set of independent results, the seed should be changed or chosen randomly.

Collect Distribution Samples
Most users will be most interested in analyzing only the output cells; the random draws and distributions of the input cells are trivial. To reduce the amount of data that is stored and analyzed by @RISK (and correspondingly, to reduce processing time), one can bypass the accumulation of the input data. Selecting “None” will speed up the generation of output reports and statistics.

 *Start the simulation*
Starting the simulation will perform the number of iterations defined in the Simulation Settings. After initializing all of the input cells, @RISK will provide an update of the timing of this simulation based on the number of iterations that have been performed. Including the time required for the projections, as well as the processing time required to display output statistics on the screen, 1,000 iterations takes about 10 minutes on a 1GHz PC. The time increases as the number of output cells increases.

Step 4: Reviewing Output and Creating Reports

*Results window*
After @RISK completes a simulation, users will be shown the results window displaying summary statistics for all of the output cells (and input cells if chosen in the simulation settings as mentioned above). If the user returns to Excel, he can get back to the @RISK results window by clicking on the button as illustrated above.



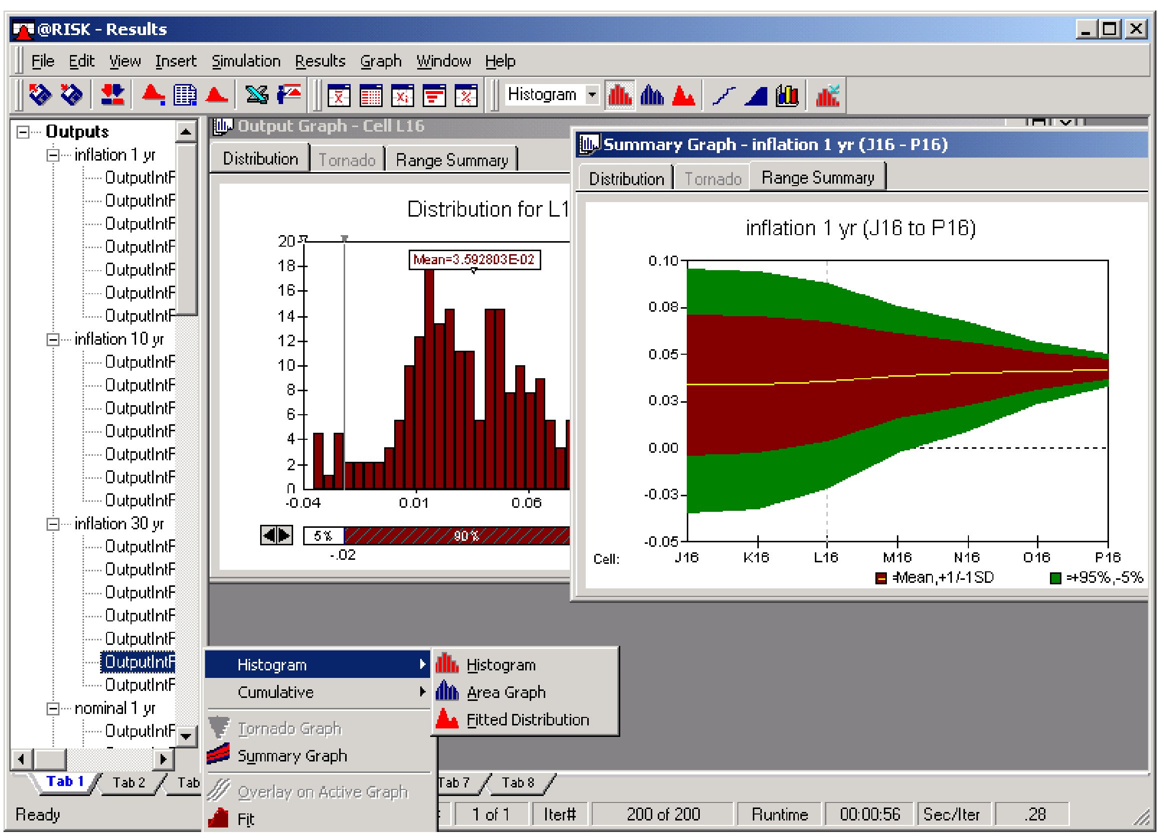
@RISK's Results Toolbar

The results window then allows users to review detailed output statistics, illustrate results using @RISK's graphing capabilities, and create reports. The following discussion is not meant to exhaust the abundant and powerful reporting capabilities of @RISK. Rather, we only try to highlight some of the basic reporting features.

Summary statistics. The default screen for the @RISK results window is summary statistics. Basic statistics for all of the output cells are shown to give the user a simple snapshot of the simulation results.

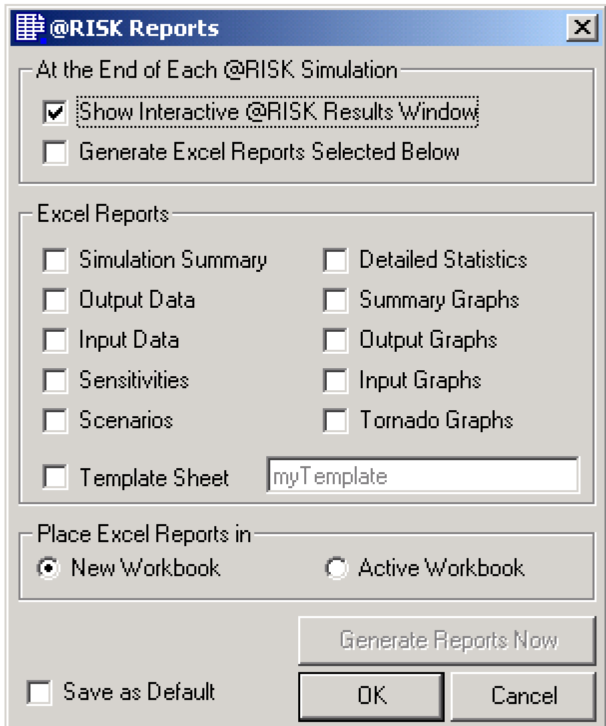
Detailed statistics. These sample statistics provide more information than in the summary window including standard deviation, variances, skewness, kurtosis, and the distribution percentiles. In addition, @RISK allows the user to ask scenario-based questions like what percentage of the time is a particular output cell negative.

Creating Graphs. Two standard graphs in @RISK are histograms and summary graphs. Summary graphs have been referred to as "funnel of doubt" graphs in the results section. The following screen shot shows the results window with these two types of graphs.



*Generating output reports*

By creating output reports, the user of the model can dump useful information directly into Excel for additional analysis, formatting, printing, or saving. The reports can dump the summary or detailed statistics in Excel, generate graphs in Excel format, or even dump all of the output for additional analysis in Excel. The following screen shot shows the @RISK Reports window:



*Saving simulations (Icons from the @RISK Toolbar or Results Window)*

Users have two options when trying to save a simulation for future reference: dump the data into an Excel file or save the simulation in @RISK's file format (.rsk files). When a simulation is saved, all of the input and output cells, data, and simulation settings are retained for later use.

SENSITIVITY ANALYSIS

An important aspect of any projection of financial results is potential model risk. The Financial Scenario Generator is not purported to be a perfect predictor of future economic conditions. To the contrary, the model explicitly recognizes the lack of predictability of future economic conditions by generating a distribution of alternate business environments. Since the magnitude of uncertainty is based on specific assumptions, actuaries need to be aware of the dependency of projections on these assumptions.

This section looks at sensitivity analysis for some of the key financial variables to give users of the model some guidance on selecting parameters and understanding their influence on simulation results. For each variable, projections are presented similar to the results and tables section of the main document. However, this user guide will focus its attention on choosing alternative values of the model's parameters to illustrate the changes. These sensitivity results are provided to gain insight into the workings of the model. It is hoped that this documentation will make the model more usable and friendly, increasing the model's ability to be customized and provide a platform for a wide variety of user-defined applications.

It should be noted that the values in this simulation do not (exactly) match the results presented in the main document since the results presented here are based on fewer iterations than were presented in the main document.

Inflation

Inflation is allowed to fluctuate around some average level of inflation based on the following mean-reverting process (an Ornstein-Uhlenbeck process):

$$dq_t = \kappa_q(\mu_q - q_t)dt + \sigma_q dB_q$$

The rate of inflation reverts toward a long-term mean μ_q. When the current level of inflation (q_t) exceeds this long-term average, the model predicts that, on average, inflation will fall in the future. This is represented in the first term of the above equation, called the drift; if inflation is high, the first term is negative. How quickly inflation moves back to its reversion level depends on the speed parameter κ_q.

While the inflation process presented above is in continuous time, the model simulates this process in monthly time steps according to the following:

$$\Delta q_t = \kappa_q(\mu_q - q_t)\frac{1}{12} + \sigma_q\sqrt{\frac{1}{12}}\varepsilon_q$$

The default parameters of the process are show below:

Base Case Inflation Parameters

0.400	mean reversion speed (κ_q)
0.040	volatility of inflation (σ_q)
0.048	long-term mean reversion level (μ_q)
-0.020	lower bound for short-rate inflation
0.025	initial inflation level (q_0)

Instead of looking only at current level of inflation, the financial scenario model develops a term structure of inflation. The term structure of inflation is important since investors pay prices for long-term bonds based on their expectations of inflation over the bond's lifetime. To determine the term structure of nominal interest rates, (expected) long-term inflation is required for determining the inflation premium on long-term bonds, which is an important part of nominal interest rates.

Using the default parameters, a simulation was performed using 200 iterations and the results of the projections are shown below:

Base Case – Inflation Statistics

Value	***Mean***	***Std. Dev.***	***5th Percentile***	***95th Percentile***
1st Projection Year				
1-Year Inflation	3.59%	3.15%	-2.15%	8.79%
10-Year Inflation	4.13%	0.94%	2.42%	5.68%
10th Projection Year				
1-Year Inflation	3.97%	3.74%	-1.77%	10.50%
10-Year Inflation	4.24%	1.11%	2.53%	6.19%

As can be seen from the statistics, the one-year inflation rate does tend toward the mean reversion level of 4.8%. The initial inflation level is 2.5%, so after one year (the start of the first projection year) the mean projected inflation rate is 3.59%, and in the 10th projection year the average level of inflation increases to 3.97%. The simulation results also show that the uncertainty in inflation increases slightly over time, since the standard deviation in the first year is 3.15% vs. the tenth projection year, where the standard deviation is 3.74%. The range in the one-year inflation rate, or the difference in the 5th and 95th percentiles, is also higher in the 10th projection year.

The 10-year inflation rate included in the table above is determined from the short-term level of inflation combined with the assumed underlying process for future inflation movements. Because the parameters of the inflation process exactly determine the expected path of inflation over the next 10 years, we can use the results of Vasicek (1977) to derive the term structure of inflation and determine long-term rates of inflation as a function of the current value and the parameters of the process.

From the above table, it is seen that the 10-year inflation rate also exhibits the tendency toward the mean reversion level of 4.8%; after the first projection year, the 10-year rate of inflation is 4.13% and in the tenth projection year, the rate is 4.24%. But because the 10-year rate at any point in time is based on an *expected* path of inflation of the next 10-year period, the volatility of the 10-year rate is lower than the 1-year rate of inflation.

The shape of the term structure of inflation is influenced by the parameters in the following ways:

- The short end of the term structure of inflation is the current inflation rate (q_t)
- Since inflation tends toward some mean reversion level, the end of the curve is closely related to this mean reversion level (but see how volatility affects the long end of the curve below)

- If mean reversion speed is high, the term structure of inflation quickly approaches the mean reversion level. If mean reversion speed is low, the term structure of inflation is much flatter.

To understand the sensitivity of results to the selected parameters, projections were also performed under several alternative scenarios. Increasing the mean reversion speed (κ_q) forces inflation back to its reversion level much faster. The following analysis is based on a simulation where the mean reversion speed was increased from 0.4 to 0.8:

Value	***Mean***	***Std. Dev.***	***5th Percentile***	***95th Percentile***
1st Projection Year				
1-Year Inflation				
κ_q = 0.4 (Base case)	3.59%	3.15%	-2.15%	8.79%
κ_q = 0.8	4.15%	2.29%	0.26%	7.90%
10-Year Inflation				
κ_q = 0.4 (Base case)	4.13%	0.94%	2.42%	5.68%
κ_q = 0.8	4.58%	0.42%	3.88%	5.26%
10th Projection Year				
1-Year Inflation				
κ_q = 0.4 (Base case)	3.97%	3.74%	-1.77%	10.50%
κ_q = 0.8	4.49%	2.46%	0.56%	8.79%
10-Year Inflation				
κ_q = 0.4 (Base case)	4.24%	1.11%	2.53%	6.19%
κ_q = 0.8	4.64%	0.45%	3.93%	5.43%

Because inflation is pulled toward the mean reversion level faster than in the base case, each of the projections of future inflation is closer to 4.8%. In addition, the volatility exhibited in projected inflation is lower than under the base case. Given the presumption of faster mean reversion, the standard deviation and the measure of range is significantly smaller than under the original parameters. In effect, when the mean reversion speed is higher, any deviation from the mean reversion level will be shorter-lived than the case where mean reversion speed is slower. Since disturbances from 4.8% tend to die out more quickly, the volatility of inflation also declines.

The second parameter of the inflation process represents the randomness of movements or *the volatility* from expected. The volatility parameter (σ_q) controls the size of the deviations from the expected movement toward the mean reversion level. The following simulation shows the effects of increasing the volatility parameter from 0.04 to 0.08:

Value	*Mean*	*Std. Dev.*	*5th Percentile*	*95th Percentile*
1st Projection Year				
1-Year Inflation				
σ_q = 0.04 (Base case)	3.59%	3.15%	-2.15%	8.79%
σ_q = 0.08	3.61%	6.30%	-7.88%	14.00%
10-Year Inflation				
σ_q = 0.04 (Base case)	4.13%	0.94%	2.42%	5.68%
σ_q = 0.08	3.20%	1.87%	-0.22%	6.30%
10th Projection Year				
1-Year Inflation				
σ_q = 0.04 (Base case)	3.97%	3.74%	-1.77%	10.50%
σ_q = 0.08	4.02%	7.49%	-7.45%	17.09%
10-Year Inflation				
σ_q = 0.04 (Base case)	4.24%	1.11%	2.53%	6.19%
σ_q = 0.08	3.32%	2.23%	-0.09%	7.22%

All the measures of volatility including standard deviation and the range of inflation are considerably higher than under the base case. Yet the lower value of the projected mean of the 10-year inflation rate (3.32%) may be a bit of a surprise. The projected one-year rate of inflation has approximately the same mean as before (with higher uncertainty), but the projection of the mean 10-year rate appears lower than before. To explain these results, let’s look at the development of the long-term rate.

Recall that long-term inflation rates are derived from the (expected) path of inflation over the investment horizon. The longer horizon inflation rates assume frequent compounding. As a result, rates of inflation over long periods (like the 10-year rate) are implied by the geometric average of inflation along the projected path.

To illustrate the effects of increasing volatility on geometric average inflation, consider the following example that calculates the two-year (geometric average) inflation rate under three assumptions of volatility.

Year 1 Inflation	*Year 2 Inflation*	*Two-Year Inflation*
4.8%	4.8%	4.80%
2.8%	6.8%	4.78%
0.8%	8.8%	4.72%

Under the first path, inflation is assumed to be constant (no volatility), so the two-year inflation rate is equal to the same realized rate over each of the two years. Under the second scenario volatility is introduced, but each year, the (arithmetic) mean inflation rate of the distribution is unaffected and equal to 4.8%. The result is that when inflation measured over two years (using geometric average), the multi-year rate is lower than the no volatility case. When volatility is increased further in the third scenario, two-year inflation rate appears even lower. The implication is that by increasing the volatility of inflation, long-term rates, which are compounded over time, appear to decline.

The next simulation decreases the mean reversion level (μ_q) from 4.8% to 3.0%:

Value	***Mean***	***Std. Dev.***	***5th Percentile***	***95th Percentile***
1st Projection Year				
1-Year Inflation				
μ_q = 0.048 (Base case)	3.59%	3.15%	-2.15%	8.79%
μ_q = 0.030	2.78%	3.15%	-2.96%	7.98%
10-Year Inflation				
μ_q = 0.048 (Base case)	4.13%	0.94%	2.42%	5.68%
μ_q = 0.030	2.62%	0.94%	0.91%	4.17%
10th Projection Year				
1-Year Inflation				
μ_q = 0.048 (Base case)	3.97%	3.74%	-1.77%	10.50%
μ_q = 0.030	2.89%	3.74%	-2.84%	9.43%
10-Year Inflation				
μ_q = 0.048 (Base case)	4.24%	1.11%	2.53%	6.19%
μ_q = 0.030	2.66%	1.11%	0.95%	4.60%

As expected, the mean of the projected inflation rates approaches 3%. However, another interesting aspect of lowering the mean reversion level is introduced. In the base case simulation, it appeared that the term structure of inflation was upward sloping. When the mean reversion level was decreased to 3.0%, the term structure of inflation is inverted. The latter case is a result of relatively higher volatility when the mean reversion level is reduced. As explained earlier, higher volatility leads to lower geometric averages for long-term rates and inflation. In the base case, the effects of volatility on the geometric average for long-term inflation was masked by the increasing tendency of inflation. But when the mean reversion level is reduced, volatility effects dominate and the long-term inflation is lower than short rates.

Finally, one last sensitivity analysis on the interest rate process shows how the mean reversion speed (κ_q) and the volatility (σ_q) can work together. The following simulation simultaneously increases κ_q to 0.8 and σ_q to 0.08:

Value	***Mean***	***Std. Dev.***	***5th Percentile***	***95th Percentile***
1st Projection Year				
1-Year Inflation				
$\kappa_q = 0.4, \sigma_q = 0.04$ (Base case)	3.59%	3.15%	-2.15%	8.79%
$\kappa_q = 0.8, \sigma_q = 0.08$	4.15%	4.58%	-3.62%	11.67%
10-Year Inflation				
$\kappa_q = 0.4, \sigma_q = 0.04$ (Base case)	4.13%	0.94%	2.42%	5.68%
$\kappa_q = 0.8, \sigma_q = 0.08$	4.29%	0.83%	2.88%	5.65%
10th Projection Year				
1-Year Inflation				
$\kappa_q = 0.4, \sigma_q = 0.04$ (Base case)	3.97%	3.74%	-1.77%	10.50%
$\kappa_q = 0.8, \sigma_q = 0.08$	4.51%	4.92%	-3.34%	13.12%
10-Year Inflation				
$\kappa_q = 0.4, \sigma_q = 0.04$ (Base case)	4.24%	1.11%	2.53%	6.19%
$\kappa_q = 0.8, \sigma_q = 0.08$	4.35%	0.89%	2.93%	5.92%

From the previous analysis, we saw that increasing mean reversion speed forces the inflation rate more quickly to the mean reversion level of 4.8%. This stronger attraction to the mean reversion level would reduce the standard deviation and the range of the projected inflation rates. However, in this simulation, the volatility parameter is also increased. The increased volatility more than offsets the decrease in uncertainty driven by stronger mean reversion. Viewed another way, the volatility parameter increases the randomness of projected inflation, but higher mean reversion dampens the effects of heightened volatility. The end result is that mean reversion speed and volatility have similar (but opposite) effects on projected inflation.

Real Interest Rates

The process for real interest rates is more complex than for inflation. Not only do short-term rates revert to some mean reversion level, the reversion level itself is stochastic. The (continuous time equivalent) model for real interest rates is as follows:

$$dr_t = \kappa_r (l_t - r_t)dt + \sigma_r dB_1$$
$$dl_t = \kappa_l (\mu_l - l_t)dt + \sigma_l dB_2$$

Short-term real interest rates (r_t) are mean-reverting to a time-dependent level l_t. Although the reversion level is random, it fluctuates around some mean value μ_l. Like the inflation process, the speed of mean reversion of the short- and long-term real interest rates are affected by speed parameters (κ_r and κ_l) and the magnitude of randomness is determined by volatility parameters (σ_r and σ_l). One alternative view of this two-factor model allows the opposite ends of the (real interest rate) term structure to move simultaneously, with some correlation between these two factors.

Base Case Real Interest Rate Parameters

1.0000	mean reversion speed for short rate process (κ_r)
0.0100	volatility of short rate process (σ_r)
0.1000	mean reversion speed for long rate process (κ_l)
0.0165	volatility of long rate process (σ_l)
0.0280	long-term mean reversion level for long rate (μ_l)
0.0100	initial short-term real interest rate (r_0)
0.0250	initial mean reversion level for real interest rate (l_0)
0.5000	correlation between long and short processes

Based on 200 iterations, here are the results of real interest rates under the base case parameters:

Value	***Mean***	***Std. Dev.***	***5th Percentile***	***95th Percentile***
1st Projection Year				
1-Year Real Rate	2.22%	1.18%	0.24%	4.28%
10-Year Real Rate	2.40%	1.02%	0.78%	4.06%
10th Projection Year				
1-Year Real Rate	2.96%	3.42%	-3.13%	8.17%
10-Year Real Rate	2.74%	2.40%	-1.53%	6.37%

Similar to the discussion of the inflation process above, the real rate of interest is pulled toward the mean reversion level. Since the initial short-term real interest rate is 1%, it tends to increase toward the initial mean reversion level of 2.5%. In subsequent projection years, the mean reversion level tends toward 2.8%. The distribution of short-term real interest rates is wider than the distribution for long-term real rates, as measured by the standard deviation and the two percentiles noted.

The sensitivity results presented here isolate individual parameters in the real interest rate process. The simulations are shown to give some introduction to the impact of the various parameters on the resulting simulation results. It should be noted that altering several parameters at once may not have the additive effects of individual parameter changes.

The following table illustrates the projection results when the reversion speed of the short process (κ_r) is increased from 1.0 to 1.5:

Value	***Mean***	***Std. Dev.***	***5th Percentile***	***95th Percentile***
1st Projection Year				
1-Year Real Rate				
κ_r = 1.0 (Base case)	2.22%	1.18%	0.24%	4.28%
κ_r = 1.5	2.40%	1.28%	0.36%	4.55%
10-Year Real Rate				
κ_r = 1.0 (Base case)	2.40%	1.02%	0.78%	4.06%
κ_r = 1.5	2.43%	1.02%	0.78%	4.03%
10th Projection Year				
1-Year Real Rate				
κ_r = 1.0 (Base case)	2.96%	3.42%	-3.13%	8.17%
κ_r = 1.5	2.96%	3.41%	-2.94%	8.09%
10-Year Real Rate				
κ_r = 1.0 (Base case)	2.74%	2.40%	-1.53%	6.37%
κ_r = 1.5	2.72%	2.34%	-1.45%	6.28%

The results show that real interest rates are driven more quickly to the mean reversion levels; note that in the first projection year, the average 1-year interest rate is 2.40% vs. 2.22% under the base case. However, perhaps surprisingly, the volatilities of the different interest rates are not dramatically affected. As with the inflation process, one may have expected that increasing the mean reversion speed would pull real rates more quickly back to their mean reversion levels and reduce volatility.

With the real interest rate process, there are two differences from the inflation process that explains the apparent lack of sensitivity between reversion speed and volatility. First, real interest rates are based on two processes (short *and* long rates), while inflation is based on only one process. Therefore, changing the mean reversion speed of the short rate process will have less of an impact given that the long rate process is unchanged. The relative impact of any parameter changes is a function of existing parameter values. (For example, note the larger impact of the mean reversion speed of the long rate process illustrated below.) Second, the short rate mean reversion speed is initially higher under the real interest rate process than it is for inflation. The impact of reversion speed on volatility is dampened as κ_r increases, especially as κ_r exceeds 1.0.

Increasing the volatility of the short real interest rate process (σ_r) from 0.01 to 0.02 yields the following simulation results:

Value	*Mean*	*Std. Dev.*	*5th Percentile*	*95th Percentile*
1st Projection Year				
1-Year Real Rate				
σ_r = 0.01 (Base case)	2.22%	1.18%	0.24%	4.28%
σ_r = 0.02	2.22%	1.52%	-0.54%	4.83%
10-Year Real Rate				
σ_r = 0.01 (Base case)	2.40%	1.02%	0.78%	4.06%
σ_r = 0.02	2.39%	1.05%	0.63%	4.11%
10th Projection Year				
1-Year Real Rate				
σ_r = 0.01 (Base case)	2.96%	3.42%	-3.13%	8.17%
σ_r = 0.02	2.98%	3.58%	-3.01%	8.64%
10-Year Real Rate				
σ_r = 0.01 (Base case)	2.74%	2.40%	-1.53%	6.37%
σ_r = 0.02	2.73%	2.42%	-1.50%	6.44%

In this case, the measures of volatility all increase for each of the real interest rate projections. But the geometric average effects that were illustrated with inflation are less prevalent when adjusting the short real interest rate process. We will see below that when adjusting the long rate volatility parameters, these effects are more evident.

When adjusting the ***long*** rate reversion speed (κ_l) from 0.1 to 0.2:

Value	*Mean*	*Std. Dev.*	*5th Percentile*	*95th Percentile*
1st Projection Year				
1-Year Real Rate				
κ_l = 0.1 (Base case)	2.22%	1.18%	0.24%	4.28%
κ_l = 0.2	2.23%	1.12%	0.31%	4.21%
10-Year Real Rate				
κ_l = 0.1 (Base case)	2.40%	1.02%	0.78%	4.06%
κ_l = 0.2	2.53%	0.71%	1.35%	3.69%
10th Projection Year				
1-Year Real Rate				
κ_l = 0.1 (Base case)	2.96%	3.42%	-3.13%	8.17%
κ_l = 0.2	2.96%	2.52%	-1.24%	6.73%
10-Year Real Rate				
κ_l = 0.1 (Base case)	2.74%	2.40%	-1.53%	6.37%
κ_l = 0.2	2.77%	1.32%	0.60%	4.85%

In this projection, the long rate moves more quickly toward its mean of 2.8%. This simulation shows that the mean of the 1-year real rate is not significantly different from the base case projections, yet the 10-year rate is closer to the long-term mean reversion level of 2.8%. More strikingly, increasing the mean reversion strength in the long rate process has a considerable effect on the volatility, particularly in the longer projections. In the 10th projection year, the standard deviation of the 1-year rate is 26% lower than the base case (2.52% vs. 3.42% in the base case) and the 10-year rate is 45% lower (1.32% vs. 2.40%).

As another illustration of the sensitivity of real interest rates to the long rate process, the following simulation statistics are calculated by raising the long rate volatility (σ_l) from 0.0165 to 0.03:

Value	***Mean***	***Std. Dev.***	***5th Percentile***	***95th Percentile***
1st Projection Year				
1-Year Real Rate				
σ_l = 0.0165 (Base case)	2.22%	1.18%	0.24%	4.28%
σ_l = 0.03	2.24%	1.90%	-0.79%	5.53%
10-Year Real Rate				
σ_l = 0.0165 (Base case)	2.40%	1.02%	0.78%	4.06%
σ_l = 0.03	2.02%	1.82%	-0.94%	4.94%
10th Projection Year				
1-Year Real Rate				
σ_l = 0.0165 (Base case)	2.96%	3.42%	-3.13%	8.17%
σ_l = 0.03	3.18%	6.12%	-8.06%	12.32%
10-Year Real Rate				
σ_l = 0.0165 (Base case)	2.74%	2.40%	-1.53%	6.37%
σ_l = 0.03	2.48%	4.35%	-5.29%	9.05%

When compared to the simulation where short rate volatility is increased, the result here shows that changes in the long rate process significantly impacts the volatilities of real interest rates. The standard deviation of each of the real interest rates is significantly higher than the base case and the range in percentiles is substantially wider. In addition, the geometric average effects that were noted when discussing the inflation are magnified. When the volatility of the short rate process was increased these effects were not as evident.

Finally, the last simulation adjusts the long-term mean reversion level for real interest rates (μ_l), from 2.8% to 1%.

Value	***Mean***	***Std. Dev.***	***5th Percentile***	***95th Percentile***
1st Projection Year				
1-Year Real Rate				
μ_l = 0.028 (Base case)	2.22%	1.18%	0.24%	4.28%
μ_l = 0.01	2.09%	1.18%	0.12%	4.16%
10-Year Real Rate				
μ_l = 0.028 (Base case)	2.40%	1.02%	0.78%	4.06%
μ_l = 0.01	1.74%	1.02%	0.12%	3.39%
10th Projection Year				
1-Year Real Rate				
μ_l = 0.028 (Base case)	2.96%	3.42%	-3.13%	8.17%
μ_l = 0.01	1.86%	3.42%	-4.23%	7.07%
10-Year Real Rate				
μ_l = 0.028 (Base case)	2.74%	2.40%	-1.53%	6.37%
μ_l = 0.01	1.40%	2.40%	-2.86%	5.03%

Each of the projected real interest rates tends toward the lower mean reversion level. The mean of the real rates is below the base case scenario, but the measures of volatility are similar. In essence, we're simply shifting the center of the distribution of the projected real rate.

Nominal interest rates

Investors demand compensation for the time value of money as well as any erosion of purchasing power. Therefore, projected nominal interest rates in the financial scenario generator are based on the combination of real interest rates and inflation. For each month during the next 50 years, the model produces nominal interest rates for seven key maturities: 1-month, 3-months, 1-year, 3-years, 5-years, 10-years, and 20-years.

Given the illustrations and sensitivities presented above for inflation and real interest rates, the following table shows how the effects of each of the changes in the parameters on nominal interest rate projections compared to the base case.

Summary of Parameter Sensitivities on Nominal Interest Rates

Value	***Mean***	***Std. Dev.***	***5th Percentile***	***95th Percentile***
1st Projection Year				
1-Year Nominal Rate				
Base case	5.81%	3.22%	0.52%	10.81%
κ_q = 0.8	6.36	2.44	2.28	10.33
σ_q = 0.08	5.83	6.26	-5.70	15.74
μ_q = 0.030	5.00	3.22	-0.29	9.99
κ_r = 1.5	5.99	3.30	0.64	11.41
σ_r = 0.02	5.82	3.23	0.57	8.87
κ_l = 0.2	5.83	3.21	0.49	10.77
σ_l = 0.03	5.83	3.55	-0.09	11.78
μ_l = 0.01	5.69	3.22	0.40	10.68
10-Year Nominal Rate				
Base case	6.53	1.38	4.30	8.86
κ_q = 0.8	6.99	1.09	5.29	8.66
σ_q = 0.08	5.60	2.12	2.04	9.22
μ_q = 0.030	5.03	1.38	2.79	7.35
κ_r = 1.5	6.56	1.38	4.34	8.88
σ_r = 0.02	6.52	1.39	4.23	8.87
κ_l = 0.2	6.66	1.17	4.66	8.61
σ_l = 0.03	6.15	2.05	2.83	9.45
μ_l = 0.01	5.87	1.38	3.64	8.20
10th Projection Year				
1-Year Nominal Rate				
Base case	6.93	5.09	-0.80	15.63
κ_q = 0.8	7.45	4.24	0.64	14.86
σ_q = 0.08	6.99	8.25	-5.77	21.28
μ_q = 0.030	5.86	5.09	-1.87	14.56
κ_r = 1.5	6.93	5.07	-0.92	15.71
σ_r = 0.02	6.95	5.23	-1.13	16.15
κ_l = 0.2	6.92	4.53	-0.32	13.96
σ_l = 0.03	7.14	7.18	-4.70	19.10
μ_l = 0.01	5.83	5.09	-1.90	14.53
10-Year Nominal Rate				
Base case	6.98	2.65	2.56	11.20
κ_q = 0.8	7.38	2.45	3.21	11.20
σ_q = 0.08	6.06	3.28	0.95	11.88
μ_q = 0.030	5.39	2.65	0.98	9.61
κ_r = 1.5	6.96	2.59	2.65	11.12
σ_r = 0.02	6.97	2.67	2.51	11.38
κ_l = 0.2	7.01	1.73	4.13	9.78
σ_l = 0.03	6.73	4.49	-1.01	13.77
μ_l = 0.01	5.64	2.65	1.23	9.86

Through casual observation of the financial markets, it is easy to recognize that interest rates of different time horizons are correlated. In the financial scenario generator, the parameter that controls this correlation is part of the real interest rate process. In the base case, the correlation between the long and short real interest rate processes is 0.5. The following table selects two nominal interest rates (proxies for the short and long nominal rates) and analyzes the sensitivity of the nominal rate correlation to the real interest rate correlation parameter.

Real Interest Correlation	***Nominal rate 3-mo / 20-yr correlation***
0.5	0.81
0.0	0.77
-0.5	0.73

Note that the effects of changing the real interest rate correlation do not have dramatic effects on the resulting nominal rate correlation. For example, when the correlation between the long and short real interest rate is dropped from 0.5 to 0.0, the nominal rate correlation between long and short rates only falls from 0.81 to 0.77. This is because nominal interest rates are also related to inflation. Given that inflation is a one-factor process, movements in the inflation rate affect all nominal rates predictably, which establish a certain level of correlation among all maturities. Therefore, only the real interest rate portion of the nominal rate is altered in the correlation shifts noted above, which is why the effects on nominal rate correlation appear muted.

Equity Returns

Movements in equity returns are composed of three separate pieces: the risk-free rate, excess equity returns (risk premia), and returns from dividends. The risk-free rate is based on the short-term nominal interest rate, which was discussed above. Returns from dividends are modeled separately, based on the same structural process for inflation (a mean-reverting Ornstein-Uhlenbeck process). This section discusses the excess return component of equity returns.

Excess equity returns are based on a regime-switching model, similar to Hardy (2001). In the financial scenario model, there are two states of the world: a low volatility regime and a high volatility regime. The return generating process is allowed to switch regimes based on a 2×2 transition probability matrix. At any moment in time, excess equity returns follow a normal distribution that is dependent upon the state of the world.

The following parameters were selected based on an analysis of excess equity returns in the U.S., shown separately for large and small stocks.

Large Stocks

0.008	Mean monthly excess return in low volatility regime
0.039	Volatility of monthly return in low volatility regime
-0.011	Mean monthly excess return in high volatility regime
0.113	Volatility of monthly return in high volatility regime
0.011	Switch from low to high regime ($P_{L,H}$)
0.059	Switch from high to low regime ($P_{H,L}$)

The annual excess return for large stocks in the low volatility regime is around 10%. In each month, there is a 1.1% probability of changing from low volatility regime to the high volatility regime. The distribution in the high volatility regime is significantly wider, but it is more likely to switch back to low regime.

Small Stocks

0.010	Mean monthly excess return in low volatility regime
0.052	Volatility of monthly return in low volatility regime
0.003	Mean monthly excess return in high volatility regime
0.166	Volatility of monthly return in high volatility regime
0.024	Switch from low to high regime ($p_{l,h}$)
0.100	Switch from high to low regime ($p_{h,l}$)

As expected, the risk and returns for small stocks tend to be higher on average than for large stocks. In addition, small stocks tend to move back and forth between regimes more readily than large stocks.

Using the base case parameters above for inflation, real interest rates, and excess equity returns, the annually compounded, geometric average equity returns are shown below:

Value	***Mean***	***Std. Dev.***	***5th Percentile***	***95th Percentile***
1st Projection Year				
Avg. Ann. Return - Large Stocks	11.71%	21.52%	-28.93%	41.79%
Avg. Ann. Return - Small Stocks	15.93%	34.46%	-40.32%	70.93%
10th Projection Year				
Avg. Ann. Return - Large Stocks	12.19%	9.32%	-5.18%	26.26%
Avg. Ann. Return - Small Stocks	14.68%	11.81%	-7.93%	31.65%

Instead of illustrating the effects of changing each of the parameters listed above, consider the unconditional probabilities of being in each of the two states. For large stocks, since it is more likely to switch regimes if you are in the high volatility regime, it follows that it is more likely that we would be in the low volatility regime. In fact, one can calculate the probability of being in low or high regime (π_L, π_H) for large stocks as:

$$\pi_L = \frac{P_{H,L}}{P_{L,H} + P_{H,L}} \approx 84\%$$

$$\pi_H = \frac{P_{L,H}}{P_{L,H} + P_{H,L}} \approx 16\%$$

Therefore, 84% of the draws for large stocks come from the high return, low volatility regime. For small stocks, the same calculation shows $\pi_l = 81\%$ and $\pi_h = 19\%$. Thus, small stocks also have a higher probability of deriving their returns from the high volatility regime.

After determining the unconditional probabilities, the user can then estimate the effects of parameter changes on the distribution of equity returns.

The correlation between small and large equity returns is driven by two separate parameters. The first correlation controls the dependency of regime switches. If both small and large stocks have a tendency to move from one regime to another simultaneously, the correlation of regime switching would be high. However, since small stocks have a higher probability of switching regimes than large stocks, even if the regime switching correlation is close to 1.0, it does *not* follow that the two classes of stocks will always switch regimes at the same time.

The second correlation between small and large stocks is the regime independent, stochastic excess return. Regardless of the existing regimes for each class of stock, there may be a tendency for all stock returns to move together. Therefore, a correlation is introduced to connect the random component of stock returns. The net result of these two correlations is illustrated in the following table.

Regime Switch Correlation	***Stochastic Excess Return***	***Correlation***
0.90	0.95	0.75
0.00	0.00	0.00
0.50	0.95	0.73
0.90	0.50	0.39
0.50	0.50	0.38

The first example in the table shows the base case scenario, where the correlation controlling for regime switches at the same time is 0.90 and the regime independent return correlation is 0.95. This produces a overall correlation between small and large stocks of 0.75 based on 200 iterations. The additional examples help illustrate that the excess return correlation is more important than the correlation for regime switches. This can be seen in the third and fourth examples. When the regime switching correlation is reduced from 0.90 to 0.50, the correlation of monthly returns is not markedly affected (from 0.75 to 0.73). But when the correlation within regimes is dropped from 0.95 to 0.50, the resulting correlation of monthly returns falls by almost 50 percent (from 0.75 to 0.39).