DISCUSSION OF PAPER PUBLISHED IN VOL. 6, NO. 1:

"The Mathematics of Excess Losses"

by Leigh J. Halliwell

Discussion by LIANG HONG

1. Introduction

My congratulations to Mr. Leigh J. Halliwell on this paper that clearly presents the mathematics of excess losses with an interesting example. I agree with him that the mathematics of excess losses is beautiful and powerful. However, the mathematics of excess losses also contains several subtle points that are not mentioned in the paper. This discussion note complements the article by clarifying some of these points. To be clear, it is not my intention to be critical of Mr. Halliwell. The purpose of this note is two-fold:

1. To clarify some important hidden points in the mathematics of excess losses;
2. To give references to some uncredited results.

For those ambitious actuaries who want to dig deeper for a full understanding of the rigorous mathematics of excess losses, this note also provides some directions for further studies.

For the convenience of readers, we will adopt the notations in Halliwell (2013). Throughout this note, X will denote a nonnegative random variable. F and G will denote the cumulative distribution function (CDF) and survival function of X, respectively.

2. The reason why we need to watch our steps

Halliwell (2013) argues that points of probability are allowed due to four properties of the CDF: (1) non-decreasing; (2) total probability; (3) continuity from the right; and (4) non-negative. Indeed, we will also need one more important property of the CDF: the left-hand limit of a CDF exists at each point (see, for example, Shiryaev 1996). This is clear from the second equality of the following equation.

$$P[X=a]=\lim_{n\to\infty}P[a-1/n<X\le a] \qquad (1)$$
$$=F(a)-F(a-).$$

Since a CDF is always non-decreasing, its left-hand limit and right-hand limit exist at each point in the domain (see, for example, Rudin 1976). This means

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$P[X = a]$ needs not be zero. Precisely, we should define X to be *continuous* if F is continuous, that is, $P[X = a] = 0$ for each a. In particular, if F admits a density, then it is continuous. Unfortunately, some elementary probability textbooks do not clarify this and simply define a random variable to be continuous if its CDF admits a density. It is possible but quite difficult to construct a continuous random variable which has no density function.[1] Equation (1) and the definition of Riemann-Stieltjes Integrals[2] show that Riemann-Stieltjes integrals with respect to a CDF integrator[3] will count probability masses at a given point (not just at zero!). However, the ordinary Riemann-integrals do not because in this case we are integrating with respect to the continuous function $F(x) = x$, i.e., $F(x) - F(x-) = x - (x-) = 0$. In words, our experience with the ordinary Riemann-integrals could be misleading when we need to deal with Riemann-Stieltjes integrals. This is why we need to watch our steps!

3. A subtle point both casualty and non-casualty actuaries might want to know

There is a subtle point hidden in the derivation of the following formula in Halliwell (2013).

$$Excess_x(r) = 0 - 0 + \int_{x=r}^{\infty} G_x(x)\,dx. \qquad (2)$$

The formula $\lim_{r\to\infty} rG_x(r) = 0$ is unjustified in the derivation. A close scrutiny reveals that the derivations of several other formulas in Halliwell (2013) will need the justification of this, too. This important formula has been frequently used in both life and nonlife insurance. See, for example, Cunningham, Herzog, and London (2008); Dickson, Hardy, and Waters (2009); Klugman, Panjer, and Willmot (1998). We point out that this formula is not as trivial as it might seem to be, because a $0 \cdot \infty$ form appears here and L'Hôpital's rule does not seem to be helpful. (We challenge readers to give a correct justification on their own.) To our best knowledge, this crucial point has been missed by the actuarial community for a long time. For example, neither Klugman, Panjer, and Willmot (1998) nor Cunningham, Herzog, and London (2008) provides a justification of this. However, Dickson et al. (2009; p. 20) is aware of the subtlety of this; but they impose cumbersome assumptions. Indeed, their assumptions 2 and 3 can be simply replaced by the one that X has finite second moment. Following Hong (2012), we give a correct and simple justification. First, note that

$$0 \le rG_x(r) = rP\{X > r\} = r\int_r^{\infty} dG_x(t) \le \int_r^{\infty} t\,dG_x(t).$$

Then we obtain the result by taking $r \to \infty$.

4. References for some uncredited results

There are many formulas derived in Halliwell (2013). It seems to us that the author might be unaware of some part of the existing literature. We respectfully point out that quite a few of these formulas are special cases of well known equations in probability theory but in some new notations. Here we give references to these known results. The formula (2) and the second moment formula on p. 35 of Halliwell (2013) are both given in Cunningham, Herzog, and London (2008); Dickson, Hardy, and Waters (2009); and Ross (2010); the formula on p. 43 and the formula about $\int_{x=0}^{\infty} Excess(x)dh(x)$ on p. 35 of Halliwell (2013) are given in Klebaner (2005); the formula of layered losses on p. 37 of Halliwell (2013) is given in Wang (1996; 2000). The result in the footnote 11 of Halliwell (2013) is also a well-known result that is documented in Klebaner (2005), Royden (1988), and Rudin (1976). We feel that the proof in Rudin (1976) is shorter and cleaner.

In addition, the derivation of the first formula on p. 36 of Halliwell (2013) is not necessary. The equation follows trivially from the definition of Riemann-Stieltjes integral (cf p. 43 of Halliwell 2013) and the fact adding a constant to a function will not change its variation, i.e., $d(h(x) + c) = dh(x)$.

[1]Interested readers can consult Royden (1988) for further details.

[2]There are two kinds of Stieltjes integrals: Riemann-Stieltjes integrals and Lebesgue-Stieltjes integrals. To be precise, the Stieltjes integrals in Halliwell (2013) are Riemann-Stieltjes integrals.

[3]More generally, this is true for a Riemann-Stieltjes integral with respect to a function of bounded variation. In particular, this holds for a non-decreasing function. See Royden (1988).

5. For ambitious actuaries

Finally, we would like to provide ambitious readers with some big picture. The Riemann-Stieltjes integral is a generalization of the ordinary Riemann-integral since the integrator is allowed to be a function F instead of the variable x. Indeed, Riemann-Stieltjes integrals can be defined for a much wider class of integrators than the class of CDFs. But the most interesting (and arguably the most useful) case is the one where the integrator F is a function of bounded variation.[4] In particular, the Riemann-integral with respect to a nondecreasing function integrator (hence CDF integrator) can be defined. Halliwell (2013) makes heavy use of Riemann-Stieltjes integrals. While Riemann-Stieltjes integrals may be a useful tool for studying CDFs, in general it is not a favorable choice in probability. One of the main reasons is it may not preserve limits of increasing sequences of loss random variables. For example, suppose $X_1, X_2, \ldots$ is an increasing sequence of loss random variables that converges to a loss random variable X with probability one, a desirable situation for an insurer would be $E[X] = \lim_{n\to\infty} E[X_n]$. However, this is not true under the framework of Riemann-Stieltjes integrals. On the other hand, Lebesgue integrals do preserve limits in such as case. This explains why most advanced monographs on probability favor Lebesgue integrals. For more details, readers can consult Billingsely (1995), Chow and Teicher (1997) and Shiryaev (1996). The mathematics of excess losses mainly addresses the probabilistic part of excess losses. In practice, an actuary will need to use loss data for his/her work. Therefore, one important direction of future research on this topic could be finding better ways to estimate various excess loss formulas. Efforts along this line are expected to involve survival analysis. Readers can consult Aalen, Borgan, and Gjessing (2008), Andersen et al. (1993), Fleming and Harrington (1991) and Klein and Moeschberger (2005).

[4]Readers without background in real analysis can just think it as a difference of two monotone functions. For more details, see Hewitt and Stromberg (1965) or Royden (1988).

6. Conclusion

A well written paper on the mathematics of excess losses is a needed service for our actuarial community. I congratulate Mr. Leigh J. Halliwell again on providing such an paper. I hope actuaries will find his paper and this discussion note useful.

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Response by the Author, Leigh J. Halliwell

This discussion is a valuable adjunct to my 2012 *Variance* paper, "The Mathematics of Excess Losses." Dr. Hong states that the value of his discussion is twofold: (1) to clarify "subtle" or "hidden" points, and (2) to provide scholarly references. The latter is especially welcome, since my formal mathematical education left off in the 1970s. Since then, I've learned on my own and from the actuarial syllabus and literature. I've never believed my work to be original, and Dr. Hong has shown where in the academic literature others have gone before me. Truly, according to Ecclesiastes, "There is nothing new under the sun."

1. My Background

My work at NCCI in the early 1990s on retrospective rating and Table M introduced me to the excess-loss function. Table M consists of ninety-nine columns, 01–99. The value of Table M at entry ratio 1.00 equals the column number as a percentage. For example, eighty percent of a loss whose distribution accords with column 80 is in excess of its expected value. Higher column numbers indicate greater variance, or greater variance in relation to expected value. Column 00, were it published, would be the distribution of a constant random variable, none of whose loss is in excess of its expected value. At that point I began to imagine what appears as Figure 1 in my paper, which then led to the double-integral proof of Section 3 that $\int_{x=0}^{\infty} Excess_X(x)\,dx = \frac{1}{2}E[X^2]$.

I've been away from Table M for nearly twenty years, and don't know how NCCI currently calculates it. But in the mid-1990s I programmed as an Excel 4 macro the then complicated Table-M function. By experiment I found that excess-loss functions of gamma distributions with appropriate parameters fairly approximated the Table-M formulas. Appendix A below shows how to do this.

About that time I was also studying for CAS Exam 5, which then covered Risk Theory. It was from the syllabus reading *Risk Theory* (Chapman and Hall, 1984), by Beard, Pentikäinen, and Pesonen, that I first learned about Stieltjes integrals, a subject on which Dr. Hong rightfully concentrates. I will take this up in Section 3. But for now I will note only that I should have proofread my paper more carefully and corrected several errors in its Appendix A. The cryptic '[1, p. 12]' in its third sentence had originally been a reference to the page of *Risk Theory* on Stieltjes integrals.

2. Two Subtleties

Dr. Hong deems two points subtle enough to deserve clarification. First, in addition to the four properties that I attributed in Section 2 to the cumulative distribution function, he adds "one more important property of the CDF: the left-hand limit of a CDF exists at each point." But this is not a separate property; rather, it is implicit in the nature of the real numbers. A fundamental theorem of the real numbers, based as they are on "Dedekind cuts" of the rational numbers, is that any upper-bounded subset of the real numbers has a least upper bound.[1] But since $F_X(x)$ is nondecreasing (property 1), $\lim_{x \to a^-} F_X(x) \le F_X(a)$. And since $F_X(a)$ is an upper bound to such limits, there must be a least upper bound, which may be symbolized as $F_X(a^-)$. Of course, if $F_X(a^-) < F_X(a)$, there is a mass of probability at $x = a$. All this is clear in a quotation from Section 2 of my paper:

> Two appealing properties of the excess-loss function are (1) that it is everywhere continuous, and (2) that if it is positive, it strictly decreases. Moreover, its derivative at r, if it exists, equals $-G_X(r)$. Even if it does not exist, at least the left

[1] Likewise, any lower-bounded subset of the real numbers has a greatest lower bound.

and right derivatives exist, and the difference of the left derivative from the right is the probability mass at *r*.

The other subtle point concerns the equation in Section 2:

$$\begin{aligned} Excess_X(r) &= \int_{x=r}^{\infty}(x-r)\,dF_X(x) \\ &= (x-r)G_X(x)\Big|_{\infty}^{r} + \int_{x=r}^{\infty} G_X(x)\,dx \\ &= 0 - \lim_{x\to\infty}(x-r)G_X(x) + \int_{x=r}^{\infty} G_X(x)\,dx \end{aligned}$$

In his Section 3 Dr. Hong notes that $\lim_{x\to\infty} xG_X(x)$, which equals $\lim_{x\to\infty}(x-r)G_X(x)$, does not necessarily equal zero. I had glossed over this, presuming the reader to understand that if $\lim_{x\to\infty}(x-r)G_X(x) > 0$, then $E[X]$ is infinite, in which case X has no excess-loss function. Dr. Hong's "justification" that $\lim_{x\to\infty} xG_X(x) = 0$ relies on the inference that if $E[X] = \lim_{M\to\infty}\int_{x=0}^{M} x\,dF_X(x)$ converges to a real number, then $\lim_{a\to\infty}\int_{x=a}^{\infty} x\,dF_X(x) = 0$. However, the proof of this inference is complicated by the need to work with nested limits. The complete proof is:

$$\begin{aligned} \lim_{a\to\infty}\int_{x=a}^{\infty} x\,dF_X(x) &= \lim_{a\to\infty}\left\{\lim_{M\to\infty}\int_{x=a}^{M} x\,dF_X(x)\right\} \\ &= \lim_{a\to\infty}\left\{\lim_{M\to\infty}\left(\int_{x=0}^{M} x\,dF_X(x) - \int_{x=0}^{a} x\,dF_X(x)\right)\right\} \\ &= \lim_{a\to\infty}\left\{\lim_{M\to\infty}\int_{x=0}^{M} x\,dF_X(x) - \int_{x=0}^{a} x\,dF_X(x)\right\} \\ &= \lim_{a\to\infty}\left\{E[X] - \int_{x=0}^{a} x\,dF_X(x)\right\} \\ &= E[X] - \lim_{a\to\infty}\int_{x=0}^{a} x\,dF_X(x) \\ &= E[X] - E[X] \\ &= 0 \end{aligned}$$

A similar argument will prove that if $E[X] = Excess_X(0)$ is finite, then $\lim_{r\to\infty} Excess_X(r) = 0$.

Nonetheless, I believe the following proof to be more elegant and insightful. And because Dr. Hong rightly says that ". . . other formulas . . . will need the justification of this too," I will generalize from $E[X]$ to $E[h(X)]$ by revisiting (and correcting) the formula for $E[h(X)]$ in my Appendix A. The usual derivation, using integration by parts, is:

$$\begin{aligned} E[h(X)] &= h(0)Prob[X=0] + \int_{x=0}^{\infty} h(x)\,dF_X(x) \\ &= h(0)Prob[X=0] - \int_{x=0}^{\infty} h(x)\,dG_X(x) \\ &= h(0)Prob[X=0] - h(x)G_X(x)\Big|_0^{\infty} \\ &\quad + \int_{x=0}^{\infty} G_X(x)\,dh(x) \\ &= h(0)Prob[X=0] - \lim_{x\to\infty} h(x)G_X(x) \\ &\quad + h(0)G_X(0) + \int_{x=0}^{\infty} G_X(x)\,dh(x) \\ &= h(0)Prob[X\geq 0] - \lim_{x\to\infty} h(x)G_X(x) \\ &\quad + \int_{x=0}^{\infty} G_X(x)\,dh(x) \\ &= h(0) - \lim_{x\to\infty} h(x)G_X(x) + \int_{x=0}^{\infty} G_X(x)\,dh(x) \end{aligned}$$

But another derivation uses the inversion-of-a-double-integral technique in my Section 3:

$$\begin{aligned} E[h(X)] &= h(0)Prob[X=0] + \int_{x=0}^{\infty} h(x)\,dF_X(x) \\ &= h(0)Prob[X=0] \\ &\quad + \int_{x=0}^{\infty}\left\{h(0) + \int_{y=0}^{x} dh(y)\right\}dF_X(x) \\ &= h(0)Prob[X=0] + h(0)\int_{x=0}^{\infty} dF_X(x) \\ &\quad + \int_{x=0}^{\infty}\int_{y=0}^{x} dh(y)\,dF_X(x) \end{aligned}$$

$$= h(0)Prob[X=0] + h(0)Prob[X>0]$$

$$+ \int_{y=0}^{\infty}\int_{x=y}^{\infty} dF_X(x)\,dh(y)$$

$$= h(0)\cdot Prob[X \geq 0] + \int_{y=0}^{\infty} Prob[X>y]\,dh(y)$$

$$= h(0) + \int_{x=0}^{\infty} G_X(x)\,dh(x)$$

Comparison of the last lines of both derivations leads to the conclusion that if the integral for $E[h(X)]$ converges, then $\lim_{x\to\infty} h(x)G_X(x)$ must be zero.

But one must not succumb to the fallacy of affirming the consequent. Even if $\lim_{x\to\infty} h(x)G_X(x) = 0$, the integral for $E[h(X)]$ will not converge, unless $h(x)G_X(x)$ approaches zero quickly enough. Assume that $\lim_{x\to\infty} h(x) = \infty$; so, for large enough M, $h(x > M)$ is positive. Restate the integral as:

$$E[h(X)] = h(0) + \int_{x=0}^{\infty} G_X(x)\,dh(x)$$

$$= h(0) + \int_{x=0}^{M} G_X(x)\,dh(x)$$

$$+ \int_{x=M}^{\infty} h(x)G_X(x)\frac{dh(x)}{h(x)}$$

If $h(x)G_X(x)$ approaches zero on the order of an inverse power curve, i.e., $h(x)G_X(x) \approx \frac{1}{h(x)^{\varepsilon>0}}$:

$$\int_{x=M}^{\infty} h(x)G_X(x)\frac{dh(x)}{h(x)} \approx \int_{x=M}^{\infty}\frac{dh(x)}{h(x)^{1+\varepsilon}} = -\frac{1}{\varepsilon}\frac{1}{h(x)^{\varepsilon}}\Bigg|_M^{\infty}$$

$$= \frac{1}{\varepsilon}\frac{1}{h(x)^{\varepsilon}}\Bigg|_{\infty}^{M} = \frac{1}{\varepsilon h(M)^{\varepsilon}}$$

Then the integral for $E[h(X)]$ converges. But if $h(x)G_X(x)$ approaches zero on the order of an inverse logarithm, i.e., $h(x)G_X(x) \approx \frac{1}{\ln h(x)}$, the integral will not converge:

$$\int_{x=M}^{\infty} h(x)G_X(x)\frac{dh(x)}{h(x)} \approx \int_{x=M}^{\infty}\frac{1}{\ln h(x)}\frac{dh(x)}{h(x)}$$

$$= \ln(\ln h(x))\Big|_M^{\infty} = \infty$$

$$-\ln(\ln h(M)) = \infty$$

Therefore, $\lim_{x\to\infty} h(x)G_X(x) = 0$ is necessary, but not sufficient, for $E[h(X)]$ to be a real number.

3. Stieltjes Integrals and Cardinality

Knowing just enough about Stieltjes integrals to be dangerous,[2] I used them in the paper only because the Stieltjes integral, unlike the classical or Riemann integral, allows for discontinuities in its integrand. In other words, the formulas using them accommodate discrete and mixed distributions. I am intrigued by Dr. Hong's claim that "It is possible but quite difficult to construct a continuous random variable which has no density function."[3]

Although the subtleties of measure theory are beyond me, the proof in my Footnote 11 was enough to justify the Stieltjes integral as a shorthand for the expectation of a mixed distribution:

$$E[g(X)] = \int_{x=-\infty}^{\infty} g(x)f_X(x)\,dx + \sum_{i=1}^{\infty} g(x_i)Prob[X = x_i]$$

But to use this formula, one must prove that the number of points at which a random variable has positive probability must be countable (hence indexable in the sigma operator). Although I knew that this was well-known to mathematicians, Dr. Hong's remark surprised me: "We feel that the proof in Rudin (1976) is shorter and cleaner." My response is: What can be

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[2]To learn what little I know, I recommend the treatment of integration and measure theory from an historical viewpoint in Chapter 27 of Kramer, Edna E., *The Nature and Growth of Modern Mathematics* (Princeton University Press, 1981).

[3]I am aware of curves that are continuous everywhere but differentiable nowhere, e.g., Brownian motion and the Koch snowflake. But these curves require both up and down movements, whereas cumulative distribution functions must be "up" only, i.e., non-decreasing (property 1). Just as intriguing is the claim in his Section 5 that for the limiting sequence of random variables $X_1, X_2, \ldots \to X$, Lebesgue integration must preserve the expectation, as opposed to Riemann-Stieltjes integration.

simpler and cleaner than Cantor's equation $\aleph_0 \times \aleph_0 = \aleph_0$? In words, a countable union of countable sets is countable. In regard to probability distributions this means that if the number of the probability masses of a random variable were uncountable, there would exist a positive value the number of points whose probability is greater than which would be uncountable. But then the total probability would be infinite, rather than the required unity. Hence no random variable may have an uncountable number of mass points. This argument is so powerful that in Appendix B I will use it to prove the theorem of analytic continuation.

In conclusion, I thank Dr. Hong for his discussion, and hope that the "ambitious actuaries" to whom at the end he appeals will continue to integrate the mathematics of excess losses into the broader wealth of modern mathematics.

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Appendix A

Gamma-Distribution Approximations to Table M

The probability density function of the Gamma (α, θ)-distributed random variable X is:

$$f_X(x) = \frac{1}{\Gamma(\alpha)} e^{-\frac{x}{\theta}} \left(\frac{x}{\theta}\right)^{\alpha-1} \frac{1}{\theta}$$

The support of the distribution is the non-negative subset of the real numbers; both the shape parameter α and the scale parameter θ must be positive real numbers. The first two moments of the gamma random variable are $E[X] = \alpha\theta$ and $Var[X] = \alpha\theta^2 = \theta E[X]$. The proportionality in θ of the variance to the mean suggests closure with respect to addition of independent random gamma variables with the same scale θ. Indeed, one can prove, granting independence, that $Gamma(\alpha_1, \theta) + Gamma(\alpha_2, \theta) = Gamma(\alpha_1 + \alpha_2, \theta)$. This property makes the gamma distribution an attractive candidate for an exposure unit. The loss distributions of risks may then be treated as convolutions of the loss distribution of a standard exposure unit.

If X is Gamma(α, θ)-distributed, then the excess-loss function of X is:

$$\begin{aligned}
Excess_X(r \geq 0) &= \int_{x=r}^{\infty} (x-r) \frac{1}{\Gamma(\alpha)} e^{-\frac{x}{\theta}} \left(\frac{x}{\theta}\right)^{\alpha-1} \frac{1}{\theta} dx \\
&= \int_{x=r}^{\infty} x \frac{1}{\Gamma(\alpha)} e^{-\frac{x}{\theta}} \left(\frac{x}{\theta}\right)^{\alpha-1} \frac{1}{\theta} dx \\
&\quad - r \int_{x=r}^{\infty} \frac{1}{\Gamma(\alpha)} e^{-\frac{x}{\theta}} \left(\frac{x}{\theta}\right)^{\alpha-1} \frac{1}{\theta} dx \\
&= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \theta \int_{x=r}^{\infty} \frac{1}{\Gamma(\alpha+1)} e^{-\frac{x}{\theta}} \left(\frac{x}{\theta}\right)^{(\alpha+1)-1} \\
&\quad \frac{1}{\theta} dx - r \int_{x=r}^{\infty} \frac{1}{\Gamma(\alpha)} e^{-\frac{x}{\theta}} \left(\frac{x}{\theta}\right)^{\alpha-1} \frac{1}{\theta} dx \\
&= \alpha\theta \left(1 - \int_{x=0}^{r} \frac{1}{\Gamma(\alpha+1)} e^{-\frac{x}{\theta}} \left(\frac{x}{\theta}\right)^{(\alpha+1)-1} \frac{1}{\theta} dx\right) \\
&\quad - r \left(1 - \int_{x=0}^{r} \frac{1}{\Gamma(\alpha)} e^{-\frac{x}{\theta}} \left(\frac{x}{\theta}\right)^{\alpha-1} \frac{1}{\theta} dx\right) \\
&= E[X] \begin{pmatrix} 1 - gamma.dist \\ (r, \alpha+1, \theta, true) \end{pmatrix} \\
&\quad - r(1 - gamma.dist(r, \alpha, \theta, true))
\end{aligned}$$

The last line of the equation is how in Excel 2013 one would calculate the expected excess loss.

Both $Excess_X(r)$ and r in Table M are scaled to $E[X] = 1$. This is most easily accommodated by fixing $\theta = E[X]/\alpha = 1/\alpha$. So, if the gamma distribution underlies column k of Table M, one solves the following equation for α:

$$\begin{aligned}
&Excess_x(r=1)\\
&\quad = 1 \cdot (1 - gamma.dist(1, \alpha+1, 1/\alpha, true))\\
&\qquad - 1 \cdot (1 - gamma.dist(1, \alpha, 1/\alpha, true))\\
&\quad = gamma.dist(1, \alpha, 1/\alpha, true)\\
&\qquad - gamma.dist(1, \alpha+1, 1/\alpha, true)\\
&\quad = k/100
\end{aligned}$$

To refer back to the example of Section 1 above, if $\alpha = 0.0716$, then the expected loss of a Gamma (0.0716, θ)-distributed random variable in excess of its expected value equals eighty percent (0.80) of its expected value. By then varying the entry ratio, one approximates the whole column 80 of Table M.

It is well known that for $0 < \alpha < 1$ the gamma distribution is J-shaped; for $1 < \alpha < \infty$ it is S-shaped. For $\alpha = 1$ the distribution is exponential, technically J-shaped but really borderline. In the exponential case:

$$\begin{aligned}
Excess_x(r=1) &= gamma.dist(1, 1, 1, true)\\
&\quad - gamma.dist(1, 2, 1, true)\\
&= 1/e \quad [\approx 0.368]\\
&= k/100
\end{aligned}$$

Therefore, if the gamma-distribution assumption fairly reproduces Table M, columns 01–36 correspond to S-shaped distributions and columns 37–99 to J-shaped.

Appendix B

Countability, Continuity, and Analytic Continuation

Analytic continuation is a theorem of complex variables. Let f be a complex function that is analytic in domain Δ, and g be analytic in domain $E \supset \Delta$. If $g(z) = f(z)$ for every $z \in \Delta$, then g is the one and only continuation of f beyond domain Δ. More accurately, if two analytic functions are equal over a domain whose Lebesgue measure is greater than zero, they must be equal everywhere. Equivalently, if two analytic functions are equal at an uncountable number of points, they must be equal everywhere.[4] This theorem is the workhorse for extending real-valued functions into the complex plane.[5] An equivalent form of the theorem is that if analytic function f has an uncountable number of roots, then $f(z) = 0$ throughout its domain.

So, as to the proof, let $f : \Delta \rightarrow C$ be analytic in complex domain $\Delta \subseteq C$. Δ may be C itself, the complex plane, in which case the function is said to be "entire." Let $Z = \{z \in \Delta : f(z) = 0\}$ and assume Z to be uncountable. Hence, the cardinality of Z is greater than $\aleph_0$, the cardinality of the natural numbers $\Im$. In words, there is no one-to-one mapping from Z into $\Im$.

Complex plane C can be partitioned into a countable number of non-empty subsets whose areas are bounded. Because the areas are bounded, the number of subsets must be countably infinite, or $\aleph_0$. Each zero of f, or each element of Z, is in one and only one of the subsets C_j. Define $Z_j = Z \cap C_j$. Then $Z_j \cap Z_{k \neq j} = \phi$ and $Z = \bigcup_{j=1}^{\infty} Z_j$. Now if each Z_j were countable (including null and finite, as well as countably infinite), then Z, being a countable union of countable sets would be countable. In Cantorian arithmetic, $\aleph_0 \times \aleph_0 = \aleph_0$. But this contradicts the assumption that Z is uncountable. Therefore, some bounded subset of uncountable Z

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[4]The Lebesgue measure of a set is the limit of the smallest area needed to envelop it. The Lebesgue measure of a countable set of points $\{z_j : j \in \Im\}$ is zero. For one can envelop z_j inside an circle of area $\mu/2^j$. Since the circles may overlap, the total area is less than or equal to μ. But μ can be made arbitrarily small; hence, the total area needed to envelop a countable number of points is arbitrarily small, and zero in the limit.

[5]The most famous application of analytic continuation is the extension of the zeta function $\zeta(s) = \sum_{j=1}^{\infty} \frac{1}{j^s}$ from $\{s \in \Re : s > 1\}$ to any complex s. Since Bernhard Riemann accomplished this in 1859 (ET, "On the Number of Prime Numbers less than a Given Quantity," www.maths.tcd.ie/pub/HistMath/People/Riemann/Zeta/EZeta.pdf), the analytically continued function has been named the Riemann zeta function. The history and the mathematics of this is ably recounted in Havil, Julian, *Gamma: Exploring Euler's Constant* (Princeton University Press, 2003), especially in its Appendices D and E. Havil's proof of analytic continuation is in Appendix D.12.

must itself be uncountable. And that bounded subset can be halved, and at least one of the halves must be uncountable. One can halve indefinitely and arrive at one element of Z arbitrarily close to which are uncountably many points of Z.

The following definition of continuity will appear strange; but reflection will prove it apt. Subset Z of the complex plane is *continuous* at $\zeta \in Z$ if and only if every neighborhood around ζ contains an uncountable number of elements of Z. Obviously, only an uncountable set can have such points of continuity. But the previous paragraph has shown that every uncountable set of complex numbers is continuous at one or more of its points.

Next, partition Z into two subsets. One subset contains the points at which Z is continuous and the other contains the points at which it is not. If the latter subset were uncountable, the argument two paragraphs above would lead to its having at least one point of continuity. Since this contradicts the nature of this set, it must be the case that the points at which Z is not continuous are countable. Hence, the former subset must be uncountable. The result so important for analytic continuation is that an uncountable subset of the complex plane must be continuous at an uncountable number of its points. So within the uncountable set of roots Z lies an uncountable subset Z_c at whose points Z is continuous.

Now consider the derivative $f'(z) = \lim_{w \to z} \frac{f(w) - f(z)}{w - z} = \lim_{w \to z} \frac{f(w) - 0}{w - z}$ for any $z \in Z_c$. Because f is analytic, the limit exists. But in any neighborhood of Z_c are other zeroes, which make $\frac{f(w) - 0}{w - z}$ equal to zero. So, the limit must be zero. Therefore, if $z \in Z_c$, $f'(z) = f(z) = 0$. The points of continuity persist through repeated differentiation; hence, if $z \in Z_c$, $f^{[j]}(z) = 0$. Cauchy's Integral Theorem is the reason why every analytic function has a Taylor-series expression (valid within its domain Δ):

$$f(z) = f(\zeta) + \frac{f'(\zeta)}{1!}(\zeta - z) + \frac{f''(\zeta)}{2!}(\zeta - z)^2 + \cdots$$

Since f and its derivatives are zero at every $\zeta \in Z_c$, f must be the zero function. A corollary of analytic continuation is that the roots of a non-constant analytic function must be countable (i.e., zero, finite, or countably infinite). Even better, since the sum of a constant and an analytic function is itself analytic, the number of times a non-constant analytic function attains to any given value is countable.