Combining Chain-Ladder and Additive Loss Reserving Methods for Dependent Lines of Business

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ABSTRACT

Often in non-life insurance, claim reserves are the largest position on the liability side of the balance sheet. Therefore, the estimation of adequate claim reserves for a portfolio consisting of several run-off subportfolios is relevant for every non-life insurance company. In the present paper we provide a framework in which we unify the multivariate chain-ladder (CL) model and the multivariate additive loss reserving (ALR) model into one model. This model allows for the simultaneous study of individual run-off subportfolios in which we use both the CL method and the ALR method for different subportfolios. Moreover, we derive an estimator for the conditional mean square error of prediction (MSEP) for the predictor of the ultimate claims of the total portfolio.

KEYWORDS

Claims reserving, multivariate chain-ladder method, multivariate additive loss reserving method, mean square error of prediction

1. Introduction and motivation

1.1. Claims reserving for several correlated run-off subportfolios

Often, claim reserves are the largest position on the liability side of the balance sheet of a non-life insurance company. Therefore, given the available information about the past development, the prediction of adequate claim reserves as well as the quantification of the uncertainties in these reserves is a major task in actuarial practice and science (e.g., Wüthrich and Merz (2008), Casualty Actuarial Society (2001), or Teugels and Sundt (2004)).

In this paper we consider the claim reserving problem in a multivariate context. More precisely, we consider a portfolio consisting of several correlated run-off subportfolios. On some subportfolios we use the chain-ladder (CL) method and on the other subportfolios we use the additive loss reserving (ALR) method to estimate the claim reserves. Since in actuarial practice the conditional mean square error of prediction (MSEP) is the most popular measure to quantify the uncertainties, we provide an MSEP estimator for the overall reserves. This means that we provide a first step towards an estimate of the overall MSEP for the predictor of the ultimate claims for aggregated subportfolios using different claims reserving methods for different subportfolios. These studies of uncertainties are crucial in the development of new solvency guidelines where one exactly quantifies the risk profiles of the different insurance companies.

1.2. Multivariate claims reserving methods

The simultaneous study of several correlated run-off subportfolios is motivated by the fact that:

1. In practice it is quite natural to subdivide a non-life run-off portfolio into several correlated subportfolios, such that each subportfolio satisfies certain homogeneity properties (e.g., the CL assumptions or the assumptions of the ALR method).
2. It addresses the problem of dependence between run-off portfolios of different lines of business (e.g., bodily injury claims in auto liability and in general liability business).
3. The multivariate approach has the advantage that by observing one run-off subportfolio we learn about the behavior of the other run-off subportfolios (e.g., subportfolios of small and large claims).
4. It resolves the problem of additivity (i.e., the estimators of the ultimate claims for the whole portfolio are obtained by summation over the estimators of the ultimate claims for the individual run-off subportfolios).

Holmberg (1994) was probably one of the first to investigate the problem of dependence between run-off portfolios of different lines of business. Braun (2004) and Merz and Wüthrich (2007; 2008) generalized the univariate CL model of Mack (1993) to the multivariate CL case by incorporating correlations between several run-off subportfolios. Another feasible multivariate claims reserving method is given by the multivariate ALR method proposed by Hess, Schmidt, and Zocher (2006) and Schmidt (2006a) which is based on a multivariate linear model. Under the assumptions of their multivariate ALR model Hess, Schmidt, and Zocher (2006) and Schmidt (2006a) derived a formula for the Gauss-Markov predictor for the nonobservable incremental claims which is optimal in terms of the classical optimality criterion of minimal expected squared loss. Merz and Wüthrich (2009) derived an estimator for the conditional MSEP in the multivariate ALR method using the Gauss-Markov predictor proposed by Hess, Schmidt, and Zocher (2006) and Schmidt (2006a).

1.3. Combination of the multivariate CL and ALR methods

In the sequel we provide a framework in which we combine the multivariate CL model and the multivariate ALR model into one multivariate model. The use of different reserving methods for different subportfolios is motivated by the fact that

1. in general not all subportfolios satisfy the same homogeneity assumptions; and/or
2. sometimes we have a priori information (e.g., premium, number of contracts, external knowledge from experts, data from similar portfolios, market statistics) for some selected subportfolios which we want to incorporate into our claims reserving analysis.

That is, we use the CL method for a subset of subportfolios on the one hand and we use the ALR method for the complementary subset of subportfolios on the other hand. From this point of view it is interesting to note that the CL method and the ALR method are very different in some aspects and therefore exploit differing features of the data belonging to the individual subportfolios:

1. The CL method is based on cumulative claims whereas the ALR method is applied to incremental claims.
2. Unlike the CL method, the ALR method combines past observations in the upper triangle with external knowledge from experts or with a priori information.
3. The ALR method is more robust to outliers in the observations than the CL method.

Organization of this paper. In Section 2 we provide the notation and data structure for our multivariate framework. In Section 3 we define the combined model and derive the properties of the estimators for the ultimate claims within the framework of the combined method. In Section 4 we give an estimation procedure for the conditional MSEP in the combined method and our main results are presented in Estimator 4.7 and Estimator 4.8. Section 5 is dedicated to the estimation of the model parameters, and, finally, in Section 6 we give an example. An interested reader will find proofs of the results in Section 7.

2. Notation and multivariate framework

We assume that the subportfolios consist of $N \geq 1$ run-off triangles of observations of the same size. However, the multivariate CL method and the multivariate ALR method can also be applied to other shapes of data (e.g., run-off trapezoids). In these N triangles the indices

$n, \quad 1 \leq n \leq N,$	refer to subportfolios (triangles),
$i, \quad 0 \leq i \leq I,$	refer to accident years (rows),
$j, \quad 0 \leq j \leq J = I,$	refer to development years (columns).

The incremental claims (i.e., incremental payments, change of reported claim amounts or number of newly reported claims) of run-off triangle n for accident year i and development year j are denoted by $X_{i,j}^{(n)}$ and cumulative claims (i.e., cumulative payments, claims incurred or total number of reported claims) are given by

$$C_{i,j}^{(n)} = \sum_{k=0}^{j} X_{i,k}^{(n)}. \qquad (1)$$

Figure 1 shows the claims data structure for N individual claims development triangles described above.

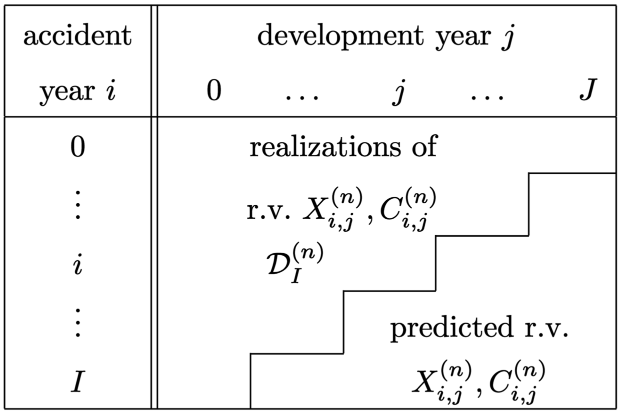
Usually, at time I, we have observations

$$\mathcal{D}_I^{(n)} = \{C_{i,j}^{(n)};\ i + j \leq I\}, \qquad (2)$$

for all run-off subportfolios $n \in \{1,\ldots,N\}$. This means that at time I (calendar year I) we have a total of observations over all subportfolios given by

$$\mathcal{D}_I^N = \bigcup_{n=1}^{N} \mathcal{D}_I^{(n)}, \qquad (3)$$

Figure 1. Claims development of triangle $n \in \{1,\ldots,N\}$



and we need to predict the random variables in its complement

$$\mathcal{D}_I^{N,c} = \{C_{i,j}^{(n)};\ i \le I,\ i+j > I,\ 1 \le n \le N\}. \tag{4}$$

In the sequel we assume without loss of generality that we use the multivariate CL method for the first K (i.e., $K \le N$) run-off triangles $n = 1,\ldots,K$ and the multivariate ALR method for the remaining $n = K+1,\ldots,N$ triangles. Therefore, we introduce the following vector notation

$$\mathbf{C}_{i,j}^{\mathrm{CL}} = \begin{pmatrix} C_{i,j}^{(1)} \\ \vdots \\ C_{i,j}^{(K)} \end{pmatrix}, \qquad \mathbf{X}_{i,j}^{\mathrm{CL}} = \begin{pmatrix} X_{i,j}^{(1)} \\ \vdots \\ X_{i,j}^{(K)} \end{pmatrix}, \tag{5}$$

$$\mathbf{C}_{i,j}^{\mathrm{AD}} = \begin{pmatrix} C_{i,j}^{(K+1)} \\ \vdots \\ C_{i,j}^{(N)} \end{pmatrix} \quad \text{and} \quad \mathbf{X}_{i,j}^{\mathrm{AD}} = \begin{pmatrix} X_{i,j}^{(K+1)} \\ \vdots \\ X_{i,j}^{(N)} \end{pmatrix}$$

for all $i \in \{0,\ldots,I\}$ and $j \in \{0,\ldots,J\}$. In particular, this means that the cumulative/incremental claims of the whole portfolio are given by the vectors

$$\mathbf{C}_{i,j} = \begin{pmatrix} \mathbf{C}_{i,j}^{\mathrm{CL}} \\ \mathbf{C}_{i,j}^{\mathrm{AD}} \end{pmatrix} \quad \text{and} \quad \mathbf{X}_{i,j} = \begin{pmatrix} \mathbf{X}_{i,j}^{\mathrm{CL}} \\ \mathbf{X}_{i,j}^{\mathrm{AD}} \end{pmatrix}. \tag{6}$$

We define the first $k+1$ columns of CL observations by

$$\mathcal{B}_k^K = \{\mathbf{C}_{i,j}^{\mathrm{CL}};\ i+j \le I \text{ and } 0 \le j \le k\} \tag{7}$$

for $k \in \{0,\ldots,J\}$. Finally, we define L-dimensional column vectors for $L = N, K, N-K$ consisting of ones by $\mathbf{1}_L = (1,\ldots,1)' \in \mathbb{R}^L$, and denote by

$$\mathrm{D}(\mathbf{a}) = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_L \end{pmatrix} \quad \text{and}$$
$$\mathrm{D}(\mathbf{c})^b = \begin{pmatrix} c_1^b & & 0 \\ & \ddots & \\ 0 & & c_L^b \end{pmatrix} \tag{8}$$

the $L \times L$-diagonal matrices of the L-dimensional vectors $\mathbf{a} = (a_1,\ldots,a_L)' \in \mathbb{R}^L$ and $(c_1^b,\ldots,c_L^b)' \in \mathbb{R}_+^L$, where $b \in \mathbb{R}$ and $\mathbf{c} = (c_1,\ldots,c_L)' \in \mathbb{R}_+^L$.

3. Combined multivariate CL and ALR method

The following model is a combination of the multivariate CL model and the multivariate ALR model presented in Merz and Wüthrich (2008) and Merz and Wüthrich (2009), respectively.

Assumptions 3.1 (Combined CL and ALR model)

- Incremental claims $\mathbf{X}_{i,j}$ of different accident years i are independent.
- There exist K-dimensional constants

$$\mathbf{f}_j = (f_j^{(1)},\ldots,f_j^{(K)})' \quad \text{and}$$
$$\boldsymbol{\sigma}_j^{\mathrm{CL}} = (\sigma_j^{(1)},\ldots,\sigma_j^{(K)})' \tag{9}$$

with $f_j^{(k)} > 0$, $\sigma_j^{(k)} > 0$ and K-dimensional random variables

$$\boldsymbol{\varepsilon}_{i,j+1}^{\mathrm{CL}} = (\varepsilon_{i,j+1}^{(1)},\ldots,\varepsilon_{i,j+1}^{(K)})', \tag{10}$$

such that for all $i \in \{0,\ldots,I\}$ and $j \in \{0,\ldots,J-1\}$ we have

$$\mathbf{C}_{i,j+1}^{\mathrm{CL}} = \mathrm{D}(\mathbf{f}_j)\cdot\mathbf{C}_{i,j}^{\mathrm{CL}} + \mathrm{D}(\mathbf{C}_{i,j}^{\mathrm{CL}})^{1/2}\cdot\mathrm{D}(\boldsymbol{\varepsilon}_{i,j+1}^{\mathrm{CL}})\cdot\boldsymbol{\sigma}_j^{\mathrm{CL}}. \tag{11}$$

- There exist $(N-K)$-dimensional constants

$$\mathbf{m}_j = (m_j^{(1)},\ldots,m_j^{(N-K)})' \quad \text{and} \quad \boldsymbol{\sigma}_{j-1}^{\mathrm{AD}} = (\sigma_{j-1}^{(K+1)},\ldots,\sigma_{j-1}^{(N)})', \tag{12}$$

with $\sigma_{j-1}^{(n)} > 0$ and $(N-K)$-dimensional random variables

$$\varepsilon_{i,j}^{\mathrm{AD}} = (\varepsilon_{i,j}^{(K+1)},\ldots,\varepsilon_{i,j}^{(N)})', \tag{13}$$

such that for all $i \in \{0,\ldots,I\}$ and $j \in \{1,\ldots,J\}$ we have

$$\mathbf{X}_{i,j}^{\mathrm{AD}} = \mathrm{V}_i \cdot \mathbf{m}_j + \mathrm{V}_i^{1/2} \cdot \mathrm{D}(\varepsilon_{i,j}^{\mathrm{AD}}) \cdot \boldsymbol{\sigma}_{j-1}^{\mathrm{AD}}, \tag{14}$$

where $\mathrm{V}_i \in \mathbb{R}^{(N-K)\times(N-K)}$ are deterministic positive definite symmetric matrices.

- The N-dimensional random variables

$$\varepsilon_{i,j+1} = \begin{pmatrix} \varepsilon_{i,j+1}^{\mathrm{CL}} \\ \varepsilon_{i,j+1}^{\mathrm{AD}} \end{pmatrix} \quad \text{and} \quad \varepsilon_{k,l+1} = \begin{pmatrix} \varepsilon_{k,l+1}^{\mathrm{CL}} \\ \varepsilon_{k,l+1}^{\mathrm{AD}} \end{pmatrix}$$

are independent for $i \neq k$ or $j \neq l$, with $E[\varepsilon_{i,j+1}] = \mathbf{0}$ and

$$\begin{aligned} \mathrm{Cov}(\varepsilon_{i,j+1},\varepsilon_{i,j+1}) &= E[\varepsilon_{i,j+1} \cdot \varepsilon_{i,j+1}'] \\ &= \begin{pmatrix} 1 & \rho_j^{(1,2)} & \cdots & \cdots & \rho_j^{(1,N)} \\ \rho_j^{(2,1)} & 1 & \cdots & \cdots & \rho_j^{(2,N)} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ \rho_j^{(N,1)} & \rho_j^{(N,2)} & \cdots & \cdots & 1 \end{pmatrix}, \end{aligned} \tag{15}$$

for fixed $\rho_j^{(n,m)} \in (-1,1)$ for $n \neq m$. □

We introduce the notation

$$\begin{aligned} \boldsymbol{\sigma}_j &= (\boldsymbol{\sigma}_j^{\mathrm{CL}},\boldsymbol{\sigma}_j^{\mathrm{AD}})', \\ \Sigma_j &= E[\mathrm{D}(\varepsilon_{i,j+1}) \cdot \boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_j' \cdot \mathrm{D}(\varepsilon_{i,j+1})], \\ \Sigma_j^{(C)} &= E[\mathrm{D}(\varepsilon_{i,j+1}^{\mathrm{CL}}) \cdot \boldsymbol{\sigma}_j^{\mathrm{CL}} \cdot (\boldsymbol{\sigma}_j^{\mathrm{CL}})' \cdot \mathrm{D}(\varepsilon_{i,j+1}^{\mathrm{CL}})], \\ \Sigma_j^{(A)} &= E[\mathrm{D}(\varepsilon_{i,j+1}^{\mathrm{AD}}) \cdot \boldsymbol{\sigma}_j^{\mathrm{AD}} \cdot (\boldsymbol{\sigma}_j^{\mathrm{AD}})' \cdot \mathrm{D}(\varepsilon_{i,j+1}^{\mathrm{AD}})], \end{aligned} \tag{16}$$

$$\begin{aligned} \Sigma_j^{(C,A)} &= E[\mathrm{D}(\varepsilon_{i,j+1}^{\mathrm{CL}}) \cdot \boldsymbol{\sigma}_j^{\mathrm{CL}} \cdot (\boldsymbol{\sigma}_j^{\mathrm{AD}})' \cdot \mathrm{D}(\varepsilon_{i,j+1}^{\mathrm{AD}})], \\ \Sigma_j^{(A,C)} &= E[\mathrm{D}(\varepsilon_{i,j+1}^{\mathrm{AD}}) \cdot \boldsymbol{\sigma}_j^{\mathrm{AD}} \cdot (\boldsymbol{\sigma}_j^{\mathrm{CL}})' \cdot \mathrm{D}(\varepsilon_{i,j+1}^{\mathrm{CL}})] \\ &= (\Sigma_j^{(C,A)})'. \end{aligned} \tag{17}$$

Thus, we have

$$\Sigma_j = \begin{pmatrix} (\sigma_j^{(1)})^2 & \sigma_j^{(1)}\sigma_j^{(2)}\rho_j^{(1,2)} & \cdots & \cdots & \sigma_j^{(1)}\sigma_j^{(N)}\rho_j^{(1,N)} \\ \sigma_j^{(2)}\sigma_j^{(1)}\rho_j^{(2,1)} & (\sigma_j^{(2)})^2 & \cdots & \cdots & \sigma_j^{(2)}\sigma_j^{(N)}\rho_j^{(2,N)} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ \sigma_j^{(N)}\sigma_j^{(1)}\rho_j^{(N,1)} & \sigma_j^{(N)}\sigma_j^{(2)}\rho_j^{(N,2)} & \cdots & \cdots & (\sigma_j^{(N)})^2 \end{pmatrix} = \begin{pmatrix} \Sigma_j^{(C)} & \Sigma_j^{(C,A)} \\ \Sigma_j^{(A,C)} & \Sigma_j^{(A)} \end{pmatrix}. \tag{18}$$

The Multivariate Model 3.1 is suitable for portfolios of N correlated subportfolios in which the first K subportfolios satisfy the homogeneity assumptions of the CL method, and the other $N-K$ subportfolios satisfy the homogeneity assumptions of the ALR method. Under Model Assumptions 3.1, the properties of the cumulative claims $\mathbf{C}_{i,j}^{\mathrm{CL}}$ and the incremental claims $\mathbf{X}_{i,j}^{\mathrm{AD}}$ are consistent with the assumptions of the multivariate CL time series model (see Merz and Wüthrich (2008)) and the multivariate ALR model (see Merz and Wüthrich (2009)). In particular for $K = N$ and $K = 0$ Model Assumptions 3.1 reduce to the model assumptions of the multivariate CL time series model and the multivariate ALR model, respectively.

Remark 3.2

- The factors $\mathbf{f}_j$ are called K-dimensional development factors, CL factors, age-to-age factors or link-ratios. The $N-K$-dimensional constants $\mathbf{m}_j$ are called incremental loss ratios and can be interpreted as a multivariate scaled expected reporting/cashflow pattern over the different development years.

- In most practical applications, V_i is chosen to be diagonal so as to represent a volume measure of accident year i, a priori known (e.g., premium, number of contracts, expected number of claims, etc.) or external knowledge from experts, similar portfolios or market statistics. Since we assume that V_i is a positive definite symmetric matrix, there is a well-defined positive definite symmetric matrix $\mathrm{V}_i^{1/2}$ (called square root of V_i) satisfying $\mathrm{V}_i = \mathrm{V}_i^{1/2} \cdot \mathrm{V}_i^{1/2}$.
- Within the CL and ALR framework, Braun (2004) and Merz and Wüthrich (2007; 2008; 2009) proposed the development year-based correlations given by (15). Often correlations between different run-off triangles are attributed to claims inflation. Under this point of view it may seem more reasonable to allow for correlation between the cumulative or incremental claims of the same calender year (diagonals of the claims development triangles). This would introduce dependencies between accident years. However, at the moment it is not mathematically tractable to treat such year-based correlations within the CL and ALR framework. That is, all calender year-based dependencies should be removed from the data before calculating the reserves with the CL or ALR method. However, after correcting the data for the calender year-based correlations, further direct and indirect sources for correlations between different run-off triangles of a portfolio exist and should be taken into account (cf. Houltram (2003)). This is exactly what our model does.
- Matrix $\Sigma_{j-1}^{(C)}$ reflects the correlation structure between the cumulative claims of development year j within the first K subportfolios and matrix $\Sigma_{j-1}^{(A)}$ the correlation structure between the incremental claims of development year j within the last $N-K$ subportfolios. The matrices $\Sigma_{j-1}^{(C,A)}$ and $\Sigma_{j-1}^{(A,C)}$ reflect the correlation structure between the cumulative claims of development year j in the first K subportfolios and the incremental claims of development year j in the last $N-K$ subportfolios.
- There may occur difficulties about positivity in the time-series definition (11), which can be solved in a mathematically correct way. We omit these derivations since they do not lead to a deeper understanding of the model. Refer to Wüthrich, Merz, and Bühlmann (2008) for more details.
- The indices for $\boldsymbol{\sigma}$ and ε differ by 1, since it simplifies the comparability with the derivations and results in Merz and Wüthrich (2008; 2009).

We obtain for the conditionally expected ultimate claim $E[\mathbf{C}_{i,J} \mid \mathcal{D}_I^N]$:

LEMMA 3.3 *Under Model Assumptions* 3.1 *we have for all* $1 \le i \le I$*:*

a)

$$E[\mathbf{C}_{i,J}^{\mathrm{CL}} \mid \mathcal{D}_I^N] = E[\mathbf{C}_{i,J}^{\mathrm{CL}} \mid \mathbf{C}_{i,I-i}] = E[\mathbf{C}_{i,J}^{\mathrm{CL}} \mid \mathbf{C}_{i,I-i}^{\mathrm{CL}}] = \prod_{j=I-i}^{J-1} \mathrm{D}(\mathbf{f}_j) \cdot \mathbf{C}_{i,I-i}^{\mathrm{CL}},$$

b)

$$E[\mathbf{C}_{i,J}^{\mathrm{AD}} \mid \mathcal{D}_I^N] = E[\mathbf{C}_{i,J}^{\mathrm{AD}} \mid \mathbf{C}_{i,I-i}] = E[\mathbf{C}_{i,J}^{\mathrm{AD}} \mid \mathbf{C}_{i,I-i}^{\mathrm{AD}}] = \mathbf{C}_{i,I-i}^{\mathrm{AD}} + \mathrm{V}_i \cdot \sum_{j=I-i+1}^{J} \mathbf{m}_j.$$

PROOF This immediately follows from Model Assumptions 3.1. □

This result motivates an algorithm for estimating the outstanding claims liabilities, given the observations $\mathcal{D}_I^N$. If the K-dimensional CL factors $\mathbf{f}_j$ and the $(N-K)$-dimensional incremental loss ratios $\mathbf{m}_j$ are known, the outstanding claims liabilities of accident year i for the first K and the last $N-K$ correlated run-off triangles are predicted by

$$E[\mathbf{C}_{i,J}^{\mathrm{CL}} \mid \mathcal{D}_I^N] - \mathbf{C}_{i,I-i}^{\mathrm{CL}} = \prod_{j=I-i}^{J-1} \mathrm{D}(\mathbf{f}_j) \cdot \mathbf{C}_{i,I-i}^{\mathrm{CL}} - \mathbf{C}_{i,I-i}^{\mathrm{CL}} \tag{19}$$

and

$$E[\mathbf{C}_{i,J}^{\mathrm{AD}} \mid \mathcal{D}_I^N] - \mathbf{C}_{i,I-i}^{\mathrm{AD}} = \mathrm{V}_i \cdot \sum_{j=I-i+1}^{J} \mathbf{m}_j, \tag{20}$$

respectively. However, in practical applications we have to estimate the parameters $\mathbf{f}_j$ and $\mathbf{m}_j$ from the data in the N upper triangles. Pröhl and Schmidt (2005) and Schmidt (2006a) proposed the multivariate CL factor estimates for $\mathbf{f}_j$ ($j = 0,\ldots,J-1$)

$$\begin{aligned}\hat{\mathbf{f}}_j &= (\hat{f}_j^{(1)},\ldots,\hat{f}_j^{(K)})' \\ &= \left(\sum_{i=0}^{I-j-1} \mathrm{D}(\mathbf{C}_{i,j}^{\mathrm{CL}})^{1/2}(\Sigma_j^{(C)})^{-1}\mathrm{D}(\mathbf{C}_{i,j}^{\mathrm{CL}})^{1/2}\right)^{-1} \\ &\quad \cdot \sum_{i=0}^{I-j-1} \mathrm{D}(\mathbf{C}_{i,j}^{\mathrm{CL}})^{1/2}(\Sigma_j^{(C)})^{-1}\mathrm{D}(\mathbf{C}_{i,j}^{\mathrm{CL}})^{-1/2} \cdot \mathbf{C}_{i,j+1}^{\mathrm{CL}}.\end{aligned} \tag{21}$$

In the framework of the multivariate ALR method Hess, Schmidt, and Zocher (2006) and Schmidt (2006a) proposed the multivariate estimates for the incremental loss ratios $\mathbf{m}_j$ ($j = 1,\ldots,J$)

$$\begin{aligned}\hat{\mathbf{m}}_j &= (\hat{m}_j^{(1)},\ldots,\hat{m}_j^{(N-K)})' \\ &= \left(\sum_{i=0}^{I-j} \mathrm{V}_i^{1/2} \cdot (\Sigma_{j-1}^{(A)})^{-1} \cdot \mathrm{V}_i^{1/2}\right)^{-1} \\ &\quad \cdot \sum_{i=0}^{I-j} \mathrm{V}_i^{1/2} \cdot (\Sigma_{j-1}^{(A)})^{-1} \cdot \mathrm{V}_i^{-1/2} \cdot \mathbf{X}_{i,j}^{\mathrm{AD}}.\end{aligned} \tag{22}$$

REMARK 3.4

- In the case $K = 1$ (i.e., only one CL run-off subportfolio) the estimator (21) coincides with the classical univariate CL estimator of Mack (1993). Analogously, in the case $N-K = 1$ (i.e., only one additive run-off subportfolio) the estimator (22) coincides with the univariate incremental loss ratio estimates

$$\hat{m}_j = \sum_{i=0}^{I-j} \frac{X_{i,j}}{\sum_{k=0}^{I-j} V_k} \tag{23}$$

  with deterministic one-dimensional weights V_i (see, e.g., Schmidt (2006a; 2006b)).
- With respect to the criterion of minimal expected squared loss the multivariate CL factor estimates (21) are optimal unbiased linear estimators for $\mathbf{f}_j$ (cf. Pröhl and Schmidt (2005) and Schmidt (2006a)) and the multivariate incremental loss ratio estimates (22) are optimal unbiased linear estimators for $\mathbf{m}_j$ (cf. Hess, Schmit, and Zocher (2006) and Schmidt (2006a)).
- For uncorrelated cumulative and incremental claims in the different run-off subportfolios (i.e., we set $\Sigma = \mathbf{I}$, where $\mathbf{I}$ denotes the identity matrix) we obtain the (unbiased) estimators for $\mathbf{f}_j$ and $\mathbf{m}_j$

$$\hat{\mathbf{f}}_j^{(0)} = \left(\sum_{i=0}^{I-j-1} \mathrm{D}(\mathbf{C}_{i,j}^{\mathrm{CL}})\right)^{-1} \cdot \sum_{i=0}^{I-j-1} \mathbf{C}_{i,j+1}^{\mathrm{CL}} \tag{24}$$

  and

$$\hat{\mathbf{m}}_j^{(0)} = \left(\sum_{i=0}^{I-j} \mathrm{V}_i\right)^{-1} \cdot \sum_{i=0}^{I-j} \mathbf{X}_{i,j}^{\mathrm{AD}}. \tag{25}$$

  For a given Σ, both $\hat{\mathbf{f}}_j$ and $\hat{\mathbf{f}}_j^{(0)}$ as well as $\hat{\mathbf{m}}_j$ and $\hat{\mathbf{m}}_j^{(0)}$ are unbiased estimators for the multivariate CL factor $\mathbf{f}_j$ and multivariate incremental loss ratio $\mathbf{m}_j$, respectively (see Lemma 3.6 below). However, only $\hat{\mathbf{f}}_j$ and $\hat{\mathbf{m}}_j$ are optimal in the sense that they have minimal expected squared loss; see the second bullet of these remarks.

In the sequel we predict the cumulative claims $\mathbf{C}_{i,j}^{\mathrm{CL}}$ of the first K run-off triangles and the cumulative claims $\mathbf{C}_{i,j}^{\mathrm{AD}}$ of the last $N-K$ run-off triangles for $i + j > I$ by the multivariate CL predictors

$$\widehat{\mathbf{C}_{i,j}}^{\mathrm{CL}} = (\widehat{C_{i,j}^{(1)}}^{\mathrm{CL}}, \ldots, \widehat{C_{i,j}^{(K)}}^{\mathrm{CL}})' = \hat{E}[\mathbf{C}_{i,j}^{\mathrm{CL}} \mid \mathcal{D}_I^N] = \prod_{l=I-i}^{j-1} \mathrm{D}(\hat{\mathbf{f}}_l) \cdot \mathbf{C}_{i,I-i}^{\mathrm{CL}} \tag{26}$$

and the multivariate ALR predictors

$$\widehat{\mathbf{C}_{i,j}}^{\mathrm{AD}} = (\widehat{C_{i,j}^{(K+1)}}^{\mathrm{AD}}, \ldots, \widehat{C_{i,j}^{(N)}}^{\mathrm{AD}})' = \hat{E}[\mathbf{C}_{i,j}^{\mathrm{AD}} \mid \mathcal{D}_I^N] = \mathbf{C}_{i,I-i}^{\mathrm{AD}} + \mathrm{V}_i \cdot \sum_{l=I-i+1}^{j} \hat{\mathbf{m}}_l. \tag{27}$$

This means that we predict the N-dimensional ultimate claims $\mathbf{C}_{i,J}$ by

$$\widehat{\mathbf{C}_{i,J}} = \begin{pmatrix} \widehat{\mathbf{C}_{i,J}}^{\mathrm{CL}} \\ \widehat{\mathbf{C}_{i,J}}^{\mathrm{AD}} \end{pmatrix}. \tag{28}$$

ESTIMATOR 3.5 (Combined CL and ALR estimator) *The combined CL and ALR estimator for* $E[\mathbf{C}_{i,j} \mid \mathcal{D}_I^N]$ *is for* $i+j>I$ *given by*

$$\widehat{\mathbf{C}_{i,j}} = \hat{E}[\mathbf{C}_{i,j} \mid \mathcal{D}_I^N] = \begin{pmatrix} \widehat{\mathbf{C}_{i,j}}^{\mathrm{CL}} \\ \widehat{\mathbf{C}_{i,j}}^{\mathrm{AD}} \end{pmatrix}.$$

The following lemma collects results from Lemma 3.5 in Merz and Wüthrich (2008) as well as from Property 3.4 and Property 3.7 in Merz and Wüthrich (2009).

LEMMA 3.6 *Under Model Assumptions 3.1 we have:*

a) $\hat{\mathbf{f}}_j$ *is, given* $\mathcal{B}_j^K$, *an unbiased estimator for* $\mathbf{f}_j$, *i.e.,* $\mathrm{E}[\hat{\mathbf{f}}_j \mid \mathcal{B}_j^K] = \mathbf{f}_j$;

b) $\hat{\mathbf{f}}_j$ *and* $\hat{\mathbf{f}}_k$ *are uncorrelated for* $j \neq k$, *i.e.,* $\mathrm{E}[\hat{\mathbf{f}}_j \cdot \hat{\mathbf{f}}_k'] = \mathbf{f}_j \cdot \mathbf{f}_k' = \mathrm{E}[\hat{\mathbf{f}}_j] \cdot \mathrm{E}[\hat{\mathbf{f}}_k]'$;

c) $\hat{\mathbf{m}}_j$ *is an unbiased estimator for* $\mathbf{m}_j$, *i.e.,* $\mathrm{E}[\hat{\mathbf{m}}_j] = \mathbf{m}_j$;

d) $\hat{\mathbf{m}}_j$ *and* $\hat{\mathbf{m}}_k$ *are independent for* $j \neq k$;

e) $\mathrm{Var}(\hat{\mathbf{m}}_j) = \left(\sum_{l=0}^{I-j} \mathrm{V}_l^{1/2} \cdot (\Sigma_{j-1}^{(A)})^{-1} \cdot \mathrm{V}_l^{1/2}\right)^{-1}$;

f) $\widehat{\mathbf{C}_{i,J}}$ *is, given* $\mathbf{C}_{i,I-i}$, *an unbiased estimator for* $E[\mathbf{C}_{i,J} \mid \mathcal{D}_I^N]$, *i.e.,* $E[\widehat{\mathbf{C}_{i,J}} \mid \mathbf{C}_{i,I-i}] = E[\mathbf{C}_{i,J} \mid \mathcal{D}_I^N] = E[\mathbf{C}_{i,J} \mid \mathbf{C}_{i,I-i}]$.

REMARK 3.7

- Note that Lemma 3.6 f) shows that we have unbiased estimators of the conditionally expected ultimate claim $E[\mathbf{C}_{i,J} \mid \mathcal{D}_I^N]$. Moreover, it implies that the estimator of the aggregated ultimate claims for accident year i

$$\sum_{n=1}^{K} \widehat{C_{i,J}^{(n)}}^{\mathrm{CL}} + \sum_{n=K+1}^{N} \widehat{C_{i,J}^{(n)}}^{\mathrm{AD}} = \mathbf{1}' \cdot \widehat{\mathbf{C}_{i,J}} = \mathbf{1}_K' \cdot \widehat{\mathbf{C}_{i,J}}^{\mathrm{CL}} + \mathbf{1}_{N-K}' \cdot \widehat{\mathbf{C}_{i,J}}^{\mathrm{AD}}$$

is, given $\mathbf{C}_{i,I-i}$, an unbiased estimator for $\sum_{n=1}^{N} E[C_{i,J}^{(n)} \mid \mathbf{C}_{i,I-i}]$.

- Note that the parameters for the CL method are estimated independently from the observations belonging to the ALR method and vice versa. That is, here we could even go one step beyond and learn from ALR method observations when estimating CL parameters and vice versa. We omit these derivations since formulas get more involved and neglect the fact that one may even improve estimators. Our goal here is to give an estimate for the overall MSEP for the parameter estimators (21) and (22).

4. Conditional MSEP

In this section we consider the prediction uncertainty of the predictors

$$\sum_{n=1}^{K} \widehat{C_{i,J}^{(n)}}^{\mathrm{CL}} + \sum_{n=K+1}^{N} \widehat{C_{i,J}^{(n)}}^{\mathrm{AD}} \quad \text{and}$$

$$\sum_{i=1}^{I} \left(\sum_{n=1}^{K} \widehat{C_{i,J}^{(n)}}^{\mathrm{CL}} + \sum_{n=K+1}^{N} \widehat{C_{i,J}^{(n)}}^{\mathrm{AD}} \right),$$

given the observations $\mathcal{D}_I^N$, for the ultimate claims. This means our goal is to derive an estimate of the conditional MSEP for single accident years $i \in \{1, \ldots, I\}$ which is defined as

$$\mathrm{msep}_{\sum_{n=1}^{N} C_{i,J}^{(n)} \mid \mathcal{D}_I^N} \left(\sum_{n=1}^{K} \widehat{C_{i,J}^{(n)}}^{\mathrm{CL}} + \sum_{n=K+1}^{N} \widehat{C_{i,J}^{(n)}}^{\mathrm{AD}} \right) = E\left[\left(\sum_{n=1}^{K} \widehat{C_{i,J}^{(n)}}^{\mathrm{CL}} + \sum_{n=K+1}^{N} \widehat{C_{i,J}^{(n)}}^{\mathrm{AD}} - \sum_{n=1}^{N} C_{i,J}^{(n)} \right)^2 \middle| \mathcal{D}_I^N \right], \tag{29}$$

as well as an estimate of the conditional MSEP for aggregated accident years given by

$$\operatorname{msep}_{\sum_{i,n} C_{i,J}^{(n)} | \mathcal{D}_I^N}\left(\sum_{i=1}^{I}\sum_{n=1}^{K}\widehat{C_{i,J}^{(n)}}^{\mathrm{CL}} + \sum_{i=1}^{I}\sum_{n=K+1}^{N}\widehat{C_{i,J}^{(n)}}^{\mathrm{AD}}\right)$$
$$= E\left[\left(\sum_{i=1}^{I}\sum_{n=1}^{K}\widehat{C_{i,J}^{(n)}}^{\mathrm{CL}} + \sum_{i=1}^{I}\sum_{n=K+1}^{N}\widehat{C_{i,J}^{(n)}}^{\mathrm{AD}} - \sum_{i=1}^{I}\sum_{n=1}^{N} C_{i,J}^{(n)}\right)^2 \middle| \mathcal{D}_I^N\right]. \tag{30}$$

4.1. Conditional MSEP for single accident years

We choose $i \in \{1,\ldots,I\}$. The conditional MSEP (29) for a single accident year i decomposes as

$$\operatorname{msep}_{\sum_{n=1}^{N} C_{i,J}^{(n)} | \mathcal{D}_I^N}\left(\sum_{n=1}^{K}\widehat{C_{i,J}^{(n)}}^{\mathrm{CL}} + \sum_{n=K+1}^{N}\widehat{C_{i,J}^{(n)}}^{\mathrm{AD}}\right)$$
$$= \operatorname{msep}_{\sum_{n=1}^{K} C_{i,J}^{(n)} | \mathcal{D}_I^N}\left(\sum_{n=1}^{K}\widehat{C_{i,J}^{(n)}}^{\mathrm{CL}}\right) + \operatorname{msep}_{\sum_{n=K+1}^{N} C_{i,J}^{(n)} | \mathcal{D}_I^N}\left(\sum_{n=K+1}^{N}\widehat{C_{i,J}^{(n)}}^{\mathrm{AD}}\right)$$
$$+ 2\cdot E\left[\left(\sum_{n=1}^{K}\widehat{C_{i,J}^{(n)}}^{\mathrm{CL}} - \sum_{n=1}^{K} C_{i,J}^{(n)}\right)\cdot\left(\sum_{n=K+1}^{N}\widehat{C_{i,J}^{(n)}}^{\mathrm{AD}} - \sum_{n=K+1}^{N} C_{i,J}^{(n)}\right) \middle| \mathcal{D}_I^N\right]. \tag{31}$$

The first two terms on the right-hand side of (31) are the conditional MSEP for single accident years i if we use the multivariate CL method for the first K run-off triangles (numbered by $n = 1,\ldots,K$) and the multivariate ALR method for the last $N-K$ run-off triangles (numbered by $n = K+1,\ldots,N$), respectively. Estimators for these two conditional MSEPs are derived in Merz and Wüthrich (2008; 2009) and are given by Estimator 4.1 and Estimator 4.2, below.

ESTIMATOR 4.1 (MSEP for single accident years, CL method, cf. Merz and Wüthrich (2008)) *Under Model Assumptions* 3.1 *we have the estimator for the conditional MSEP of the ultimate claims in the first K run-off triangles for a single accident year $i \in \{1,\ldots,I\}$*

$$\widehat{\operatorname{msep}}_{\sum_{n=1}^{K} C_{i,J}^{(n)} | \mathcal{D}_I^N}\left(\sum_{n=1}^{K}\widehat{C_{i,J}^{(n)}}^{\mathrm{CL}}\right)$$
$$= \mathbf{1}_K' \cdot \left(\sum_{l=I-i+1}^{J}\prod_{k=l}^{J-1}\mathrm{D}(\hat{\mathbf{f}}_k)\cdot\hat{\Sigma}_{i,l-1}^{\mathbf{C}}\cdot\prod_{k=l}^{J-1}\mathrm{D}(\hat{\mathbf{f}}_k)\right)\cdot\mathbf{1}_K$$
$$+ \mathbf{1}_K'\cdot\mathrm{D}(\mathbf{C}_{i,I-i}^{\mathrm{CL}})\cdot(\hat{\Delta}_{i,J}^{(n,m)})_{1\le n,m\le K}\cdot\mathrm{D}(\mathbf{C}_{i,I-i}^{\mathrm{CL}})\cdot\mathbf{1}_K, \tag{32}$$

with

$$\hat{\Sigma}_{i,l-1}^{\mathbf{C}} = \mathrm{D}(\widehat{\mathbf{C}_{i,l-1}}^{\mathrm{CL}})^{1/2}\cdot\hat{\Sigma}_{l-1}^{(C)}\cdot\mathrm{D}(\widehat{\mathbf{C}_{i,l-1}}^{\mathrm{CL}})^{1/2}, \tag{33}$$

$$\hat{\Delta}_{i,J}^{(n,m)} = \prod_{l=I-i}^{J-1}\left(\hat{f}_l^{(n)}\cdot\hat{f}_l^{(m)} + \sum_{k=0}^{I-l-1}\hat{\mathbf{a}}_{n|l}^k\cdot\hat{\Sigma}_l^{(C)}\cdot(\hat{\mathbf{a}}_{m|l}^k)'\right) - \prod_{l=I-i}^{J-1}\hat{f}_l^{(n)}\cdot\hat{f}_l^{(m)}, \tag{34}$$

where $\hat{\mathbf{a}}_{n|l}^k$ and $\hat{\mathbf{a}}_{m|l}^k$ are the nth and mth row of

$$\hat{\mathbf{A}}_l^k = \left(\sum_{i=0}^{I-l-1}\mathrm{D}(\mathbf{C}_{i,l}^{\mathrm{CL}})^{1/2}\cdot(\hat{\Sigma}_l^{(C)})^{-1}\cdot\mathrm{D}(\mathbf{C}_{i,l}^{\mathrm{CL}})^{1/2}\right)^{-1}\cdot\mathrm{D}(\widehat{\mathbf{C}_{k,l}}^{\mathrm{CL}})^{1/2}\cdot(\hat{\Sigma}_l^{(C)})^{-1} \tag{35}$$

and the parameter estimates $\hat{\Sigma}_{l-1}^{(C)}$ are given in Section 5.

ESTIMATOR 4.2 (MSEP for single accident years, ALR method, cf. Merz and Wüthrich (2009)) *Under Model Assumptions* 3.1 *we have the estimator for the conditional MSEP of the ultimate claims in the last $N-K$ run-off triangles for a single accident year $i \in \{1,\ldots,I\}$*

$$\widehat{\operatorname{msep}}_{\sum_{n=K+1}^{N} C_{i,J}^{(n)} | \mathcal{D}_I^N}\left(\sum_{n=K+1}^{N}\widehat{C_{i,J}^{(n)}}^{\mathrm{AD}}\right)$$
$$= \mathbf{1}_{N-K}'\cdot\mathrm{V}_i^{1/2}\cdot\sum_{j=I-i+1}^{J}\hat{\Sigma}_{j-1}^{(A)}\cdot\mathrm{V}_i^{1/2}\cdot\mathbf{1}_{N-K}$$
$$+ \mathbf{1}_{N-K}'\cdot\mathrm{V}_i\cdot\sum_{j=I-i+1}^{J}\left(\sum_{l=0}^{I-j}\mathrm{V}_l^{1/2}\cdot(\hat{\Sigma}_{j-1}^{(A)})^{-1}\cdot\mathrm{V}_l^{1/2}\right)^{-1}\cdot\mathrm{V}_i\cdot\mathbf{1}_{N-K}, \tag{36}$$

where the parameter estimates $\hat{\Sigma}_{j-1}^{(A)}$ *are given in Section* 5.

REMARK 4.3

- The first terms on the right-hand side of (32) and (36) are the estimators of the conditional process variances and the second terms are the estimators of the conditional estimation errors, respectively.
- For $K = 1$ Estimator 4.1 reduces to the estimator of the conditional MSEP for a single run-off triangle in the univariate CL time series model of Buchwalder et al. (2006).
- For $N - K = 1$ Estimator 4.2 reduces to the estimator of the conditional MSEP for a single run-off triangle in the univariate ALR model (see Mack (2002)).

In addition to Estimators 4.1 and 4.2 we have to estimate the cross product terms between the CL estimators and the ALR method estimators, namely (see (31))

$$E\left[\left(\sum_{n=1}^{K}\widehat{C_{i,J}^{(n)}}^{\mathrm{CL}} - \sum_{n=1}^{K} C_{i,J}^{(n)}\right)\cdot\left(\sum_{n=K+1}^{N}\widehat{C_{i,J}^{(n)}}^{\mathrm{AD}} - \sum_{n=K+1}^{N} C_{i,J}^{(n)}\right)\middle|\, \mathcal{D}_I^N\right]$$
$$= \mathbf{1}_K'\cdot\mathrm{Cov}(\mathbf{C}_{i,J}^{\mathrm{CL}},\mathbf{C}_{i,J}^{\mathrm{AD}}\mid\mathcal{D}_I^N)\cdot\mathbf{1}_{N-K} + \mathbf{1}_K'\cdot(\widehat{\mathbf{C}_{i,J}}^{\mathrm{CL}} - E[\mathbf{C}_{i,J}^{\mathrm{CL}}\mid\mathcal{D}_I^N])\cdot(\widehat{\mathbf{C}_{i,J}}^{\mathrm{AD}} - E[\mathbf{C}_{i,J}^{\mathrm{AD}}\mid\mathcal{D}_I^N])'\cdot\mathbf{1}_{N-K}. \tag{37}$$

That is, this cross product term, again, decouples into a process error part and an estimation error part (first and second term on the right-hand side of (37)).

4.1.1. Conditional cross process variance

In this subsection we provide an estimate of the conditional cross process variance. The following result holds:

LEMMA 4.4 (Cross process variance for single accident years) *Under Model Assumptions* 3.1 *the conditional cross process variance for the ultimate claims* $\mathbf{C}_{i,J}$ *of accident year* $i\in\{1,\ldots,I\}$*, given the observations* $\mathcal{D}_I^N$*, is given by*

$$\mathbf{1}_K'\cdot\mathrm{Cov}(\mathbf{C}_{i,J}^{\mathrm{CL}},\mathbf{C}_{i,J}^{\mathrm{AD}}\mid\mathcal{D}_I^N)\cdot\mathbf{1}_{N-K} = \mathbf{1}_K'\cdot\sum_{j=I-i+1}^{J}\prod_{l=j}^{J-1}\mathrm{D}(\mathbf{f}_l)\cdot\Sigma_{i,j-1}^{\mathbf{CA}}\cdot\mathbf{1}_{N-K}, \tag{38}$$

where

$$\Sigma_{i,j-1}^{\mathbf{CA}} = E[\mathrm{D}(\mathbf{C}_{i,j-1}^{\mathrm{CL}})^{1/2}\cdot\Sigma_{j-1}^{(C,A)}\mid\mathbf{C}_{i,I-i}]\cdot\mathrm{V}_i^{1/2}. \tag{39}$$

PROOF See appendix, Section 7.1. □

If we replace the parameters $\mathbf{f}_l$ and $\Sigma_{i,j-1}^{\mathbf{CA}}$ in (38) by their estimates (cf. Section 5), we obtain an estimator of the conditional cross process variance for a single accident year.

4.1.2. Conditional cross estimation error

In this subsection we deal with the second term on the right-hand side of (37). Using Lemma 3.3 as well as definitions (26) and (27), we obtain for the cross estimation error of accident year $i\in\{1,\ldots,I\}$ the representation

$$\mathbf{1}_K'\cdot(\widehat{\mathbf{C}_{i,J}}^{\mathrm{CL}} - E[\mathbf{C}_{i,J}^{\mathrm{CL}}\mid\mathcal{D}_I^N])\cdot(\widehat{\mathbf{C}_{i,J}}^{\mathrm{AD}} - E[\mathbf{C}_{i,J}^{\mathrm{AD}}\mid\mathcal{D}_I^N])'\cdot\mathbf{1}_{N-K}$$
$$= \mathbf{1}_K'\cdot\left(\prod_{j=I-i}^{J-1}\mathrm{D}(\hat{\mathbf{f}}_j) - \prod_{j=I-i}^{J-1}\mathrm{D}(\mathbf{f}_j)\right)\cdot\mathbf{C}_{i,I-i}^{\mathrm{CL}}\cdot\left(\sum_{j=I-i+1}^{J}(\widehat{\mathbf{X}_{i,j}}^{\mathrm{AD}} - E[\mathbf{X}_{i,j}^{\mathrm{AD}}])\right)'\cdot\mathbf{1}_{N-K}$$
$$= \mathbf{1}_K'\cdot\mathrm{D}(\mathbf{C}_{i,I-i}^{\mathrm{CL}})\cdot(\hat{\mathbf{g}}_{i|J} - \mathbf{g}_{i|J})\cdot\left(\sum_{j=I-i+1}^{J}(\hat{\mathbf{m}}_j - \mathbf{m}_j)\right)'\cdot\mathrm{V}_i\cdot\mathbf{1}_{N-K}, \tag{40}$$

where $\hat{\mathbf{g}}_{i|J}$ and $\mathbf{g}_{i|J}$ are defined by

$$\hat{\mathbf{g}}_{i|J} = \mathrm{D}(\hat{\mathbf{f}}_{I-i})\cdot\ldots\cdot\mathrm{D}(\hat{\mathbf{f}}_{J-1})\cdot\mathbf{1}_K,$$
$$\mathbf{g}_{i|J} = \mathrm{D}(\mathbf{f}_{I-i})\cdot\ldots\cdot\mathrm{D}(\mathbf{f}_{J-1})\cdot\mathbf{1}_K. \tag{41}$$

In order to derive an estimator of the conditional cross estimation error we would like to calculate the right-hand side of (40). Observe that the realizations of the estimators $\hat{\mathbf{f}}_{I-i},\ldots,\hat{\mathbf{f}}_{J-1}$ and $\hat{\mathbf{m}}_{I-i+1},\ldots,\hat{\mathbf{m}}_J$ are known at time I, but the "true" CL factors $\mathbf{f}_{I-i},\ldots,\mathbf{f}_{J-1}$ and the incremental loss ratios $\mathbf{m}_{I-i+1},\ldots,\mathbf{m}_J$ are unknown. Hence (40) cannot be calculated explicitly. In order to determine the conditional cross estimation error we analyze how much the "possible" CL factor estimators and the incremental loss ratio estimators fluctuate around their "true" mean values $\mathbf{f}_j$ and $\mathbf{m}_j$. In the following, analogously to Merz and Wüthrich (2008), we measure these volatilities of the estimators $\hat{\mathbf{f}}_j$ and $\hat{\mathbf{m}}_j$ by means of resampled observations for $\hat{\mathbf{f}}_j$ and $\hat{\mathbf{m}}_j$. For this purpose we use the conditional resampling approach presented in Buchwalder et al. (2006), Section 4.1.2, to get an estimate for the term (40). By conditionally resampling the observations for $\hat{\mathbf{f}}_{I-i},\ldots,\hat{\mathbf{f}}_{J-1}$ and $\hat{\mathbf{m}}_{I-i+1},\ldots,\hat{\mathbf{m}}_J$, given the upper triangles $\mathcal{D}_I^N$, we take into account the possibility that the observations for $\hat{\mathbf{f}}_j$ and $\hat{\mathbf{m}}_j$ could have been different from the observed values. This means that, given $\mathcal{D}_I^N$, we generate "new" observations $\tilde{\mathbf{C}}_{i,j+1}^{\mathrm{CL}}$ and $\tilde{\mathbf{X}}_{i,j+1}^{\mathrm{AD}}$ for $i\in\{0,\ldots,I\}$ and $j\in\{0,\ldots,J-1\}$ using the formulas (conditional resampling)

$$\tilde{\mathbf{C}}_{i,j+1}^{\mathrm{CL}} = \mathrm{D}(\mathbf{f}_j)\cdot\mathbf{C}_{i,j}^{\mathrm{CL}} + \mathrm{D}(\mathbf{C}_{i,j}^{\mathrm{CL}})^{1/2}\cdot\mathrm{D}(\tilde{\varepsilon}_{i,j+1}^{\mathrm{CL}})\cdot\boldsymbol{\sigma}_j^{\mathrm{CL}} \tag{42}$$

and

$$\tilde{\mathbf{X}}_{i,j+1}^{\mathrm{AD}} = \mathrm{V}_i\cdot\mathbf{m}_{j+1} + \mathrm{V}_i^{1/2}\cdot\mathrm{D}(\tilde{\varepsilon}_{i,j+1}^{\mathrm{AD}})\cdot\boldsymbol{\sigma}_j^{\mathrm{AD}}, \tag{43}$$

with

$$\tilde{\varepsilon}_{i,j+1} = \begin{pmatrix}\tilde{\varepsilon}_{i,j+1}^{\mathrm{CL}}\\ \tilde{\varepsilon}_{i,j+1}^{\mathrm{AD}}\end{pmatrix}, \qquad \varepsilon_{i,j+1} = \begin{pmatrix}\varepsilon_{i,j+1}^{\mathrm{CL}}\\ \varepsilon_{i,j+1}^{\mathrm{AD}}\end{pmatrix} \tag{44}$$

are independent and identically distributed copies.

We define

$$\mathrm{W}_j = \left(\sum_{k=0}^{I-j-1}\mathrm{D}(\mathbf{C}_{k,j}^{\mathrm{CL}})^{1/2}(\Sigma_j^{(C)})^{-1}\mathrm{D}(\mathbf{C}_{k,j}^{\mathrm{CL}})^{1/2}\right)^{-1}$$

and

$$\mathrm{U}_j = \left(\sum_{k=0}^{I-j-1}\mathrm{V}_k^{1/2}(\Sigma_j^{(A)})^{-1}\mathrm{V}_k^{1/2}\right)^{-1}.$$

The resampled representations for the estimates of the multivariate CL factors and the incremental loss ratios are then given by (see (21) and (22))

$$\hat{\mathbf{f}}_j = \mathbf{f}_j + \mathrm{W}_j\sum_{i=0}^{I-j-1}\mathrm{D}(\mathbf{C}_{i,j}^{\mathrm{CL}})^{1/2}(\Sigma_j^{(C)})^{-1}\mathrm{D}(\tilde{\varepsilon}_{i,j+1}^{\mathrm{CL}})\boldsymbol{\sigma}_j^{\mathrm{CL}}, \tag{45}$$

and

$$\hat{\mathbf{m}}_{j+1} = \mathbf{m}_{j+1} + \mathrm{U}_j\sum_{i=0}^{I-j-1}\mathrm{V}_i^{1/2}(\Sigma_j^{(A)})^{-1}\mathrm{D}(\tilde{\varepsilon}_{i,j+1}^{\mathrm{AD}})\boldsymbol{\sigma}_j^{\mathrm{AD}}. \tag{46}$$

Note, in (45) and (46) as well as in the following exposition, we use the previous notations $\hat{\mathbf{f}}_j$ and $\hat{\mathbf{m}}_{j+1}$ for the resampled estimates of the multivariate CL factors $\mathbf{f}_j$ and the incremental loss ratios $\mathbf{m}_{j+1}$, respectively, to avoid an overloaded notation. Furthermore, given the observations $\mathcal{D}_I^N$, we denote the conditional probability measure of these resampled multivariate estimates by $P^*_{\mathcal{D}_I^N}$. For a more detailed discussion of this conditional resampling approach we refer to Merz and Wüthrich (2008). We obtain the following lemma:

LEMMA 4.5 *Under Model Assumptions* 3.1 *and resampling assumptions* (42)–(44) *we have:*

a) $\hat{\mathbf{f}}_0,\ldots,\hat{\mathbf{f}}_{J-1}$ *are independent under* $P^*_{\mathcal{D}_I^N}$, $\hat{\mathbf{m}}_1,\ldots,\hat{\mathbf{m}}_J$ *are independent under* $P^*_{\mathcal{D}_I^N}$, *and* $\hat{\mathbf{f}}_j$ *and* $\hat{\mathbf{m}}_k$ *are independent under* $P^*_{\mathcal{D}_I^N}$ *if* $k\neq j+1$,

b) $E^*_{\mathcal{D}_I^N}[\hat{\mathbf{f}}_j]=\mathbf{f}_j$ *and* $E^*_{\mathcal{D}_I^N}[\hat{\mathbf{m}}_{j+1}]=\mathbf{m}_{j+1}$ *for* $0\le j\le J-1$ *and*

c) $E^*_{\mathcal{D}_I^N}[\hat{f}_j^{(m)}\hat{m}_{j+1}^{(n)}] = f_j^{(m)}m_{j+1}^{(n)} + \mathrm{T}_j(m,n)$,

where $\mathrm{T}_j(m,n)$ *is the entry* (m,n) *of the* $K\times(N-K)$*-matrix*

$$\mathrm{T}_j = \mathrm{W}_j\sum_{i=0}^{I-j-1}\mathrm{D}(\mathbf{C}_{i,j}^{\mathrm{CL}})^{1/2}(\Sigma_j^{(C)})^{-1}\Sigma_j^{(C,A)}\cdot(\Sigma_j^{(A)})^{-1}\mathrm{V}_i^{1/2}\mathrm{U}_j. \tag{47}$$

PROOF See appendix, Section 7.2. □

Using Lemma 4.5 we choose for the conditional cross estimation error (40) the estimator

$$\mathbf{1}'_K \cdot \mathrm{D}(\mathbf{C}^{\mathrm{CL}}_{i,I-i}) \cdot E^*_{\mathcal{D}^N_I}\left[(\hat{\mathbf{g}}_{i|J} - \mathbf{g}_{i|J}) \cdot \left(\sum_{j=I-i+1}^{J} (\hat{\mathbf{m}}_j - \mathbf{m}_j)\right)'\right] \cdot \mathrm{V}_i \cdot \mathbf{1}_{N-K}$$

$$= \mathbf{1}'_K \cdot \mathrm{D}(\mathbf{C}^{\mathrm{CL}}_{i,I-i}) \cdot \mathrm{Cov}^*_{\mathcal{D}^N_I}\left(\hat{\mathbf{g}}_{i|J}, \sum_{j=I-i+1}^{J} \hat{\mathbf{m}}_j\right) \cdot \mathrm{V}_i \cdot \mathbf{1}_{N-K}. \quad (48)$$

We define the matrix

$$\Psi_{k,i} = (\Psi^{(m,n)}_{k,i})_{m,n} = \mathrm{Cov}^*_{\mathcal{D}^N_I}\left(\hat{\mathbf{g}}_{k|J}, \sum_{j=I-i+1}^{J} \hat{\mathbf{m}}_j\right) = \sum_{j=I-i+1}^{J} \mathrm{Cov}^*_{\mathcal{D}^N_I}(\hat{\mathbf{g}}_{k|J}, \hat{\mathbf{m}}_j) \quad (49)$$

for all $k, i \in \{1, \ldots, I\}$. The following result holds for its components $\Psi^{(m,n)}_{k,i}$:

LEMMA 4.6 *Under Model Assumptions* 3.1 *and resampling assumptions* (42)–(44) *we have for* $m = 1, \ldots, K$ *and* $n = 1, \ldots, N-K$

$$\Psi^{(m,n)}_{k,i} = \sum_{j=(I-i+1)\vee(I-k+1)}^{J} \prod_{r=I-k}^{J-1} f^{(m)}_r \frac{1}{f^{(m)}_{j-1}} \mathrm{T}_{j-1}(m,n).$$

PROOF See appendix, Section 7.3. □

Putting (31), (37), (38) and (48) together and replacing the parameters by their estimates we motivate the following estimator for the conditional MSEP of a single accident year in the multivariate combined method:

ESTIMATOR 4.7 (MSEP for single accident years, combined method) *Under Model Assumptions* 3.1 *we have the estimator for the conditional MSEP of the ultimate claims for a single accident year* $i \in \{1, \ldots, I\}$

$$\widehat{\mathrm{msep}}_{\sum_{n=1}^N C^{(n)}_{i,J}|\mathcal{D}^N_I}\left(\sum_{n=1}^{K} \widehat{C^{(n)}_{i,J}}^{\mathrm{CL}} + \sum_{n=K+1}^{N} \widehat{C^{(n)}_{i,J}}^{\mathrm{AD}}\right)$$

$$= \widehat{\mathrm{msep}}_{\sum_{n=1}^K C^{(n)}_{i,J}|\mathcal{D}^N_I}\left(\sum_{n=1}^{K} \widehat{C^{(n)}_{i,J}}^{\mathrm{CL}}\right) + \widehat{\mathrm{msep}}_{\sum_{n=K+1}^N C^{(n)}_{i,J}|\mathcal{D}^N_I}\left(\sum_{n=K+1}^{N} \widehat{C^{(n)}_{i,J}}^{\mathrm{AD}}\right)$$

$$+ 2 \cdot \mathbf{1}'_K \cdot \sum_{j=I-i+1}^{J} \prod_{l=j}^{J-1} \mathrm{D}(\hat{\mathbf{f}}_l) \cdot \hat{\Sigma}^{\mathbf{CA}}_{i,j-1} \cdot \mathbf{1}_{N-K}$$

$$+ 2 \cdot \mathbf{1}'_K \cdot \mathrm{D}(\mathbf{C}^{\mathrm{CL}}_{i,I-i}) \cdot (\hat{\Psi}^{(m,n)}_{i,i})_{m,n} \cdot \mathrm{V}_i \cdot \mathbf{1}_{N-K}, \quad (50)$$

with

$$\hat{\Sigma}^{\mathbf{CA}}_{i,j-1} = \mathrm{D}(\widehat{\mathbf{C}_{i,j-1}}^{\mathrm{CL}})^{1/2} \cdot \hat{\Sigma}^{(C,A)}_{j-1} \cdot \mathrm{V}^{1/2}_i, \quad (51)$$

$$\hat{\Psi}^{(m,n)}_{k,i} = \hat{g}^{(m)}_{k|J} \sum_{j=(I-i+1)\vee(I-k+1)}^{J} \frac{1}{\hat{f}^{(m)}_{j-1}} \hat{\mathrm{T}}_{j-1}(m,n). \quad (52)$$

Thereby, the first two terms on the right-hand side of (50) *are given by* (32) *and* (36), $\hat{g}^{(m)}_{k|J}$ *denotes the mth coordinate of* $\hat{\mathbf{g}}_{k|J}$ *(cf.* (41)*) and the parameter estimates* $\hat{\Sigma}^{(C,A)}_{j-1}$ *as well as* $\hat{\mathrm{T}}_{j-1}(m,n)$ *(entry* (m,n) *of the estimate* $\hat{\mathrm{T}}_{j-1}$ *for the* $K \times (N-K)$*-matrix* T_{j-1}*) are given in Section* 5.

4.2. Conditional MSEP for aggregated accident years

Now, we derive an estimator of the conditional MSEP (30) for aggregated accident years. To this end we consider two different accident years $1 \le i < l \le I$. We know that the ultimate claims $\mathbf{C}_{i,J}$ and $\mathbf{C}_{l,J}$ are independent but we also know that we have to take into account the dependence of the estimators $\widehat{\mathbf{C}_{i,J}}$ and $\widehat{\mathbf{C}_{l,J}}$. The conditional MSEP for two aggregated accident years i and l

is given by

$$\operatorname{msep}_{\sum_{n=1}^N (C_{i,J}^{(n)}+C_{l,J}^{(n)})|\mathcal{D}_I^N}\left(\sum_{n=1}^K \widehat{C_{i,J}^{(n)}}^{\mathrm{CL}} + \sum_{n=K+1}^N \widehat{C_{i,J}^{(n)}}^{\mathrm{AD}} + \sum_{n=1}^K \widehat{C_{l,J}^{(n)}}^{\mathrm{CL}} + \sum_{n=K+1}^N \widehat{C_{l,J}^{(n)}}^{\mathrm{AD}}\right)$$
$$= \operatorname{msep}_{\sum_{n=1}^N C_{i,J}^{(n)}|\mathcal{D}_I^N}\left(\sum_{n=1}^K \widehat{C_{i,J}^{(n)}}^{\mathrm{CL}} + \sum_{n=K+1}^N \widehat{C_{i,J}^{(n)}}^{\mathrm{AD}}\right) + \operatorname{msep}_{\sum_{n=1}^N C_{l,J}^{(n)}|\mathcal{D}_I^N}\left(\sum_{n=1}^K \widehat{C_{l,J}^{(n)}}^{\mathrm{CL}} + \sum_{n=K+1}^N \widehat{C_{l,J}^{(n)}}^{\mathrm{AD}}\right)$$
$$+ 2\cdot E\left[\left(\sum_{n=1}^K \widehat{C_{i,J}^{(n)}}^{\mathrm{CL}} + \sum_{n=K+1}^N \widehat{C_{i,J}^{(n)}}^{\mathrm{AD}} - \sum_{n=1}^N C_{i,J}^{(n)}\right)\cdot\left(\sum_{n=1}^K \widehat{C_{l,J}^{(n)}}^{\mathrm{CL}} + \sum_{n=K+1}^N \widehat{C_{l,J}^{(n)}}^{\mathrm{AD}} - \sum_{n=1}^N C_{l,J}^{(n)}\right)\middle|\, \mathcal{D}_I^N\right]. \quad (53)$$

The first two terms on the right-hand side of (53) are the conditional prediction errors for the two single accident years $1 \le i < l \le I$, respectively, which we estimate by Estimator 4.7. For the third term on the right-hand side of (53) we obtain the decomposition

$$E\left[\left(\sum_{n=1}^K \widehat{C_{i,J}^{(n)}}^{\mathrm{CL}} + \sum_{n=K+1}^N \widehat{C_{i,J}^{(n)}}^{\mathrm{AD}} - \sum_{n=1}^N C_{i,J}^{(n)}\right)\right.$$
$$\left.\cdot\left(\sum_{n=1}^K \widehat{C_{l,J}^{(n)}}^{\mathrm{CL}} + \sum_{n=K+1}^N \widehat{C_{l,J}^{(n)}}^{\mathrm{AD}} - \sum_{n=1}^N C_{l,J}^{(n)}\right)\middle|\, \mathcal{D}_I^N\right]$$
$$= E\left[\left(\sum_{n=1}^K \widehat{C_{i,J}^{(n)}}^{\mathrm{CL}} - \sum_{n=1}^K C_{i,J}^{(n)}\right)\cdot\left(\sum_{n=K+1}^N \widehat{C_{l,J}^{(n)}}^{\mathrm{AD}} - \sum_{n=K+1}^N C_{l,J}^{(n)}\right)\middle|\, \mathcal{D}_I^N\right]$$
$$+ E\left[\left(\sum_{n=K+1}^N \widehat{C_{i,J}^{(n)}}^{\mathrm{AD}} - \sum_{n=K+1}^N C_{i,J}^{(n)}\right)\cdot\left(\sum_{n=1}^K \widehat{C_{l,J}^{(n)}}^{\mathrm{CL}} - \sum_{n=1}^K C_{l,J}^{(n)}\right)\middle|\, \mathcal{D}_I^N\right]$$
$$+ E\left[\left(\sum_{n=1}^K \widehat{C_{i,J}^{(n)}}^{\mathrm{CL}} - \sum_{n=1}^K C_{i,J}^{(n)}\right)\cdot\left(\sum_{n=1}^K \widehat{C_{l,J}^{(n)}}^{\mathrm{CL}} - \sum_{n=1}^K C_{l,J}^{(n)}\right)\middle|\, \mathcal{D}_I^N\right]$$
$$+ E\left[\left(\sum_{n=K+1}^N \widehat{C_{i,J}^{(n)}}^{\mathrm{AD}} - \sum_{n=K+1}^N C_{i,J}^{(n)}\right)\cdot\left(\sum_{n=K+1}^N \widehat{C_{l,J}^{(n)}}^{\mathrm{AD}} - \sum_{n=K+1}^N C_{l,J}^{(n)}\right)\middle|\, \mathcal{D}_I^N\right]. \quad (54)$$

Using the independence of different accident years we obtain for the first two terms on the right-hand side of (54)

$$E\left[\left(\sum_{n=1}^K \widehat{C_{i,J}^{(n)}}^{\mathrm{CL}} - \sum_{n=1}^K C_{i,J}^{(n)}\right)\cdot\left(\sum_{n=K+1}^N \widehat{C_{l,J}^{(n)}}^{\mathrm{AD}} - \sum_{n=K+1}^N C_{l,J}^{(n)}\right)\middle|\, \mathcal{D}_I^N\right]$$
$$= \mathbf{1}_K' \cdot (\widehat{\mathbf{C}_{i,J}}^{\mathrm{CL}} - E[\mathbf{C}_{i,J}^{\mathrm{CL}} \mid \mathcal{D}_I^N]) \cdot (\widehat{\mathbf{C}_{l,J}}^{\mathrm{AD}} - E[\mathbf{C}_{l,J}^{\mathrm{AD}} \mid \mathcal{D}_I^N])' \cdot \mathbf{1}_{N-K}$$
$$= \mathbf{1}_K' \cdot \left(\prod_{j=I-i}^{J-1} \mathrm{D}(\hat{\mathbf{f}}_j) - \prod_{j=I-i}^{J-1} \mathrm{D}(\mathbf{f}_j)\right) \cdot \mathbf{C}_{i,I-i}^{\mathrm{CL}} \cdot \left(\sum_{j=I-l+1}^{J} (\widehat{\mathbf{X}_{l,j}}^{\mathrm{AD}} - E[\mathbf{X}_{l,j}^{\mathrm{AD}}])\right)' \cdot \mathbf{1}_{N-K}$$
$$= \mathbf{1}_K' \cdot \mathrm{D}(\mathbf{C}_{i,I-i}^{\mathrm{CL}}) \cdot (\hat{\mathbf{g}}_{i|J} - \mathbf{g}_{i|J}) \cdot \left(\sum_{j=I-l+1}^{J} (\hat{\mathbf{m}}_j - \mathbf{m}_j)\right)' \cdot \mathrm{V}_l \cdot \mathbf{1}_{N-K}, \quad (55)$$

and analogously

$$E\left[\left(\sum_{n=K+1}^N \widehat{C_{i,J}^{(n)}}^{\mathrm{AD}} - \sum_{n=K+1}^N C_{i,J}^{(n)}\right)\cdot\left(\sum_{n=1}^K \widehat{C_{l,J}^{(n)}}^{\mathrm{CL}} - \sum_{n=1}^K C_{l,J}^{(n)}\right)\middle|\, \mathcal{D}_I^N\right]$$
$$= \mathbf{1}_K' \cdot \mathrm{D}(\mathbf{C}_{l,I-l}^{\mathrm{CL}}) \cdot (\hat{\mathbf{g}}_{l|J} - \mathbf{g}_{l|J}) \cdot \left(\sum_{j=I-i+1}^{J} (\hat{\mathbf{m}}_j - \mathbf{m}_j)\right)' \cdot \mathrm{V}_i \cdot \mathbf{1}_{N-K}. \quad (56)$$

Under the conditional resampling measure $P^*_{\mathcal{D}_I^N}$ these two terms are estimated by (see also Lemma 4.6), $s = i,l$ and $t = l,i$,

$$\mathbf{1}'_K \cdot \mathrm{D}(\mathbf{C}^{\mathrm{CL}}_{s,I-s}) \cdot E^*_{\mathcal{D}_I^N}\left[(\hat{\mathbf{g}}_{s|J} - \mathbf{g}_{s|J}) \cdot \left(\sum_{j=I-t+1}^{J} (\hat{\mathbf{m}}_j - \mathbf{m}_j)\right)'\right]$$
$$\cdot \mathrm{V}_t \cdot \mathbf{1}_{N-K}$$
$$= \mathbf{1}'_K \cdot \mathrm{D}(\mathbf{C}^{\mathrm{CL}}_{s,I-s}) \cdot (\Psi^{(m,n)}_{s,t})_{m,n} \cdot \mathrm{V}_t \cdot \mathbf{1}_{N-K}.$$

Now we consider the third term on the right-hand side of (54). Again, using the independence of different accident years we obtain

$$E\left[\left(\sum_{n=1}^{K}\widehat{C^{(n)}_{i,J}}^{\mathrm{CL}} - \sum_{n=1}^{K} C^{(n)}_{i,J}\right)\right.$$
$$\left.\cdot\left(\sum_{n=1}^{K}\widehat{C^{(n)}_{l,J}}^{\mathrm{CL}} - \sum_{n=1}^{K} C^{(n)}_{l,J}\right) \,\middle|\, \mathcal{D}_I^N\right]$$
$$= \mathbf{1}'_K \cdot (\widehat{\mathbf{C}_{i,J}}^{\mathrm{CL}} - E[\mathbf{C}^{\mathrm{CL}}_{i,J} \mid \mathcal{D}_I^N])$$
$$\cdot (\widehat{\mathbf{C}_{l,J}}^{\mathrm{CL}} - E[\mathbf{C}^{\mathrm{CL}}_{l,J} \mid \mathcal{D}_I^N])' \cdot \mathbf{1}_K$$
$$= \mathbf{1}'_K \cdot \mathrm{D}(\mathbf{C}^{\mathrm{CL}}_{i,I-i}) \cdot (\hat{\mathbf{g}}_{i|J} - \mathbf{g}_{i|J})$$
$$\cdot (\hat{\mathbf{g}}_{l|J} - \mathbf{g}_{l|J})' \cdot \mathrm{D}(\mathbf{C}^{\mathrm{CL}}_{l,I-l}) \cdot \mathbf{1}_K. \qquad (57)$$

This term is estimated by

$$\mathbf{1}'_K \cdot \mathrm{D}(\mathbf{C}^{\mathrm{CL}}_{i,I-i}) \cdot E^*_{\mathcal{D}_I^N}[(\hat{\mathbf{g}}_{i|J} - \mathbf{g}_{i|J})$$
$$\cdot (\hat{\mathbf{g}}_{l|J} - \mathbf{g}_{l|J})'] \cdot \mathrm{D}(\mathbf{C}^{\mathrm{CL}}_{l,I-l}) \cdot \mathbf{1}_K$$
$$= \mathbf{1}'_K \cdot \mathrm{D}(\mathbf{C}^{\mathrm{CL}}_{i,I-i}) \cdot (\Delta^{(n,m)}_{i,J})_{1\le n,m\le K}$$
$$\cdot \mathrm{D}(\mathbf{C}^{\mathrm{CL}}_{l,I-l}) \cdot \prod_{k=I-l}^{I-i-1} \mathrm{D}(\mathbf{f}_k) \cdot \mathbf{1}_K, \qquad (58)$$

where $\Delta^{(n,m)}_{i,J}$ is estimated by

$$\hat{\Delta}^{(n,m)}_{i,J} = \prod_{l=I-i}^{J-1}\left(\hat{f}^{(n)}_l \cdot \hat{f}^{(m)}_l + \sum_{k=0}^{I-l-1} \hat{\mathbf{a}}^k_{n|l} \cdot \hat{\Sigma}^{(C)}_l \cdot (\hat{\mathbf{a}}^k_{m|l})'\right)$$
$$- \prod_{l=I-i}^{J-1} \hat{f}^{(n)}_l \cdot \hat{f}^{(m)}_l. \qquad (59)$$

The parameter estimates $\hat{\mathbf{a}}^k_{n|l}$ and $\hat{\mathbf{a}}^k_{m|l}$ are the nth and mth row of (35) and the parameter estimate $\hat{\Sigma}^{(C)}_l$ is given in Section 5 (see also Merz and Wüthrich (2008)).

Finally, we obtain for the last term on the right-hand side of (54)

$$E\left[\left(\sum_{n=K+1}^{N}\widehat{C^{(n)}_{i,J}}^{\mathrm{AD}} - \sum_{n=K+1}^{N} C^{(n)}_{i,J}\right)\right.$$
$$\left.\cdot\left(\sum_{n=K+1}^{N}\widehat{C^{(n)}_{l,J}}^{\mathrm{AD}} - \sum_{n=K+1}^{N} C^{(n)}_{l,J}\right) \,\middle|\, \mathcal{D}_I^N\right]$$
$$= \mathbf{1}'_{N-K} \cdot (\widehat{\mathbf{C}_{i,J}}^{\mathrm{AD}} - E[\mathbf{C}^{\mathrm{AD}}_{i,J} \mid \mathcal{D}_I^N])$$
$$\cdot (\widehat{\mathbf{C}_{l,J}}^{\mathrm{AD}} - E[\mathbf{C}^{\mathrm{AD}}_{l,J} \mid \mathcal{D}_I^N])' \cdot \mathbf{1}_{N-K}, \qquad (60)$$

which is estimated by (see also Merz and Wüthrich (2009))

$$\mathbf{1}'_{N-K} \cdot E[(\widehat{\mathbf{C}_{i,J}}^{\mathrm{AD}} - E[\mathbf{C}^{\mathrm{AD}}_{i,J} \mid \mathcal{D}_I^N])$$
$$\cdot (\widehat{\mathbf{C}_{l,J}}^{\mathrm{AD}} - E[\mathbf{C}^{\mathrm{AD}}_{l,J} \mid \mathcal{D}_I^N])'] \cdot \mathbf{1}_{N-K}$$
$$= \mathbf{1}'_{N-K} \cdot \mathrm{V}_i$$
$$\cdot \left[\sum_{j=I-i+1}^{J}\left(\sum_{k=0}^{I-j} \mathrm{V}_k^{1/2} \cdot (\Sigma^{(A)}_{j-1})^{-1} \cdot \mathrm{V}_k^{1/2}\right)^{-1}\right]$$
$$\cdot \mathrm{V}_l \cdot \mathbf{1}_{N-K}. \qquad (61)$$

Putting all the terms together and replacing the parameters by their estimates we obtain the following estimator for the conditional MSEP of aggregated accident years in the multivariate combined method:

Estimator 4.8 (MSEP for aggregated accident years, combined method) *Under Model Assumptions* 3.1 *we have the estimator for the conditional MSEP of the ultimate claims for aggregated accident years*

$$\widehat{\mathrm{msep}}_{\sum_i\sum_n C^{(n)}_{i,J}|\mathcal{D}_I^N}\left(\sum_{i=1}^{I}\sum_{n=1}^{K}\widehat{C^{(n)}_{i,J}}^{\mathrm{CL}} + \sum_{i=1}^{I}\sum_{n=K+1}^{N}\widehat{C^{(n)}_{i,J}}^{\mathrm{AD}}\right)$$
$$= \sum_{i=1}^{I}\widehat{\mathrm{msep}}_{\sum_n C^{(n)}_{i,J}|\mathcal{D}_I^N}\left(\sum_{n=1}^{K}\widehat{C^{(n)}_{i,J}}^{\mathrm{CL}} + \sum_{n=K+1}^{N}\widehat{C^{(n)}_{i,J}}^{\mathrm{AD}}\right)$$
$$+ 2\cdot\sum_{1\le i<l\le I} \mathbf{1}'_K \cdot \mathrm{D}(\mathbf{C}^{\mathrm{CL}}_{i,I-i}) \cdot (\hat{\Psi}^{(m,n)}_{i,l})_{m,n} \cdot \mathrm{V}_l \cdot \mathbf{1}_{N-K}$$

$$+2\cdot\sum_{1\le i<l\le I}\mathbf{1}'_K\cdot D(\mathbf{C}^{\mathrm{CL}}_{l,I-l})\cdot(\hat{\Psi}^{(m,n)}_{l,i})_{m,n}\cdot V_i\cdot\mathbf{1}_{N-K}$$

$$+2\cdot\sum_{1\le i<l\le I}\mathbf{1}'_K\cdot D(\mathbf{C}^{\mathrm{CL}}_{i,I-i})\cdot(\hat{\Delta}^{(m,n)}_{i,J})_{m,n}$$

$$\cdot D(\mathbf{C}^{\mathrm{CL}}_{l,I-l})\cdot\prod_{j=I-l}^{I-i-1}D(\hat{\mathbf{f}}_j)\cdot\mathbf{1}_K$$

$$+2\cdot\sum_{1\le i<l\le I}\mathbf{1}'_{N-K}\cdot V_i$$

$$\cdot\sum_{j=I-i+1}^{J}\left(\sum_{k=0}^{I-j}V_k^{1/2}\cdot(\hat{\Sigma}^{(A)}_{j-1})^{-1}\cdot V_k^{1/2}\right)^{-1}$$

$$\cdot V_l\cdot\mathbf{1}_{N-K}. \tag{62}$$

4.3. Conditional MSEP with $\hat{\mathbf{f}}_j^{(0)}$ and $\hat{\mathbf{m}}_j^{(0)}$

In some cases, it may be more convenient to use estimators (24) and (25) to estimate $\mathbf{f}_j$ and $\mathbf{m}_j$, respectively, instead of (21) and (22). Estimators (24) and (25) do not reflect the correlation among subportfolios and are thus simpler to calculate, but being less than optimal, will have greater MSEP than estimators (21) and (22).

The changes that occur when estimators (24) and (25) are used are noted here. In Estimator 4.1, (35) becomes

$$\hat{\mathbf{A}}_l^k=\left(\sum_{i=0}^{I-l-1}D(\mathbf{C}^{\mathrm{CL}}_{i,l})\right)^{-1}\cdot D(\widehat{\mathbf{C}_{k,l}}^{\mathrm{CL}})^{1/2}.$$

In Estimator 4.2, the last term of (36) becomes

$$\mathbf{1}'_{N-K}\cdot V_i\cdot\left[\sum_{j=I-i+1}^{J}\left(\sum_{l=0}^{I-j}V_l\right)^{-1}\right.$$
$$\left.\cdot\left(\sum_{l=0}^{I-j}V_l^{1/2}\cdot\hat{\Sigma}^{(A)}_{j-1}\cdot V_l^{1/2}\right)\cdot\left(\sum_{l=0}^{I-j}V_l\right)^{-1}\right]$$
$$\cdot V_i\cdot\mathbf{1}_{N-K}.$$

W_j becomes

$$\left(\sum_{k=0}^{I-j-1}D(\mathbf{C}^{\mathrm{CL}}_{k,j})\right)^{-1},$$

U_j becomes

$$\left(\sum_{k=0}^{I-j-1}V_k\right)^{-1},$$

and T_j becomes

$$W_j\sum_{i=0}^{I-j-1}D(\mathbf{C}^{\mathrm{CL}}_{i,j})^{1/2}\Sigma_j^{(C,A)}V_i^{1/2}U_j,$$

with analogous changes to their estimators. The right-hand side of (61) and the expression to the right of the first summation sign in the last term of (62) become

$$\mathbf{1}'_{N-K}\cdot V_i\cdot\left[\sum_{j=I-i+1}^{J}\left(\sum_{k=0}^{I-j}V_k\right)^{-1}\right.$$
$$\left.\cdot\left(\sum_{k=0}^{I-j}V_k^{1/2}\cdot\hat{\Sigma}^{(A)}_{j-1}\cdot V_k^{1/2}\right)\cdot\left(\sum_{k=0}^{I-j}V_k\right)^{-1}\right]$$
$$\cdot V_l\cdot\mathbf{1}_{N-K}.$$

5. Parameter estimation

For the estimation of the claim reserves and the conditional MSEP we need estimates of the K-dimensional parameters $\mathbf{f}_j$, the $(N-K)$-dimensional parameters $\mathbf{m}_j$ as well as the covariance matrices $\Sigma_j^{(C)}$, $\Sigma_j^{(A)}$ and $\Sigma_j^{(C,A)}$. Observe the fact that the multivariate CL factor estimates and incremental loss ratio estimates $\hat{\mathbf{f}}_j$ and $\hat{\mathbf{m}}_j$, respectively, can only be calculated if the covariance matrices $\Sigma_j^{(C)}$ and $\Sigma_j^{(A)}$ are known (cf. (21) and (22)). On the other hand, the covariance matrices $\Sigma_j^{(C)}$, $\Sigma_j^{(A)}$ and $\Sigma_j^{(C,A)}$ are estimated by means of $\hat{\mathbf{f}}_j$ and $\hat{\mathbf{m}}_j$. Therefore, as in the multivariate CL method (cf. Merz and Wüthrich (2008)) and the multivariate ALR method (cf. Merz and Wüthrich (2009)), in the following we propose an iterative estimation of these parameters. In this spirit, the "true" estimation error is slightly larger because it should also involve the uncertainties in the estimates of the variance parameters. However, in order to obtain a feasible MSEP formula we neglect this term of uncertainty.

Estimation of $\mathbf{f}_j$ and $\mathbf{m}_j$. As starting values for the iteration we use the unbiased estimators $\hat{\mathbf{f}}_{j-1}^{(0)}$ and $\hat{\mathbf{m}}_j^{(0)}$ defined by (24) and (25) for $j = 1,\ldots,J$. From $\hat{\mathbf{f}}_{j-1}^{(0)}$ and $\hat{\mathbf{m}}_j^{(0)}$ we derive the estimates $\hat{\Sigma}_{j-1}^{(C)(1)}$ and $\hat{\Sigma}_{j-1}^{(A)(1)}$ of the covariance matrices $\Sigma_{j-1}^{(C)}$ and $\Sigma_{j-1}^{(A)}$ for $j = 1,\ldots,J$ (see estimators (64) and (67) below). Then these estimates $\hat{\Sigma}_{j-1}^{(C)(1)}$ and $\hat{\Sigma}_{j-1}^{(A)(1)}$ are used to determine $\hat{\mathbf{f}}_{j-1}^{(1)}$ and $\hat{\mathbf{m}}_j^{(1)}$ via ($s \geq 1$)

$$\hat{\mathbf{f}}_{j-1}^{(s)} = \left(\sum_{i=0}^{I-j} D(\mathbf{C}_{i,j-1}^{\mathrm{CL}})^{1/2}(\hat{\Sigma}_{j-1}^{(C)(s)})^{-1} D(\mathbf{C}_{i,j-1}^{\mathrm{CL}})^{1/2}\right)^{-1} \cdot \sum_{i=0}^{I-j} D(\mathbf{C}_{i,j-1}^{\mathrm{CL}})^{1/2}(\hat{\Sigma}_{j-1}^{(C)(s)})^{-1} D(\mathbf{C}_{i,j-1}^{\mathrm{CL}})^{-1/2} \cdot \mathbf{C}_{i,j}^{\mathrm{CL}} \tag{63}$$

and

$$\hat{\mathbf{m}}_j^{(s)} = \left(\sum_{i=0}^{I-j} V_i^{1/2} \cdot (\hat{\Sigma}_{j-1}^{(A)(s)})^{-1} \cdot V_i^{1/2}\right)^{-1} \cdot \sum_{i=0}^{I-j} V_i^{1/2} \cdot (\hat{\Sigma}_{j-1}^{(A)(s)})^{-1} \cdot V_i^{-1/2} \cdot \mathbf{X}_{i,j}^{\mathrm{AD}}.$$

This algorithm is then iterated until it has sufficiently converged.

Estimation of $\Sigma_{j-1}^{(C)}$, $\Sigma_{j-1}^{(A)}$ and $\Sigma_{j-1}^{(C,A)}$. The covariance matrices $\Sigma_{j-1}^{(C)}$ and $\Sigma_{j-1}^{(A)}$ are estimated iteratively from the data for $j = 1,\ldots,J$. For the covariance matrices $\Sigma_{j-1}^{(C)}$ we use the estimator proposed by Merz and Wüthrich (2008) ($s \geq 1$)

$$\hat{\Sigma}_{j-1}^{(C)(s)} = Q_j \odot \sum_{i=0}^{I-j} D(\mathbf{C}_{i,j-1}^{\mathrm{CL}})^{1/2} \cdot (\mathbf{F}_{i,j}^{\mathrm{CL}} - \hat{\mathbf{f}}_{j-1}^{(s-1)}) \cdot (\mathbf{F}_{i,j}^{\mathrm{CL}} - \hat{\mathbf{f}}_{j-1}^{(s-1)})' \cdot D(\mathbf{C}_{i,j-1}^{\mathrm{CL}})^{1/2}, \tag{64}$$

where $\odot$ denotes the Hadamard product (entrywise product of two matrices),

$$\mathbf{F}_{i,j}^{\mathrm{CL}} = D(\mathbf{C}_{i,j-1}^{\mathrm{CL}})^{-1} \cdot \mathbf{C}_{i,j}^{\mathrm{CL}} \quad \text{and} \quad Q_j = \left(\frac{1}{I-j-1+w_j^{(n,m)}}\right)_{1\leq n,m\leq K} \tag{65}$$

with

$$w_j^{(n,m)} = \frac{\left(\sum_{l=0}^{I-j}\sqrt{C_{l,j-1}^{(n)}} \cdot \sqrt{C_{l,j-1}^{(m)}}\right)^2}{\sum_{l=0}^{I-j} C_{l,j-1}^{(n)} \cdot \sum_{l=0}^{I-j} C_{l,j-1}^{(m)}}. \tag{66}$$

For more details on this estimator see Merz and Wüthrich (2008), Section 5.

For the covariance matrices $\Sigma_{j-1}^{(A)}$ we use the iterative estimation procedure suggested by Merz and Wüthrich (2009) ($s \geq 1$)

$$\hat{\Sigma}_{j-1}^{(A)(s)} = \frac{1}{I-j} \cdot \sum_{i=0}^{I-j} V_i^{-1/2} \cdot (\mathbf{X}_{i,j}^{\mathrm{AD}} - V_i \cdot \hat{\mathbf{m}}_j^{(s-1)}) \cdot (\mathbf{X}_{i,j}^{\mathrm{AD}} - V_i \cdot \hat{\mathbf{m}}_j^{(s-1)})' \cdot V_i^{-1/2}. \tag{67}$$

For more details on this estimator see Merz and Wüthrich (2009), Section 5.

Motivated by estimators (64) and (67) for matrices $\Sigma_{j-1}^{(C)}$ and $\Sigma_{j-1}^{(A)}$, we propose for the covariance matrix $\Sigma_{j-1}^{(C,A)} = (\Sigma_{j-1}^{(A,C)})'$ estimator

$$\hat{\Sigma}_{j-1}^{(C,A)} = \frac{1}{I-j} \cdot \sum_{i=0}^{I-j} D(\mathbf{C}_{i,j-1}^{\mathrm{CL}})^{1/2} \cdot (\mathbf{F}_{i,j}^{\mathrm{CL}} - \hat{\mathbf{f}}_{j-1}) \cdot (\mathbf{X}_{i,j}^{\mathrm{AD}} - V_i \cdot \hat{\mathbf{m}}_j)' \cdot V_i^{-1/2}. \tag{68}$$

Estimation of $\Sigma_{i,j}^{\mathbf{CA}}$ and T_j. With these estimates we obtain as estimates of the matrices $\Sigma_{i,j}^{\mathbf{CA}}$ and T_j

$$\widehat{\Sigma_{i,j}^{\mathbf{CA}}} = D(\widehat{\mathbf{C}_{i,j}^{\mathrm{CL}}})^{1/2}\hat{\Sigma}_j^{(C,A)} V_i^{1/2},$$

$$\hat{T}_j = \hat{W}_j \sum_{i=0}^{I-j-1} D(\mathbf{C}_{i,j}^{\mathrm{CL}})^{1/2} \cdot (\hat{\Sigma}_j^{(C)})^{-1}\hat{\Sigma}_j^{(C,A)}(\hat{\Sigma}_j^{(A)})^{-1} V_i^{1/2}\hat{U}_j,$$

where

$$\hat{W}_j = \left(\sum_{k=0}^{I-j-1} D(\mathbf{C}_{k,j}^{\mathrm{CL}})^{1/2}(\hat{\Sigma}_j^{(C)})^{-1} D(\mathbf{C}_{k,j}^{\mathrm{CL}})^{1/2}\right)^{-1}$$

and

$$\hat{U}_j = \left(\sum_{k=0}^{I-j-1} V_k^{1/2}(\hat{\Sigma}_j^{(A)})^{-1} V_k^{1/2}\right)^{-1}.$$

The matrices $\hat{\Sigma}_j^{(C)}$ and $\hat{\Sigma}_j^{(A)}$ are the resulting estimates in the iterative estimation procedure for the parameters $\Sigma_j^{(C)}$ and $\Sigma_j^{(A)}$ (cf. (64) and (67)).

REMARK 5.1

- For a more detailed motivation of the estimates for the different covariance matrices see Merz and Wüthrich (2008; 2009) and Sections 8.2.5 and 8.3.5 in Wüthrich and Merz (2008).
- If we have enough data (i.e., $I > J$), we are able to estimate the parameters $\Sigma_{J-1}^{(C)}$, $\Sigma_{J-1}^{(A)}$ and $\Sigma_{J-1}^{(C,A)} = (\Sigma_{J-1}^{(A,C)})'$ by (64), (67) and (68) respectively. Otherwise, if $I = J$, we do not have enough data to estimate the last covariance matrices. In such cases we can use the estimates $\hat{\varphi}_{j-1}^{(m,n)}$ of the elements $\varphi_{j-1}^{(m,n)}$ of $\Sigma_{j-1}^{(C)}$ for $j \le J-1$ (i.e., $\hat{\varphi}_{j-1}^{(m,n)}$ is an estimate of $\varphi_{j-1}^{(m,n)} = \sigma_{j-1}^{(m)} \cdot \sigma_{j-1}^{(n)} \cdot \rho_{j-1}^{(m,n)}$, cf. (16)) to derive estimates $\hat{\varphi}_{J-1}^{(m,n)}$ of the elements $\varphi_{J-1}^{(m,n)}$ of $\Sigma_{J-1}^{(C)}$ for all $1 \le m, n \le K$. For example, this can be done by extrapolating the usually decreasing series
$$|\hat{\varphi}_0^{(m,n)}|, \ldots, |\hat{\varphi}_{J-2}^{(m,n)}| \tag{69}$$
by one additional member $\hat{\varphi}_{J-1}^{(m,n)}$ for $1 \le m, n \le K$. Analogously, we can derive estimates for $\Sigma_{J-1}^{(A)}$, $\Sigma_{J-1}^{(C,A)}$ and $\Sigma_{J-1}^{(A,C)} = (\Sigma_{J-1}^{(C,A)})'$ (see Merz and Wüthrich (2008; 2009) and the example below). However, in all cases it is important to verify that the estimated covariance matrices are positive definite.
- Observe that the $K \times K$-dimensional estimate $\hat{\Sigma}_{j-1}^{(C)(s)}$ is singular if $j \ge I - K + 2$ since in this case the dimension of the linear space generated by any realizations of the $(I - j + 1)$ K-dimensional random vectors
$$D(\mathbf{C}_{i,j-1}^{\mathrm{CL}})^{1/2} \cdot (\mathbf{F}_{i,j}^{\mathrm{CL}} - \hat{\mathbf{f}}_{j-1}^{(s-1)}) \quad \text{with} \quad i \in \{0, \ldots, I-j\} \tag{70}$$
is at most $I - j + 1 \le I - (I - K + 2) + 1 = K - 1$. Analogously, the $(N-K) \times (N-K)$-dimensional estimate $\hat{\Sigma}_{j-1}^{(A)(s)}$ is singular when $j \ge I - (N-K) + 2$. Furthermore, the random matrix $\hat{\Sigma}_{j-1}^{(C)(s)}$ and/or $\hat{\Sigma}_{j-1}^{(A)(s)}$ may be ill-conditioned for some $j < I - K + 2$ and $j < I - (N-K) + 2$, respectively. Therefore, in practical application it is important to verify whether the estimates $\hat{\Sigma}_{j-1}^{(C)(s)}$ and $\hat{\Sigma}_{j-1}^{(A)(s)}$ are well-conditioned or not and to modify those estimates (e.g., by extrapolation as in the example below) which are ill-conditioned (see also Merz and Wüthrich (2008; 2009)).

6. Example

To illustrate the methodology, we consider two correlated run-off portfolios A and B (i.e., $N = 2$) which contain data of general and auto liability business, respectively. The data is given in Tables 1 and 2 in incremental and cumulative form, respectively. This is the data used in Braun (2004) and Merz and Wüthrich (2007; 2008; 2009). The assumption that there is a positive correlation between these two lines of business is justified by the fact that both run-off portfolios contain liability business; that is, certain events (e.g., bodily injury claims) may influence both run-off portfolios, and we are able to learn from the observations from one portfolio about the behavior of the other portfolio.

In contrast to Merz and Wüthrich (2008) (multivariate CL method for both portfolios) and Merz and Wüthrich (2009) (multivariate ALR method for both portfolios) we use different claims reserving methods for the two portfolios A and B. We now assume that we only have estimates V_i of the ultimate claims for portfolio A and use the ALR method for portfolio A. The CL method is applied for portfolio B. This means we have $K = N - K = 1$, and the parameters $\mathbf{f}_j$, $\mathbf{m}_j$, $\Sigma_j^{(C)}$, $\Sigma_j^{(A)}$, $\Sigma_j^{(C,A)}$ as well as the a priori estimates V_i of the ultimate claims in the different accident years i in portfolio A are now scalars. Moreover, it holds that $\Sigma_j^{(C)} = (\boldsymbol{\sigma}_j^{\mathrm{CL}})^2 = (\sigma_j^{(1)})^2$, $\Sigma_j^{(A)} = (\boldsymbol{\sigma}_j^{\mathrm{AD}})^2 = (\sigma_j^{(2)})^2$ and $\Sigma_j^{(C,A)} = \Sigma_j^{(A,C)} = \boldsymbol{\sigma}_j^{\mathrm{CL}} \cdot \boldsymbol{\sigma}_j^{\mathrm{AD}} \cdot \rho_j^{(1,2)} = \sigma_j^{(1)} \cdot \sigma_j^{(2)} \cdot \rho_j^{(1,2)}$.

Table 1. Portfolio A (incremental claims $X_{i,j}^{(2)}$), source Braun (2004)

	General Liability Run-Off Triangle													
AY/DY	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	59,966	103,186	91,360	95,012	83,741	42,513	37,882	6,649	7,669	11,061	−1,738	3,572	6,823	1,893
1	49,685	103,659	119,592	110,413	75,442	44,567	29,257	18,822	4,355	879	4,173	2,727	−776	
2	51,914	118,134	149,156	105,825	78,970	40,770	14,706	17,950	10,917	2,643	10,311	1,414		
3	84,937	188,246	134,135	139,970	74,450	65,401	49,165	21,136	596	24,048	2,548			
4	98,921	179,408	170,201	113,161	79,641	80,364	20,414	10,324	16,204	−265				
5	71,708	173,879	171,295	144,076	93,694	72,161	41,545	25,245	17,497					
6	92,350	193,157	180,707	153,816	121,196	86,753	45,547	23,202						
7	95,731	217,413	240,558	202,276	101,881	104,966	59,416							
8	97,518	245,700	232,223	193,576	165,086	85,200								
9	173,686	285,730	262,920	232,999	186,415									
10	139,821	297,137	372,968	364,270										
11	154,965	373,115	504,604											
12	196,124	576,847												
13	204,325													

Table 2. Portfolio B (cumulative claims $C_{i,j}^{(1)}$), source Braun (2004)

	Auto Liability Run-Off Triangle													
AY/DY	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	114,423	247,961	312,982	344,340	371,479	371,102	380,991	385,468	385,152	392,260	391,225	391,328	391,537	391,428
1	152,296	305,175	376,613	418,299	440,308	465,623	473,584	478,427	478,314	479,907	480,755	485,138	483,974	
2	144,325	307,244	413,609	464,041	519,265	527,216	535,450	536,859	538,920	539,589	539,765	540,742		
3	145,904	307,636	387,094	433,736	463,120	478,931	482,529	488,056	485,572	486,034	485,016			
4	170,333	341,501	434,102	470,329	482,201	500,961	504,141	507,679	508,627	507,752				
5	189,643	361,123	446,857	508,083	526,562	540,118	547,641	549,605	549,693					
6	179,022	396,224	497,304	553,487	581,849	611,640	622,884	635,452						
7	205,908	416,047	520,444	565,721	600,609	630,802	648,365							
8	210,951	426,429	525,047	587,893	640,328	663,152								
9	213,426	509,222	649,433	731,692	790,901									
10	249,508	580,010	722,136	844,159										
11	258,425	686,012	915,109											
12	368,762	909,066												
13	394,997													

Table 3 shows the estimates of the ultimate claims for the two subportfolios A and B as well as the estimates for the whole portfolio consisting of both subportfolios.

Since $I = J = 13$ we do not have enough data to derive estimates of the parameters $\Sigma_{12}^{(C)}$, $\Sigma_{12}^{(A)}$ and $\Sigma_{12}^{(C,A)} = \Sigma_{12}^{(A,C)}$ by means of the proposed estimators. Therefore, we use the extrapolations

$$\hat{\Sigma}_{12}^{(C)} = \min\{\hat{\Sigma}_{10}^{(C)}, (\hat{\Sigma}_{11}^{(C)})^2/\hat{\Sigma}_{10}^{(C)}\},$$

$$\hat{\Sigma}_{12}^{(A)} = \min\{\hat{\Sigma}_{10}^{(A)}, (\hat{\Sigma}_{11}^{(A)})^2/\hat{\Sigma}_{10}^{(A)}\}, \qquad \text{and} \qquad (71)$$

$$\hat{\Sigma}_{12}^{(C,A)} = \hat{\Sigma}_{12}^{(A,C)} = \min\{|\hat{\Sigma}_{10}^{(C,A)}|, (\hat{\Sigma}_{11}^{(C,A)})^2/|\hat{\Sigma}_{10}^{(C,A)}|\}$$

to derive estimates of $\Sigma_{12}^{(C)}$, $\Sigma_{12}^{(A)}$ and $\Sigma_{12}^{(C,A)} = \Sigma_{12}^{(A,C)}$. Moreover, so that $\hat{\Sigma}_{11}$ and $\hat{\Sigma}_{12}$ are positive definite, we estimate $\Sigma_{11}^{(A)}$ and $\Sigma_{11}^{(C,A)} = \Sigma_{11}^{(A,C)}$ by

$$\hat{\Sigma}_{11}^{(A)} = \min\{\hat{\Sigma}_{9}^{(A)}, (\hat{\Sigma}_{10}^{(A)})^2/\hat{\Sigma}_{9}^{(A)}\}, \qquad \text{and} \qquad (72)$$

$$\hat{\Sigma}_{11}^{(C,A)} = \hat{\Sigma}_{11}^{(A,C)} = \min\{|\hat{\Sigma}_{9}^{(C,A)}|, (\hat{\Sigma}_{10}^{(C,A)})^2/|\hat{\Sigma}_{9}^{(C,A)}|\}.$$

Table 4 shows the estimates for the parameters. The one-dimensional estimates $\hat{\mathbf{m}}_j$ and $(\hat{\Sigma}_j^{(A)})^{1/2}$ are the parameter estimates used in the univariate ALR method applied to the individual subportfolio A. Analogously, the one-dimensional estimates $\hat{\mathbf{f}}_j$ and $(\hat{\Sigma}_j^{(C)})^{1/2}$ are the parameter estimates used in the univariate CL method applied

Table 3. Estimates of the ultimate claims for subportfolio A, subportfolio B, and the whole portfolio

i	Subportfolio A V_i	Subportfolio A $\widehat{C_{i,J}}^{CL}$	Subportfolio B $\widehat{C_{i,J}}^{CL}$	Portfolio Total
0	510,301	549,589	391,428	941,017
1	632,897	564,740	483,839	1,048,579
2	658,133	608,104	540,002	1,148,107
3	723,456	795,248	486,227	1,281,475
4	709,312	783,593	508,744	1,292,337
5	845,673	837,088	552,825	1,389,913
6	904,378	938,861	639,113	1,577,973
7	1,156,778	1,098,200	658,410	1,756,610
8	1,214,569	1,154,902	684,719	1,839,620
9	1,397,123	1,431,409	845,543	2,276,952
10	1,832,676	1,735,433	962,734	2,698,167
11	2,156,781	2,065,991	1,169,260	3,235,251
12	2,559,345	2,660,561	1,474,514	4,135,075
13	2,456,991	2,274,941	1,426,060	3,701,001
Total	17,758,413	17,498,658	10,823,418	28,322,077

to the individual subportfolio B. From the estimates $\hat{\Sigma}_j^{(C,A)}$ of the covariances $\Sigma_j^{(C,A)} = \Sigma_j^{(A,C)}$ we obtain estimates $\hat{\rho}_j^{(1,2)}$ of the correlation coefficients $\rho_j^{(1,2)}$ by $\hat{\Sigma}_j^{(C,A)}/\sqrt{\hat{\Sigma}_j^{(A)} \cdot \hat{\Sigma}_j^{(C)}}$.

Note: Since both the CL method and the ALR method are applied to one-dimensional triangles, the parameter estimates $\hat{f}_j$ and $\hat{m}_j$ can be calculated directly (using the univariate methods) and one can omit the iteration described in Section 5.

The first two columns of Table 5 show for each accident year the reserves for subportfolios A and B estimated by the (univariate) ALR method and the (univariate) CL method, respectively. The last column, denoted by "Portfolio Reserves Total," shows the estimated reserves for the entire portfolio.

Table 5. Estimated reserves

i	Subportfolio A Reserves ALR Method	Subportfolio B Reserves CL Method	Portfolio Reserves Total
1	2,348	−135	2,213
2	5,923	−740	5,183
3	9,608	1,211	10,819
4	13,717	992	14,709
5	26,386	3,132	29,518
6	40,906	3,661	44,567
7	80,946	10,045	90,991
8	143,915	21,567	165,482
9	283,823	54,642	338,465
10	594,362	118,575	712,937
11	1,077,515	254,151	1,331,666
12	1,806,833	565,448	2,372,281
13	2,225,221	1,031,063	3,256,284
Total	6,311,503	2,063,612	8,375,115

Table 6 shows for each accident year the estimates for the conditional process standard deviations and the corresponding estimates for the coefficients of variation. The first two columns contain the values for the individual subportfolios A and B calculated by the (univariate) ALR method and the (univariate) CL method, respectively. The last column, denoted by "Portfolio Total," shows the values for the entire portfolio.

The same overview is generated for the square roots of the estimated conditional estimation errors in Table 7.

Table 4. Parameter estimates for the parameters $\mathbf{m}_j$, $\mathbf{f}_j$, $(\Sigma_j^{(A)})^{1/2}$, $(\Sigma_j^{(C)})^{1/2}$ and $\Sigma_j^{(C,A)}$

Portfolio A/B	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$\hat{\mathbf{m}}_j$		0.19969	0.20638	0.17528	0.12117	0.08466	0.04852	0.02474	0.01403	0.01186	0.00606	0.00428	0.00529	0.00371
$\hat{\mathbf{f}}_j$	2.22582	1.26945	1.12036	1.06676	1.03542	1.01677	1.00968	1.00006	1.00374	0.99946	1.00387	0.99891	0.99972	
$(\hat{\Sigma}_j^{(A)})^{1/2}$	31.58	20.03	14.42	18.92	13.64	13.91	5.79	7.15	12.21	6.09	1.84	0.56	0.17	
$(\hat{\Sigma}_j^{(C)})^{1/2}$	105.38	24.64	17.94	19.07	12.50	5.55	4.52	2.13	5.14	1.40	3.21	1.37	0.58	
$\hat{\Sigma}_j^{(C,A)}$	−661.28	349.61	148.48	117.50	46.70	24.65	−2.15	11.39	20.71	5.62	−0.84	0.13	0.02	
$\hat{\rho}_j^{(1,2)}$	−0.19874	0.70835	0.57411	0.32569	0.27382	0.31925	−0.08215	0.74851	0.32998	0.66028	−0.14250	0.16613	0.19367	

Table 6. Estimated conditional process standard deviations

i	Subportfolio A ALR Method		Subportfolio B CL Method		Portfolio Total	
1	133	5.7%	404	−299.8%	449	20.3%
2	471	7.9%	1,091	−147.5%	1,258	24.3%
3	1,640	17.1%	2,461	203.2%	2,815	26.0%
4	5,381	39.2%	2,708	273.1%	6,498	44.2%
5	12,669	48.0%	4,750	151.7%	14,769	50.0%
6	14,763	36.1%	5,384	147.1%	17,415	39.1%
7	17,819	22.0%	6,577	65.5%	20,535	22.6%
8	23,840	16.6%	8,127	37.7%	27,258	16.5%
9	30,227	10.6%	14,609	26.7%	36,849	10.9%
10	43,067	7.2%	24,366	20.5%	55,163	7.7%
11	51,294	4.8%	33,227	13.1%	70,155	5.3%
12	64,413	3.6%	47,888	8.5%	96,211	4.1%
13	80,204	3.6%	117,293	11.4%	144,183	4.4%
Total	131,444	2.1%	134,676	6.5%	202,746	2.4%

Table 7. Square roots of estimated conditional estimation errors

i	Subportfolio A ALR Method		Subportfolio B CL Method		Portfolio Total	
1	149	6.3%	449	−333.3%	500	22.6%
2	375	6.3%	934	−126.3%	1,064	20.5%
3	1,074	11.2%	1,556	128.5%	1,823	16.8%
4	2,916	21.3%	1,708	172.2%	3,607	24.5%
5	6,710	25.4%	2,606	83.2%	7,798	26.4%
6	7,859	19.2%	3,115	85.1%	9,294	20.9%
7	10,490	13.0%	3,570	35.5%	11,902	13.1%
8	12,953	9.0%	4,144	19.2%	14,614	8.8%
9	16,473	5.8%	6,980	12.8%	19,467	5.8%
10	24,583	4.1%	11,022	9.3%	29,528	4.1%
11	30,469	2.8%	15,669	6.2%	38,363	2.9%
12	38,904	2.2%	23,625	4.2%	52,727	2.2%
13	42,287	1.9%	47,683	4.6%	66,271	2.0%
Total	172,174	2.7%	91,599	4.4%	214,339	2.6%

And finally the first three columns in Table 8 give the same overview for the estimated prediction standard errors.

Moreover, the last two columns in Table 8 contain the results for the estimated prediction standard errors assuming no correlation and perfect positive correlation between the corresponding claims reserves of the two subportfolios A and B. These values are calculated by

$$\widehat{\mathrm{msep}}_{\mathbf{C}_{i,J}|\mathcal{D}_I^N} = \widehat{\mathrm{msep}}_{C_{i,J}^{(1)}|\mathcal{D}_I^N} + \widehat{\mathrm{msep}}_{C_{i,J}^{(2)}|\mathcal{D}_I^N} + 2\,c\,\widehat{\mathrm{msep}}_{C_{i,J}^{(1)}|\mathcal{D}_I^N}^{1/2}\,\widehat{\mathrm{msep}}_{C_{i,J}^{(2)}|\mathcal{D}_I^N}^{1/2} \tag{73}$$

with $c = 0$ and $c = 1$, respectively. Except for accident year 3, for all single accident years and aggregated accident years, we observe that the estimates in the third column are between the ones assuming no correlation and perfect positive correlation. Note that accounting for the correlation between subportfolios adds about 9% to the estimated prediction standard error for the entire portfolio (295,038 vs. 271,015).

7. Appendix: Proofs

In this section we present the proofs for Lemmas 4.4, 4.5, and 4.6.

Table 8. Estimated prediction standard errors

i	Subportfolio A ALR Method		Subportfolio B CL Method		Portfolio Total		Portfolio Correlation = 0		Portfolio Correlation = 1	
1	200	8.5%	604	−448.2%	672	30.4%	636	28.7%	804	36.3%
2	602	10.2%	1,436	−194.2%	1,648	31.8%	1,557	30.0%	2,038	39.3%
3	1,961	20.4%	2,912	240.4%	3,353	31.0%	3,510	32.4%	4,872	45.0%
4	6,120	44.6%	3,202	322.8%	7,432	50.5%	6,907	47.0%	9,322	63.4%
5	14,337	54.3%	5,418	173.0%	16,701	56.6%	15,326	51.9%	19,755	66.9%
6	16,724	40.9%	6,221	169.9%	19,740	44.3%	17,844	40.0%	22,945	51.5%
7	20,677	25.5%	7,483	74.5%	23,735	26.1%	21,990	24.2%	28,160	30.9%
8	27,131	18.9%	9,123	42.3%	30,928	18.7%	28,624	17.3%	36,254	21.9%
9	34,424	12.1%	16,191	29.6%	41,675	12.3%	38,041	11.2%	50,615	15.0%
10	49,589	8.3%	26,742	22.6%	62,569	8.8%	56,340	7.9%	76,331	10.7%
11	59,660	5.5%	36,737	14.5%	79,959	6.0%	70,064	5.3%	96,397	7.2%
12	75,250	4.2%	53,399	9.4%	109,712	4.6%	92,271	3.9%	128,649	5.4%
13	90,670	4.1%	126,615	12.3%	158,684	4.9%	155,731	4.8%	217,284	6.7%
Total	216,613	3.4%	162,874	7.9%	295,038	3.5%	271,015	3.2%	379,488	4.5%

7.1. Proof of Lemma 4.4

By induction we prove that

$$\mathrm{Cov}(\mathbf{C}_{i,k}^{\mathrm{CL}}, \mathbf{X}_{i,j}^{\mathrm{AD}} \mid \mathbf{C}_{i,I-i}) = \prod_{l=j}^{k-1} \mathrm{D}(\mathbf{f}_l) \cdot \Sigma_{i,j-1}^{\mathbf{CA}}, \quad (74)$$

where $\Sigma_{i,j-1}^{\mathbf{CA}}$ is defined by (39) for all $k \geq j \geq I-i+1$ and $i = 1,\ldots,I$.

a) Assume $k = j$. Then, using (17), we have

$$\begin{aligned}
&\mathrm{Cov}(\mathbf{C}_{i,j}^{\mathrm{CL}}, \mathbf{X}_{i,j}^{\mathrm{AD}} \mid \mathbf{C}_{i,I-i}) \\
&\quad = E[\mathrm{D}(\mathbf{C}_{i,j-1}^{\mathrm{CL}})^{1/2} \cdot \mathrm{D}(\varepsilon_{i,j}^{\mathrm{CL}}) \\
&\qquad \cdot \boldsymbol{\sigma}_{j-1}^{\mathrm{CL}} \cdot (\mathrm{V}_i^{1/2} \cdot \mathrm{D}(\varepsilon_{i,j}^{\mathrm{AD}}) \cdot \boldsymbol{\sigma}_{j-1}^{\mathrm{AD}})' \mid \mathbf{C}_{i,I-i}] \\
&\quad = E[\mathrm{D}(\mathbf{C}_{i,j-1}^{\mathrm{CL}})^{1/2} \cdot E[\mathrm{D}(\varepsilon_{i,j}^{\mathrm{CL}}) \cdot \boldsymbol{\sigma}_{j-1}^{\mathrm{CL}} \\
&\qquad \cdot (\mathrm{V}_i^{1/2} \cdot \mathrm{D}(\varepsilon_{i,j}^{\mathrm{AD}}) \cdot \boldsymbol{\sigma}_{j-1}^{\mathrm{AD}})' \mid \mathbf{C}_{i,j-1}] \mid \mathbf{C}_{i,I-i}] \\
&\quad = E[\mathrm{D}(\mathbf{C}_{i,j-1}^{\mathrm{CL}})^{1/2} \cdot \Sigma_{j-1}^{(C,A)} \mid \mathbf{C}_{i,I-i}] \\
&\qquad \cdot \mathrm{V}_i^{1/2} = \Sigma_{i,j-1}^{\mathbf{CA}}. \qquad (75)
\end{aligned}$$

This completes the proof for $k = j$.

b) Induction step. Assume that the claim is true for $k \geq j$. We prove that it is also true for $k+1$. Using the induction step, we have conditional on $\mathbf{C}_{i,l}$, $l \leq k$,

$$\begin{aligned}
&\mathrm{Cov}(\mathbf{C}_{i,k+1}^{\mathrm{CL}}, \mathbf{X}_{i,j}^{\mathrm{AD}} \mid \mathbf{C}_{i,I-i}) \\
&\quad = \mathrm{D}(\mathbf{f}_k) \cdot \mathrm{Cov}(\mathbf{C}_{i,k}^{\mathrm{CL}}, \mathbf{X}_{i,j}^{\mathrm{AD}} \mid \mathbf{C}_{i,I-i}) + \mathbf{0} \\
&\quad = \prod_{l=j}^{k} \mathrm{D}(\mathbf{f}_l) \cdot \Sigma_{i,j-1}^{\mathbf{CA}}.
\end{aligned}$$

This finishes the proof of claim (74). Using result (74) leads to the proof of Lemma 4.4.

7.2. Proof of Lemma 4.5

a) Follows from (45) and (46) and the fact that $\tilde{\varepsilon}_{i,j+1}, \tilde{\varepsilon}_{i,k+1}$ are independent for $j \neq k$.

b) Follows from (45) and (46) and the fact that $E^*_{\mathcal{D}_I^N}[\tilde{\varepsilon}_{i,j+1}] = \mathbf{0}$.

c) Using the independence of different accident years we obtain

$$\begin{aligned}
&\mathrm{Cov}^*_{\mathcal{D}_I^N}(\hat{\mathbf{f}}_j, \hat{\mathbf{m}}_{j+1}) \\
&\quad = \mathrm{W}_j \sum_{i=0}^{I-j-1} \mathrm{D}(\mathbf{C}_{i,j}^{\mathrm{CL}})^{1/2} (\Sigma_j^{(C)})^{-1} \\
&\qquad \cdot \mathrm{Cov}^*_{\mathcal{D}_I^N}(\mathrm{D}(\tilde{\varepsilon}_{i,j+1}^{\mathrm{CL}})\boldsymbol{\sigma}_j^{\mathrm{CL}}, \mathrm{D}(\tilde{\varepsilon}_{i,j+1}^{\mathrm{AD}})\boldsymbol{\sigma}_j^{\mathrm{AD}}) \\
&\qquad \cdot (\Sigma_j^{(A)})^{-1} \mathrm{V}_i^{1/2} \mathrm{U}_j \\
&\quad = \mathrm{W}_j \sum_{i=0}^{I-j-1} \mathrm{D}(\mathbf{C}_{i,j}^{\mathrm{CL}})^{1/2} \\
&\qquad \cdot (\Sigma_j^{(C)})^{-1} \Sigma_j^{(C,A)} (\Sigma_j^{(A)})^{-1} \mathrm{V}_i^{1/2} \mathrm{U}_j = \mathrm{T}_j.
\end{aligned}$$

Hence,

$$\begin{aligned}
E^*_{\mathcal{D}_I^N}[\hat{f}_j^{(m)} \hat{m}_{j+1}^{(n)}] &= f_j^{(m)} m_{j+1}^{(n)} + \mathrm{Cov}^*_{\mathcal{D}_I^N}(\hat{f}_j^{(m)}, \hat{m}_{j+1}^{(n)}) \\
&= f_j^{(m)} m_{j+1}^{(n)} + \mathrm{T}_j(m,n),
\end{aligned}$$

where $\mathrm{T}_j(m,n)$ is the entry (m,n) of the $K \times (N-K)$-matrix T_j. This completes the proof of Lemma 4.5.

7.3. Proof of Lemma 4.6

The components $\Psi_{k,i}^{(m,n)}$ are defined by (49). Hence, we calculate the terms

$$\mathrm{Cov}^*_{\mathcal{D}_I^N}(\hat{\mathbf{g}}_{k|J}, \hat{\mathbf{m}}_j) = E^*_{\mathcal{D}_I^N}[\hat{\mathbf{g}}_{k|J} \hat{\mathbf{m}}_j'] - E^*_{\mathcal{D}_I^N}[\hat{\mathbf{g}}_{k|J}] E^*_{\mathcal{D}_I^N}[\hat{\mathbf{m}}_j'].$$

This expression is equal to 0 (i.e., the $K \times (N-K)$-matrix consisting of zeros) for $j-1 < I-k$. Hence

$$\Psi_{k,i} = (\Psi_{k,i}^{(m,n)})_{m,n} = \sum_{j=(I-i+1)\vee(I-k+1)}^{J} \mathrm{Cov}^*_{\mathcal{D}_I^N}(\hat{\mathbf{g}}_{k|J}, \hat{\mathbf{m}}_j).$$

For $j-1 \geq I-k$ we have, using Lemma 4.5, that the (m,n)-component of the covariance matrix on the right-hand side of the above equality is equal to

$$\begin{aligned}
&\prod_{r=I-k}^{j-2} f_r^{(m)} (f_{j-1}^{(m)} m_j^{(n)} + \mathrm{T}_{j-1}(m,n)) \prod_{r=j}^{J-1} f_r^{(m)} - \prod_{r=I-k}^{J-1} f_r^{(m)} m_j^{(n)} \\
&\quad = \prod_{r=I-k}^{J-1} f_r^{(m)} \frac{1}{f_{j-1}^{(m)}} \mathrm{T}_{j-1}(m,n).
\end{aligned}$$

This completes the proof of Lemma 4.6.

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