A Continuous Version of Sherman's Inverse Power Curve Model with Simple Cumulative Development Factor Formulas

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ABSTRACT

A continuous version of Sherman's discrete inverse power curve model for loss development is defined. This continuous version, apparently unlike its discrete counterpart, has simple formulas for cumulative development factors, including tail factors. The continuous version has the same tail convergence conditions and basic analytical properties as the discrete version. Parameter fitting and numerical comparisons between the discrete and continuous model versions are explored.

KEYWORDS

Tail factor, inverse power curve

1. Introduction

The inverse power curve model for loss development factors was introduced by Sherman (1984). Several papers have commented on the lack of a closed form representation and/or tail convergence information (Boor 2006; CAS Tail Factor WP 2013; Lowe and Mohrman 1985). The conditions for tail factor convergence and estimates of the rate of tail factor convergence have been determined (Evans 2014). However, simple or closed-form formulas for finite and infinite products of the discrete incremental development factors still appear elusive.

This paper will demonstrate that a continuous version of the inverse power curve model captures the relevant properties of the discrete model. At the same time, this continuous model leads to simple cumulative development factor, including tail factor, formulas. These simple formulas facilitate various practical calculations, such as fitting or calculating development factors using intermediate time values or fitting the model to a preselected tail factor. Basic real analysis is used throughout this paper, as described in standard textbooks such as Rudin (1976).

Section 2.1 defines the continuous version of the model. Convergence conditions are proved in Section 2.2. Several basic analytical properties are proven in Section 2.3. Empirical fitting, and a comparison with the discrete model, is discussed in Section 2.4. Further numerical comparisons are shown in Sections 2.5. Section 3 contains concluding remarks. Appendix A contains the proof of two lemmas and Theorem 2 from Section 2.3.

2. Analysis and results

In the remainder of this paper t is used for age or time, whether discrete or continuous.

2.1. Continuous inverse curve model definition

The discrete version of the inverse power curve model can be defined in terms of a cumulative development factor $F_t(a, b, c)$ from time 1 to t.

Definition: For real numbers a, b, and c, where $a > 0$ and $c \geq 0$,

(i) If $t \geq 2$ is an integer, then

$$F_t(a,b,c) = \prod_{k=1}^{t-1}\left(1 + a(k+c)^b\right).$$

(ii) If $t = 1$, then $F_1(a, b, c) = 1$.

$F_t(a, b, c)$ obeys the finite difference equation

$$F_{t+1}(a,b,c) - F_t(a,b,c) = a(t+c)^b F_t(a,b,c).$$

If the boundary value of $F_1(a, b, c) = 1$ is included, then this equation, along with the previous parameter restrictions, is an equivalent definition of $F_t(a, b, c)$.

A corresponding continuous version of the inverse power curve model can similarly be defined in terms of a cumulative development factor $F_t^*(a, b, c)$ from time 1 to t, but with much simpler closed-form expressions.

Definition: For $a > 0$, b, $c \geq 0$, and $t \geq 0$

(i) $F_t^*(a,b,c) = \exp\left(\dfrac{a(c+t)^{1+b} - a(c+1)^{1+b}}{1+b}\right)$

if $b \neq -1$

(ii) $F_t^*(a,b,c) = \left(\dfrac{c+t}{c+1}\right)^a$ if $b = -1$

Using an analogous boundary value $F_1^*(a, b, c) = 1$, $F_t^*(a, b, c)$ is the solution to a differential equation in continuous time t,

$$\frac{dF_t^*(a,b,c)}{dt} = a(t+c)^b F_t^*(a,b,c),$$

that is analogous to the finite difference equation satisfied by the discrete model.

2.2. Tail convergence

The tail of $F_t^*(a, b, c)$ converges when $b < -1$ and diverges when $b \geq -1$, which are exactly the same as the conditions for convergence of $F_t(a, b, c)$ as shown in Evans (2014).

Theorem 1

(i) If $b \geq -1$, $\lim_{t\to+\infty} F_t^*(a,b,c) = +\infty$.

(ii) If $b < -1$, $\lim_{t\to+\infty} F_t^*(a,b,c) = \exp\left(\frac{-a(c+1)^{1+b}}{1+b}\right)$.

Proof:

(i) If $b = -1$, clearly $\lim_{t\to+\infty} \frac{c+t}{c+1} = +\infty$ and since

$a > 0$ by Lemma **A.1** $\lim_{t\to+\infty}\left(\frac{c+t}{c+1}\right)^a = +\infty$.

If $b > -1$, then $b + 1 > 0$, so again by Lemma **A.1**

$\lim_{t\to+\infty} \frac{a(c+t)^{1+b}}{1+b} = +\infty$ and since $\exp(x)$ is an

increasing function of x,

$$\lim_{t\to+\infty} \exp\left(\frac{a(c+t)^{1+b} - a(c+1)^{1+b}}{1+b}\right) = +\infty.$$

(ii) If $b < -1$, then $b + 1 < 0$, so by Lemma **A.1**

$\lim_{t\to+\infty} \frac{a(c+t)^{1+b}}{1+b} = 0$ and consequently since

$\exp(x)$ is continuous

$$\lim_{t\to+\infty} \exp\left(\frac{a(c+t)^{1+b} - a(c+1)^{1+b}}{1+b}\right)$$

$$= \exp\left(\frac{-a(c+1)^{1+b}}{1+b}\right).$$

2.3. Some basic analytical properties

For convenience we will first set up notational definitions of the one-period development factors, $f_t(a, b, c)$ for the discrete model and $f_t^*(a, b, c)$ for the continuous model.

Definition: $f_t(a,b,c) = \frac{F_{t+1}(a,b,c)}{F_t(a,b,c)} = 1 + a(c+t)^b$

Definition:

(i) For $b \neq -1$, $f_t^*(a,b,c) = \frac{F_{t+1}^*(a,b,c)}{F_t^*(a,b,c)}$

$$= \exp\left(\frac{a(c+t+1)^{1+b} - a(c+t)^{1+b}}{1+b}\right)$$

(ii) For $b = -1$, $f_t^*(a,b,c) = \frac{F_{t+1}^*(a,b,c)}{F_t^*(a,b,c)}$

$$= \left(\frac{c+t+1}{c+t}\right)^a$$

Lowe and Mohrman (1985) list several analytical properties for a curve of one-period loss development factors to be "well-behaved."

Definition: A curve of one-period loss development factors, $f(t)$ for $t \geq 0$, is said to be *well-behaved* if it has all of the following properties:

(i) $f(t) \geq 1$

(ii) $\lim_{t\to+\infty} f(t) = 1$

(iii) $f'(t) < 0$

(iv) $\lim_{t\to+\infty} f'(t) = 0$

(v) $f''(t) > 0$

(vi) $\lim_{t\to+\infty} f''(t) = 0$

Theorem 2

If $b < 0$ then $f_t(a, b, c)$ and $f_t^*(a, b, c)$ are both well-behaved.

See Appendix A for details of the proof.

2.4. Fitting to empirical data

Table 1 includes an example from Sherman's original paper (1984) of the discrete model fit to empirical data. Also shown are one-period development factors from the continuous model, first using the same parameter values as the discrete model and then using another set of parameter values refit for the continuous model itself. The continuous model development factors using the discrete model parameter values are fairly close to the discrete model development factors. When the parameters values are refit, the resulting development factors for the continuous model are very close to the discrete model development factors.

Definition: $f^a(t)$ will denote an empirical observation of a one-period development factor from time t to $t + 1$.

Table 1. Comparison of discrete and continuous models fit to general liability data (actual and discrete fit are from Exhibit 2 in Sherman's original paper (1984). Time convention is shifted by −1 from the original paper.)

		Parameter Values		
Parameters		Discrete Fit	Discrete Fit	Continuous Fit
a		0.88614	0.88614	1.20154
b		–1.7338	–1.7338	–1.8306
c		0	0	0
	One-Period Development Factors From *t* to *t* + 1			
t	Actual	Discrete Model	Continuous Model	Continuous Model
1	1.839	1.886	1.618	1.884
2	1.279	1.266	1.205	1.262
3	1.185	1.132	1.108	1.131
4	1.077	1.080	1.068	1.080
5	1.039	1.054	1.048	1.055
6	1.033	1.040	1.035	1.040
7	1.029	1.030	1.027	1.031
8	1.030	1.024	1.022	1.024
9	1.019	1.020	1.018	1.020
10	1.014	1.016	1.015	1.016
11	1.016	1.014	1.013	1.014
12	1.013	1.012	1.011	1.012
13	1.012	1.010	1.010	1.010
14	1.008	1.009	1.009	1.009
Goodness of Fit (R^2)		98.3%	97.8%	98.2%

The fits and goodness-of-fit (R^2) numbers in Table 1 are determined using the squared error function

$$\sum_{t=1}^{n}\left(\log\left(f^a(t)-1\right)-\log\left(f_t(a,b,c)-1\right)\right)^2$$
$$=\sum_{t=1}^{n}\left(\log\left(f^a(t)-1\right)-\log(a)-b\log(c+t)\right)^2.$$

for the discrete model.

Correspondingly, for the continuous model,

$$\sum_{t=1}^{n}\left(\log\left(f^a(t)-1\right)-\log\left(f_t^*(a,b,c)-1\right)\right)^2$$
$$=\sum_{t=1}^{n}\left(\log\left(f^a(t)-1\right)-\log\left(\exp\left(\frac{a(c+t+1)^{1+b}-a(c+t)^{1+b}}{1+b}\right)-1\right)\right)^2.$$

A simpler squared error function for the continuous model would be

$$\sum_{t=1}^{n}\left(\log\left(f^a(t)\right)-\log\left(f_t^*(a,b,c)\right)\right)^2$$
$$=\sum_{t=1}^{n}\left(\log\left(f^a(t)\right)-\left(\frac{a(c+t+1)^{1+b}-a(c+t)^{1+b}}{1+b}\right)\right)^2.$$

This is still a fairly complicated function, likely not having a simple formulaic solution to minimize the parameters *a, b,* and *c*. One of the few apparent advantages of the discrete model is a somewhat simpler error function. It may be advantageous to use a numerical optimization program (like Solver in Excel) to first optimize *a, b,* and *c* for the discrete model. Those values can then be used as a starting point, or initial values, for the optimizer to search for values optimal for the continuous model.

2.5. More numerical comparisons with the discrete model

Table 2 follows the basic layout of Table 1, except that the fitting targets a one-period development factor of 1.01 from time 1 to 2 and a development factor of 1.30 from time 1 to 101, rather than fitting to a set of actual empirical development factors. The *b* parameter runs through the set of values {−2.0, −1.5, −1.1, −1.0, −0.9, −0.6} in these examples. Similar to what happened in Table 1, in Table 2 the continuous model is fairly close to the discrete model using the same parameter values and very close—identical up to 3 digits past the decimal in some examples—when the parameter values continuous model are refit. Note,

Table 2. Some numerical comparisons of discrete and continuous models

Parameter Values			
Parameters	Discrete Fit	Discrete Fit	Continuous Fit
a	12.121	12.121	12.1528
b	–2.0	–2.0	–2.0
c	33.8152	33.8152	33.4513
Cumulative Development Factors From 1 to *t*			
t	Discrete Model	Continuous Model	Continuous Model
2	1.010	1.010	1.010
11	1.083	1.081	1.083
101	1.300	1.295	1.300
1,001	1.406	1.400	1.406
10,001	1.421	1.415	1.421
100,001	1.423	1.416	1.423
1,000,001	1.423	1.416	1.423
10,000,001	1.423	1.416	1.423
100,000,001	1.423	1.416	1.423
1,000,000,001	1.423	1.416	1.423
Parameter Values			
Parameters	Discrete Fit	Discrete Fit	Continuous Fit
a	1.07747	1.07747	1.07894
b	–1.5	–1.5	–1.5
c	21.6432	21.6432	21.2437
Cumulative Development Factors From 1 to *t*			
t	Discrete Model	Continuous Model	Continuous Model
2	1.010	1.010	1.010
11	1.081	1.079	1.081
101	1.300	1.295	1.300
1,001	1.477	1.470	1.477
10,001	1.546	1.539	1.546
100,001	1.569	1.562	1.569
1,000,001	1.576	1.569	1.577
10,000,001	1.579	1.572	1.579
100,000,001	1.580	1.572	1.580
1,000,000,001	1.580	1.573	1.580

(continued on next page)

Table 2. Some numerical comparisons of discrete and continuous models *(continued)*

Parameter Values			
Parameters	Discrete Fit	Discrete Fit	Continuous Fit
a	0.174452	0.174452	0.174523
b	–1.1	–1.1	–1.1
c	12.4523	12.4523	12.0248
Cumulative Development Factors From 1 to *t*			
t	Discrete Model	Continuous Model	Continuous Model
2	1.010	1.010	1.010
11	1.078	1.075	1.078
101	1.300	1.295	1.300
1,001	1.610	1.603	1.611
10,001	1.926	1.917	1.926
100,001	2.221	2.211	2.222
1,000,001	2.488	2.477	2.489
10,000,001	2.723	2.711	2.723
100,000,001	2.925	2.912	2.926
1,000,000,001	3.096	3.082	3.097
Parameter Values			
Parameters	Discrete Fit	Discrete Fit	Continuous Fit
a	0.112892	0.112892	0.112913
b	–1.0	–1.0	–1.0
c	10.2892	10.2892	9.85493
Cumulative Development Factors From 1 to *t*			
t	Discrete Model	Continuous Model	Continuous Model
2	1.010	1.010	1.010
11	1.077	1.074	1.077
101	1.300	1.295	1.300
1,001	1.668	1.661	1.669
10,001	2.161	2.152	2.162
100,001	2.803	2.790	2.803
1,000,001	3.635	3.618	3.635
10,000,001	4.714	4.693	4.715
100,000,001	6.113	6.086	6.115
1,000,000,001	7.928	7.892	7.930
Parameter Values			
Parameters	Discrete Fit	Discrete Fit	Continuous Fit
a	0.0737384	0.0737384	0.0737367
b	–0.9	–0.9	–0.9
c	8.2067	8.2067	7.7661

(continued on next page)

Table 2. Some numerical comparisons of discrete and continuous models *(continued)*

Cumulative Development Factors From 1 to *t*			
t	Discrete Model	Continuous Model	Continuous Model
2	1.010	1.010	1.010
11	1.075	1.073	1.075
101	1.300	1.295	1.300
1,001	1.744	1.737	1.744
10,001	2.550	2.539	2.550
100,001	4.119	4.101	4.119
1,000,001	7.534	7.500	7.534
10,000,001	16.111	16.039	16.110
100,000,001	41.946	41.759	41.944
1,000,000,001	139.919	139.293	139.906
Parameter Values			
Parameters	Discrete Fit	Discrete Fit	Continuous Fit
a	0.021923	0.021923	0.021913
b	–0.6	–0.6	–0.6
c	2.69971	2.69971	2.24551
Cumulative Development Factors From 1 to *t*			
t	Discrete Model	Continuous Model	Continuous Model
2	1.010	1.009	1.010
11	1.069	1.066	1.068
101	1.300	1.295	1.300
1,001	2.185	2.176	2.185
10,001	8.118	8.083	8.113
100,001	219.782	218.839	219.322
1,000,001	8.72E+05	8.69E+05	8.67E+05
10,000,001	9.55E+14	9.51E+14	9.40E+14
100,000,001	4.86E+37	4.84E+37	4.67E+37
1,000,000,001	5.27E+94	5.24E+94	4.76E+94

in the table the time values extend upward to very large numbers, irrelevant to any realistic actuarial application, simply to illustrate asymptotic properties of the models.

3. Conclusions

The continuous inverse power curve model presented in this paper has the same tail convergence conditions and "well-behaved" analytical properties as the discrete model. Unlike the discrete model, it is known to have very simple closed formulas for cumulative development factors, including tail factors. It tends to produce numerical values extremely close to the discrete value when fit to the same data. Squared error functions for fitting the parameters of the continuous model tend to be more complex, but fits borrowed from the discrete model can be used as initial values to facilitate fitting the continuous model.

Appendix A

Lemma A.1

(*i*) If p > 0 then $\lim_{x \to +\infty} x^p = +\infty$.

(*ii*) If p < 0 then $\lim_{x \to +\infty} x^p = 0$.

Proof:

(i) For any $\varepsilon > 0$, choose $x > \varepsilon^{1/p}$ to make $x > \varepsilon$.

(ii) For any $\varepsilon > 0$, choose $x > \varepsilon^{-1/p}$ to make $x < \varepsilon$.

Lemma A.2

If $p < 1$ and $q \geq 0$ then $\lim_{x\to+\infty}\left((x+q)^p - x^p\right) = 0$.

Proof:

If $p < 0$ then by **Lemma A.1** $\lim_{x\to+\infty}((x+q)^p - x^p) = 0 - 0 = 0$. If $p = 0$ then trivially $\lim_{x\to+\infty}((x+q)^p - x^p) = 1 - 1 = 0$. For $1 > p > 0$, since $\frac{d(x+q)^p}{dq} = p(x+q)^{p-1} > 0$ but $\frac{d^2(x+q)^p}{dq^2} = p(p-1)(x+q)^{p-2} < 0$, it follows regarding the tangential approximation from $q = 0$ that $(x+q)^p \leq x^p + q\left(\left.\frac{d(x+q)^p}{dq}\right|_{q=0}\right) = x^p + qpx^{p-1}$. So $(x+q)^p - x^p \leq qpx^{p-1}$ and consequently $0 \leq \lim_{x\to+\infty}((x+q)^p - x^p) \leq \lim_{x\to+\infty} qpx^{p-1} = 0$ by **Lemma A.1** since $p - 1 < 0$.

Proof of Theorem 2 from Section 2.3:

For $f_t(a, b, c)$:

(i) $a(c+t)^b > 0$ and consequently $1 + a(c+t)^b > 1$.

(ii) If $b < 0$ then by **Lemma A.1** $\lim_{t\to+\infty} a(c+t)^b = 0$ and consequently $\lim_{t\to+\infty}(1 + a(c+t)^b) = 1$.

(iii) $\frac{df_t(a,b,c)}{dt} = ba(c+t)^{b-1}$. Since $b < 0$, clearly $ba(c+t)^{b-1} < 0$.

(iv) Since $b - 1 < b < 0$, by **Lemma A.1** it follows that $\lim_{t\to+\infty} ba(c+t)^{b-1} = 0$.

(v) $\frac{d^2 f_t(a,b,c)}{dt^2} = b(b-1)a(c+t)^{b-2}$. Since $b - 1 < b < 0$, obviously $b(b-1) > 0$, and consequently $b(b-1)a(c+t)^{b-2} > 0$.

(vi) Since $b - 2 < b < 0$, by **Lemma A.1** it follows that $\lim_{t\to+\infty} b(b-1)a(c+t)^{b-2} = 0$.

For $f_t^*(a, b, c)$:

(i) If $b + 1 < 0$ then $a(c+t+1)^{1+b} \leq a(c+t)^{1+b}$, or if $b + 1 > 0$ then $a(c+t+1)^{1+b} \geq a(c+t)^{1+b}$. Either way, it follows when taking the ratio that $\frac{a(c+t+1)^{1+b} - a(c+t)^{1+b}}{1+b} \geq 0$ and consequently that $\exp\left(\frac{a(c+t+1)^{1+b} - a(c+t)^{1+b}}{1+b}\right) \geq 1$.

Since $t \geq 1$, $c + t > c + 1$, and consequently $\left(\frac{c+t+1}{c+t}\right)^a \geq 1$ for the case of $b + 1 = 0$.

(ii) If $b < 0$ then $b + 1 < 1$ and by **Lemma A.2** $\lim_{t\to+\infty}(a(c+t+1)^{1+b} - a(c+t)^{1+b}) = 0$.

Since $\exp(x)$ is continuous

$$\lim_{t\to+\infty}\exp\left(\frac{a(c+t+1)^{1+b} - a(c+t)^{1+b}}{1+b}\right) = 1.$$

(iii) $\frac{df_t^*(a,b,c)}{dt} = (a(c+t+1)^b - a(c+t)^b)f_t^*(a, b, c)$.

Since $b < 0$, $a(c+t+1)^b < a(c+t)^b$ and $a(c+t+1)^b - a(c+t)^b < 0$, and since $f_t^*(a, b, c) > 0$ it follows that $(a(c+t+1)^b - a(c+t)^b)\, f_t^*(a, b, c) < 0$.

(iv) $\lim_{t\to+\infty}(a(c+t+1)^b - a(c+t)^b) = \lim_{t\to+\infty} a(c+t+1)^b - \lim_{t\to+\infty} a(c+t)^b = 0 - 0 = 0$ by applying **Lemma A.1.** Therefore, since $\lim_{t\to+\infty} f_t^*(a, b, c) = 1$ it follows that $\lim_{t\to+\infty}((a(c+t+1)^b - a(c+t)^b)\, f_t^*(a, b, c)) = 0$.

(v)

$$\frac{d^2 f_t^*(a,b,c)}{dt^2} = \left(ba(c+t+1)^{b-1} - ba(c+t)^{b-1}\right) f_t^*(a,b,c) + \left(a(c+t+1)^b - a(c+t)^b\right)\frac{df_t^*(a,b,c)}{dt}$$

$$= \left(ba(c+t+1)^{b-1} - ba(c+t)^{b-1}\right) f_t^*(a,b,c) + \left(a(c+t+1)^b - a(c+t)^b\right)^2 f_t^*(a,b,c).$$

Clearly $(a(c+t+1)^b - a(c+t)^b)^2 > 0$. Since $b < 0$ and $a(c+t+1)^{b-1} < a(c+t)^{b-1}$ it follows that $ba(c+t+1)^{b-1} - ba(c+t)^{b-1} > 0$. Since $f_t^*(a, b, c) \geq 1$, altogether it follows that $(ba(c+t+1)^{b-1} - ba(c+t)^{b-1})\, f_t^*(a, b, c) + (a(c+t+1)^b - a(c+t)^b)^2 f_t^*(a, b, c) > 0$.

(vi) Since $b - 1 < b < 0$ by **Lemma A.2** it follows that $\lim_{t\to+\infty}(ba(c+t+1)^{b-1} - ba(c+t)^{b-1}) = 0$ and

$\lim_{t\to+\infty}(a(c+t+1)^b - a(c+t)^b)^2 = 0$. Since $\lim_{t\to+\infty} f_t^*(a, b, c) = 1$ and $\lim_{t\to+\infty}\frac{df_t^*(a,b,c)}{dt} = 0$, it follows that $\lim_{t\to+\infty}((ba(c+t+1)^{b-1} - ba(c+t)^{b-1}) f_t^*(a, b, c) + (a(c+t+1)^b - a(c+t)^b)^2 f_t^*(a, b, c)) = 0$.

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