A Family of Chain-Ladder Factor Models for Selected Link Ratios

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ABSTRACT

The models of Mack (1993) and Murphy (1994) are expanded to a continuously indexed family of chain-ladder models by broadening the variance structure of the error term. It is shown that, subject to certain restrictions, an actuary's selected report-to-report factor can be considered the best linear unbiased estimate for some member of this family. The approach given in Murphy (1994) yields a mean square error estimate of the unpaid claim liability that is consistent with the actuary's selections.

KEYWORDS

Chain ladder, Mack, Murphy, variance, mean square error, reserve risk, regression

1. Introduction

The chain-ladder variance formulas first proposed by Dr. Thomas Mack (1993) are based upon all-year volume-weighted average report-to-report factors ("link ratios" or "factors") and an assumed variance structure that is proportional to the development period's initial loss. Under the regression approach of Daniel Murphy (1994) it was shown that the proportional variance structure assumption is sufficient for the weighted average link ratio to be considered the best linear unbiased estimate (BLUE) of such a chain-ladder model.[1]

In practice, however, the actuary selects factors. Factor selection is an important component of actuarial analysis[2] that utilizes actuarial judgment in its consideration of those—and other—averages as well as additional information gleaned from benchmark link ratios, industry trends, discussions with company management, etc. Although much research has been dedicated to framing the chain-ladder method within a statistical structure,[3] little ground is devoted to the treatment of the uncertainty of the unpaid claim estimates when the selected factors differ from some prescribed formula. The few treatments on the subject tend to adopt a bifurcated approach, that is, one which supplements the expected value estimates from one model with variability estimates from a different model.

A Bayesian perspective can be exploited to combine point and uncertainty estimates derived from bifurcated models. For example, Verrall (2007) assumes the actuary selects volume-weighted average link ratios from the most recent five years but derives variation estimates that reflect information from all years, not just the most recent five. Verrall's approach holds promise as actuaries become more comfortable with the Bayesian perspective, which can be useful for combining statistics and judgment but which requires "prior" distributions and sophisticated statistical software.

An approach with which actuaries do appear comfortable is based on *scaling*. Panning (2006) argues that loss reserve uncertainty under his method is "scalable." By that he means that his method's coefficient of variation (CV) "is applicable to reserves that have been estimated in different ways" (Panning 2006). Scaling is an actuarial technique utilized in a wide variety of applications. In stochastic analysis the authors are aware that it is common practice to apply a CV based on the Mack method to a chain-ladder point estimate that is based on selected factors other than the all-year volume-weighted average. The authors are concerned that bifurcated point and variability estimates may underestimate the volatility of the underlying claims process.

This paper takes a more direct approach. We show how, under certain restrictions on the selected link ratio, a chain ladder model can be formulated such that the actuary's selection can be considered a "consistent unbiased estimate" of the model. Our chain ladder models are similar to those of Mack and Murphy, but allow for a broader set of "weights" by expanding the domain of the exponent of the beginning value of loss to the entire real line. Using classical regression analysis, variability estimates fall out of the same model. This overcomes the scaling disconnect alluded to above. We also believe our approach is more accessible to practicing actuaries than Verrall's Bayesian approach. Although a drawback of our approach is that our mean square error formulas are more complicated than those of Mack and Murphy, this should not be unexpected for models that allow for a continuum of selected factors rather than just the standard averages. Despite the higher degree of difficulty, our formulas can be calculated in a spreadsheet.

To the authors' knowledge, this is the first paper to posit models that reflect the chain-ladder method

[1]An alternative variance assumption for which the simple average link ratio is the BLUE solution was also provided.

[2]For a mandate on the requirement to exercise judgement in selecting link ratios, see, for example, Friedland (2009). For a survey of how a group of actuaries selected factors under "test conditions" see Blumsohn and Laufer (2009).

[3]For stochastic research related to the chain-ladder method, see Bardis, Majidi, and Murphy (2008), Buchwalder et al. (2006), Mack (1993, 1994), Mack, Quarg, and Braun (2006), Mack (1999), Murphy (1994), Venter (2006), Wright (1990) and Barnett and Zenwirth (2000) in the references. Other prominent research includes Christofides (1997), Panning (2006), Rehman and Klugman (2009) (regression); England and Verrall (2002) (bootstrapping); Verrall (2004, 2007) (Bayesian).

in practice, i.e., when selected factors are other than the volume-weighted or simple averages. The authors believe that by associating the actuary's choice with a model, the selected link ratio can better be back-tested against the observable data, which can add more insight into the reserving exercise. We caution, however, that it is not necessarily possible to identify a chain-ladder model in our framework that is consistent with every potential selected factor. Restrictions are defined in the paper. Of course, the results of our chain ladder model are subject to model error. As with all stochastic models, the actuary must assess the applicability of the indications relative to his or her understanding of the model's assumptions, familiarity with the triangle and other data, and the judgment underlying the factor selections.

The remainder of this paper is organized as follows. In Section 2 we present a family of models that generalizes those in Mack (1993; 1999) and Murphy (1994) and is consistent with the practical implementation of the chain ladder method, because it allows for conformance with a broad set of judgmentally selected factors. In Section 3 we give formulas for the expected value and mean square error of chain ladder projections from selected factors. In Section 4 we demonstrate the concepts and calculations in a worked-through, spreadsheet-based example. Section 5 is a summary. Appendix A includes proofs of our results. Appendix B compares our model's recursive formulas with those of Mack (1999).

2. A chain-ladder model for judgmentally selected link ratios

Adopting notation commonly found in the literature, we denote the observed triangle of positive cumulative losses[4] by $D = \{C_{i,j} \mid 1 \le i \le I,\ 1 \le j \le I\}$. A model equivalent to the chain-ladder method is

$$C_{i,j+1} = f_j C_{i,j} + C_{i,j}^{\alpha_j/2}\,\sigma_j\,\varepsilon_{i,j} \qquad (1)$$

independent random variables $\varepsilon_{i,j}$ have mean 0 and variance 1

[4]"Losses" can refer cumulative paid or case incurred amounts, cumulative counts, or any triangular array of data subject to the given assumptions.

for $1 \le i \le I$ and $1 \le j \le I$. Under these assumptions it is well known (first shown by Aitken 1935) that the best linear unbiased estimate (BLUE) of the link ratio f_j from age j to age $j + 1$ given triangle D, denoted $\hat{f}_j$, is a weighted average of the observed link ratios:

$$\hat{f}_j := \hat{f}_j(\alpha_j) := \sum_{i=1}^{I-j} w_{i,j}^{(\alpha_j)} F_{i,j} \qquad (2)$$

where the weights

$$w_{i,j}^{(\alpha_j)} = \frac{C_{i,j}^{2-\alpha_j}}{\sum_{k=1}^{I-j} C_{k,j}^{2-\alpha_j}} \qquad (3)$$

are functions of the α_j and

$$F_{i,j} = \frac{C_{i,j+1}}{C_{i,j}} \qquad (4)$$

are the observed link ratios based on the triangle.

Model (1) describes a family of models indexed by a continuous parameter $\alpha_j \in \mathbb{R}$. This family contains the models given in Mack (1993; 1999) and Murphy (1994) as special cases, where those authors propose that the α_j indices assume the values 0, 1, and 2, at most.[5] Murphy (1994) demonstrated that for the member indexed by $\alpha_j = 1$ the weighted average link ratio is the best linear unbiased estimate consistent with the model's parameter f; for the $\alpha_j = 2$ member, the simple average link ratio is a consistent estimator; for $\alpha_j = 0$, a consistent link ratio is the slope of a simple regression line through the origin. Model (1) allows the domain of possible values for α to encompass the entire real line rather than just the values 0, 1, and 2. As a result, a continuum of selected factors has the potential to be consistent with Model (1). Put another way, Model (1) allows for an actuary's selected link ratio that is

[5]See also Barnett and Zehnwirth (2000). Murphy considers 0, 1, and 2. Barnett and Zehnwirth consider 1 and 2, denoting the exponent by delta (δ). In his original paper (1993) Mack only considered $\alpha = 1$. Mack (1999) reframed his model in terms of link ratios rather than cumulative loss and extended α to also include 0 and 1; given the new model formulation, the simple average corresponds to $\alpha = 0$ in Mack (1999).

different from the simple or volume-weighted average to be, nevertheless, a linear unbiased estimate of a statistical model consistent with the chain-ladder method.

We refer to Model (1) as the chain-ladder factor model (CLFM). With that as background, there remain the following questions:

- When a selected link ratio is not one of the usual averages, how does one find a member of the CLFM family for which it could be considered consistent?
- How does one calculate the value and the risk of a point estimate under the CLFM framework, and what additional assumptions are needed?

To help answer these questions we introduce the *link ratio function,* a concept fundamental to CLFM theory and results.

2.1. The link ratio function

Definition: Given observations of loss at the beginning and end of development period *j*, the *link ratio function* $\mathrm{LR}_j(\alpha)$ is a mapping on the real line given by

$$\mathrm{LR}_j(\alpha) := \sum_{i=1}^{I-j} w_{i,j}^{(\alpha)} F_{i,j}, (\alpha \in \mathrm{R}) \qquad (5)$$

where $w_{i,j}^{(\alpha)}$ and $F_{i,j}$ are defined in (3) and (4) above.[6] The link ratio function calculates weighted averages of the observed link ratios, where the weights depend on the exponent of loss at the beginning of the period. We begin our investigation of the link ratio function by considering its asymptotic properties as $\alpha \to \pm\infty$.

Lemma 1: Asymptotic properties of the link ratio function

Consider for a given triangle D and development period *j* the set of all possible values of linear estimates (2) as a function of a real valued parameter $\alpha \in \mathbb{R}$. Let $aymin_j$ and $aymax_j$ denote the accident years with the smallest and largest values of loss, respectively, as of the beginning of development period *j*:

$$aymin_j = \min_i \{C_{i,j}\} \text{ and}$$

$$aymax_j = \max_i \{C_{i,j}\} \left(1 \le i \le I - j\right).$$

Then $\lim_{\alpha\to\infty} \mathrm{LR}_j(\alpha) = F_{aymin_j,j}$ and $\lim_{\alpha\to-\infty} \mathrm{LR}_j(\alpha) = F_{aymax_j,j}$. In the case of "ties" for accident years having the smallest or largest beginning value $C_{i,j}$, $\lim_{\alpha\to\infty} \mathrm{LR}_j(\alpha) = \mathrm{mean}\{F_{i,j} | i \in aymin_j\}$ and $\lim_{\alpha\to-\infty} \mathrm{LR}_j(\alpha) = \mathrm{mean}\{F_{i,j} | i \in aymax_j\}$.

The proof can be found in the appendix.

Lemma 1 says that the best linear unbiased estimate of a link ratio for a given development period approaches the link ratio experienced by the accident year with the smallest/largest value of loss at the beginning of the development period as index α approaches +∞/–∞.

To illustrate, suppose losses as of the beginning and end of development period 1 for five accident years are as shown in Table 1. The largest and smallest values of loss as of the beginning of the period are highlighted in yellow.

The link ratio function corresponding to these losses is graphed in Figure 1.

As predicted by Lemma 1, the graph is asymptotic to the line y = 2.500, the link ratio corresponding to accident year 3, and to the line y = 2.101, the link ratio corresponding to accident year 5. The blue line corresponds to the volume-weighted average link ratio (α = 1); the red line to the simple average (α = 2).

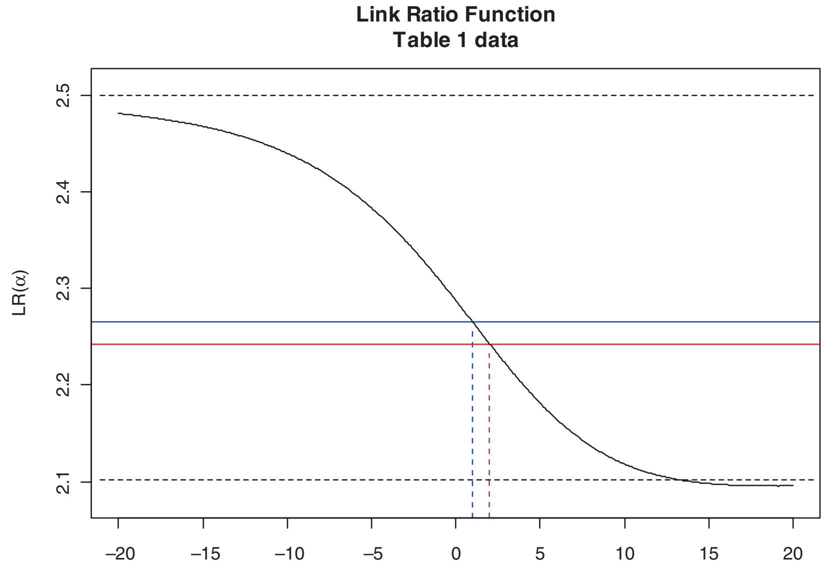
The link ratio function need not be monotonic. Indeed, change the ending value of accident year 5 to

Table 1. Development period 1 losses

$C_{i,j}$	$j = 1$	$j = 2$	$F_{i,1}$
$i = 1$	280	680	2.429
$i = 2$	250	550	2.200
$i = 3$	300	750	2.500
$i = 4$	235	466	1.983
$i = 5$	207	435	2.101
	volume weighted avg.		**2.265**
	simple avg.		**2.243**

[6]We may sometimes omit the subscript *j* when the context of development period *j* is understood.

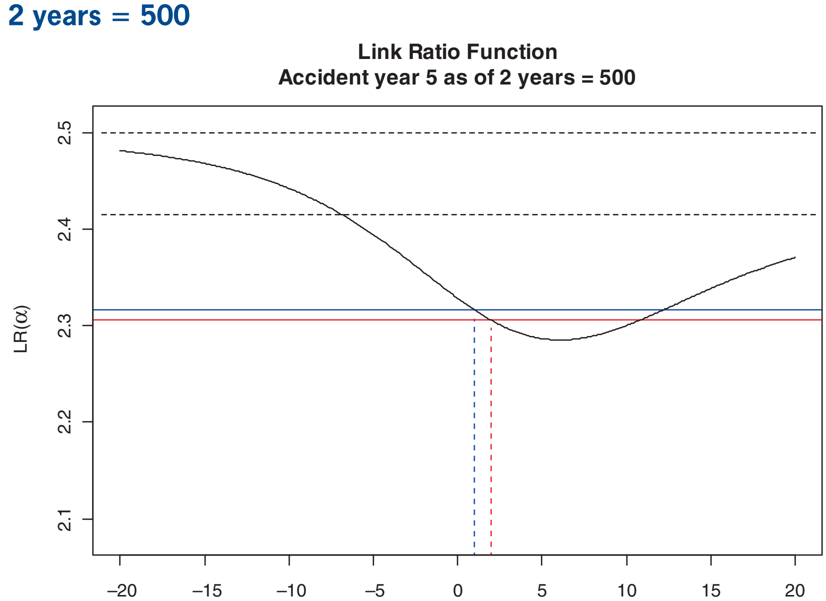
Figure 1. Link ratio function



500. Year 5 would still have the smallest beginning value so its link ratio, now 2.415, would still be the asymptote. The new non-monotonic link ratio function, graphed in Figure 2, has a minimum somewhere in the vicinity of $\alpha = 6$.

From Figures 1 and 2 it should be clear that not all possible link ratios (abscissa) are achievable from a given triangle. In fact, the maximum or minimum empirical link ratio may not even be achievable (the 1.983 link ratio for accident year 4 is literally "off the chart" in Figure 2). Mathematically stated, the image of the link ratio function is not the entire real line. In other words, many link ratio selections would be inconsistent for any member of the CLFM family

Figure 2. Link ratio function: Accident year 5 as of 2 years = 500



relative to a given triangle D.[7] This brings us to our next definition, that of a *reasonable link ratio*.

Definition: A link ratio lr is *reasonable* with respect to a given triangle D if there exists a member of the α-indexed CLFM family for which lr can be calculated as in (5). We denote the set of all reasonable development period j link ratios by $\mathrm{LR}_j(D)$:

$$\mathrm{LR}_j(D) = \left\{ \begin{array}{l} lr \,|\, lr = \mathrm{LR}_j(\alpha) \text{ for some } \alpha \in \mathbb{R}, \\ \text{given triangle } D \end{array} \right\}.$$

Noting that large values of α may lead to impractically large factors $C^{\alpha/2}$ in the error term of (1), we recommend limiting α to a prudently bounded interval; we selected [−8, 8] judgmentally.

A selected link ratio may be associated with more than one value of α (e.g., in Figure 2 the blue, volume-weighted line crosses the graph at more than one point). That is to say, there may be more than one member of the CLFM family whose best linear unbiased estimate is the selected factor. We suggest the following procedure for selecting the *selection-consistent alpha* value.

Definition: The *selection-consistent alpha* of a reasonable link ratio lr_j is the smallest positive solution $\alpha \in [-8, 8]$ of the equation $lr_j = \mathrm{LR}_j(\alpha)$, or, if no positive solution exists, the smallest solution in absolute value. Mathematically this is expressed as

$$\hat{\alpha}_j \equiv \max \left(\begin{array}{l} \min\left(\alpha > 0 \,|\, lr_j = \mathrm{LR}_j(\alpha)\right), \\ \max\left(\alpha \le 0 \,|\, lr_j = \mathrm{LR}_j(\alpha)\right) \end{array} \right).$$

By convention, if the selected link ratio lr_j is the volume-weighted average we set $\hat{\alpha}_j = 1$; for the simple average we set $\hat{\alpha}_j = 2$.

Given a selected link ratio lr_j, the *selection-consistent* member of the CLFM family can be determined by finding positive and negative solutions α of the equation

$$lr_j = lr_j \cdot \sum_{i=1}^{I-j} C_{i,j}^{2-\alpha} - \sum_{i=1}^{I-j} C_{i,j}^{1-\alpha} C_{i,j+1} \qquad (6)$$

[7]It is hoped that the actuary would rely on information beyond the triangle to justify such a selection.

and selecting the smallest positive value if one exists or the negative value closest to the origin.

According to traditional actuarial thinking, the variability of projected loss increases as the beginning value of loss increases, i.e., the value of α in the exponent of C_{ij} in model (1) should be positive. A negative value of α would say that the variability of projected losses is *inversely* proportional to the beginning value, a seemingly counterintuitive result. However, we have found contexts in which such a counterintuitive result is not unreasonable. For example, given a book of first party business with low policy limits, case reserves for "obvious limits losses" would tend to be more certain than reserves on smaller claims. For that situation it would not be unreasonable to find the variability of losses at the end of a calendar period to be inversely proportional to the beginning value of loss. We only suggest that actuaries stay open to the story that data have to tell.

3. CLFM chain-ladder projection formulas

CLFM formulas are recursive because that allows for maximum flexibility in selecting different family members from one period to the next.

3.1. Expected value formulas

We adopt the usual chain ladder convention of developing the current diagonal. For accident year i with current diagonal value $C_{i,j}$ and a selected link ratio $\hat{f}_j$, the expected value at the end of the first future development period is $\hat{C}_{i,j+1} = \hat{C}_{i,j}\hat{f}_j$. This estimate is clearly unbiased if $\hat{f}_j$ is unbiased because $C_{i,j}$ is a scalar. The expected value at the end of the next development period is $\hat{C}_{i,j+2} = \hat{C}_{i,j+1}\hat{f}_{j+1}$. Expected value estimates for subsequent development periods are iterated in a similar fashion.

The estimate $\hat{C}_{i,j+2}$ will be unbiased if we assume that the product of the two estimates $\hat{f}_j$ and $\hat{f}_{j+1}$ equals the product of the two underlying parameters f_j and f_{j+1}. Note that this assumption is implicit in chain ladder calculations where, say, a higher than average link ratio on the current diagonal has no bearing on the factors selected to develop that year going forward.[8]

The expected value of the sum of all accident years combined at development age j is the sum of the estimates of the individual accident years at the same age.

3.2. Standard error formulas

The first step in working with loss variation over a given development period is estimating the scale parameters σ_j, which can easily be found using weighted least squares available in virtually all popular statistical packages. Equivalently, for each development period the data can be transformed into ordinary least squares (OLS) form by dividing the beginning and ending values of loss by the beginning value raised to the power $\alpha/2$. As transformed, model (1) is

$$C_{i,j+1}\big/C_{i,j}^{\alpha_j/2} = f_j C_{i,j}\big/C_{i,j}^{\alpha_j/2} + \sigma_j \varepsilon_{i,j}. \qquad (7)$$

The formula for calculating an estimate $\hat{\sigma}_j^2$ of σ_j^2 can be found in any good statistical text. In the example we illustrate this approach using the LINEST function in Excel.

The next step is to estimate the variability of the selected factors $\hat{f}_j$. The estimate of the conditional variance of those factors, which we denote by Δ,[9] is by definition the quantity $\Delta^2(f_j) := \mathrm{E}\left(\left(\hat{f}_j - \mathrm{E}\left(\hat{f}_j \middle| D\right)\right)^2 \middle| D\right)$. As with the estimates of σ_j^2, these estimates are also standard outputs of regression software.[10]

[8]Mack (1993) proved that weighted average loss development factors are uncorrelated. His proof is an *unconditional* result, however, that does not necessarily hold *conditionally* for a specific triangle. Indeed, it is possible to simulate triangles that have correlated development factors, yet where all assumptions in (1) are satisfied.

[9]We use the delta operator Δ to denote parameter risk and the gamma operator Γ for process risk.

[10]We also use Excel's LINEST function for this estimate. Alternatively one could use the formula (Mack 1999, p. 363) $\Delta^2\left(\hat{f}_j\right) = \frac{\sigma_j^2}{\sum_{i=1}^{n-j} c_{i,j}^{\alpha_j}}$ where weights $w_{i,j} \equiv 1$.

3.2.1. Standard error formulas for an individual accident year

Consider an individual accident year i and its estimate $\hat{C}_{i,j}$ at age j. The mean square error of the estimate $\hat{C}_{i,j}$ is the sum of parameter risk and process risk:

$$\begin{aligned}\mathrm{mse}\left(\hat{C}_{i,j}\right) &= \mathrm{E}\left(\left(\hat{C}_{i,j} - C_{i,j}\right)^2 \middle| D\right)\\ &= \mathrm{E}\left(\left(\hat{C}_{i,I} - C_{i,I}\right)^2 \middle| D\right)\\ &= \mathrm{E}\left(\left(\hat{C}_{i,I} - \mathrm{E}\left(C_{i,I} \middle| D\right)\right)^2 \middle| D\right)\\ &\quad + \mathrm{E}\left(\left(C_{i,I} - \mathrm{E}\left(C_{i,I} \middle| D\right)\right)^2 \middle| D\right)\\ &:= \Delta^2\left(C_{i,I}\right) + \Gamma^2\left(C_{i,I}\right)\end{aligned}$$

Parameter risk (denoted Δ^2) and process risk (denoted Γ^2), notation borrowed from the literature, can be calculated recursively according to the formulas shown next.[11]

3.2.1.1. Parameter risk: Variance of the estimate of the mean future value of loss

For the first period after the current diagonal (s = 1),

$$\Delta^2\left(C_{i,j+1}\right) = C_{i,j}^2 \Delta^2\left(f_j\right) \qquad (8)$$

because $C_{i,j}^2$ is a constant. For $s = 2, 3, \ldots$

$$\begin{aligned}\Delta^2\left(C_{i,j+s}\right) &= \mu_{i,j+s-1}^2 \Delta^2\left(f_{j+s-1}\right) + \hat{f}_{j+s-1}^2 \Delta^2\left(C_{i,j+s-1}\right)\\ &\quad + \Delta^2\left(f_{j+s-1}\right)\Delta^2\left(C_{i,j+s-1}\right)\end{aligned} \qquad (9)$$

where $\mu_{i,j+s-1}^2 := \mathrm{E}\left(C_{i,j+s-1} \middle| D\right)$. Formula (9) is consistent with the formula in Mack (1999) for $\alpha = 1, 2$ except for the third term, which Mack excludes.[12]

[11]Derived in Appendix A.

[12]See Appendix B for more information.

3.2.1.2. Process risk: Variance of the deviation of future value of loss from its mean

For the first period after the current diagonal

$$\Gamma^2\left(C_{i,j+1}\right) = C_{i,j}^{\alpha_j}\hat{\sigma}_j^2. \quad (10)$$

For subsequent periods

$$\Gamma^2\left(C_{i,j+s}\right) = \mathrm{E}\left(C_{i,j+s-1}\middle|D\right)^{\alpha_{j+s-1}} \cdot \Psi\left(\alpha_{j+s-1}, \frac{\Gamma\left(C_{i,j+s-1}\right)}{\mathrm{E}\left(C_{i,j+s-1}\right)}\right) \cdot \sigma_{j+s-1}^2 + f_{j+s-1}^2\Gamma^2\left(C_{i,j+s-1}\right). \quad (11)$$

As noted in the proof in Appendix A, the process risk calculation, drawing upon the Law of Total Variation, involves the expectation $\mathrm{E}(C^\alpha)$ which is not the same as $\mathrm{E}(C)^\alpha$. Since $\mathrm{E}(C)$ is a readily available quantity, Ψ is our "helper" function which, when multiplied by $\mathrm{E}(C)^\alpha$, yields $\mathrm{E}(C^\alpha)$. For example, since $\mathrm{E}(X^2) = \mathrm{E}^2(X) + \mathrm{Var}(X)$, $\mathrm{E}(X^2)/\mathrm{E}^2(X) = 1 + \mathrm{cv}^2(X)$, so $\Psi(2, \kappa) = 1 + \kappa^2$. Clearly $\Psi(1, \kappa) = 1$; and $\Psi(0, \kappa) = 1$ as well. For higher raw moments, the ratio of $\mathrm{E}(C^\alpha)$ to $\mathrm{E}(C)^\alpha$ depends on the distribution; for the normal distributions it is a polynomial in κ. We adopt that simplification for our purposes. Therefore, for non-negative integer values n of alpha we define Ψ as

$$\Psi(\alpha,\kappa) = \sum_{\substack{j=0 \\ j\ even}}^{n} \frac{1 \cdot n \cdot (n-1)\cdots(n-(j-1))}{2^{j/2}\left(\frac{j}{2}\right)!}\kappa^j.$$

For $\alpha > 0$ but not an integer, we define $\Psi(\alpha,\kappa)$ to be the linear interpolation between $\Psi([\alpha],\kappa)$ and $\Psi([\alpha] + 1,\kappa)$ where $[x]$ denotes the greatest integer function. For negative values of α we recommend approximating Ψ using simulation.[13]

3.2.2. Standard error formulas for all accident years combined

Recursive variance formulas for all accident years combined become slightly more complicated because at each new age an additional accident year is included.

For ages $j = 2, 3, \ldots$, let $X_j = \sum_{i=I-j+2}^{I} C_{i,j}$ be the sum of the future losses for accident years that have not yet matured to age j. Let $M_j := \sum_{i=I-j+2}^{I} \mu_{i,j}$ denote the expected value of X_j and let $\hat{X}_j = \sum_{i=I-j+2}^{I} \hat{C}_{i,j}$ be its chain ladder estimate.

3.2.2.1. Parameter risk: Variance of the estimate of the mean future value of total loss

For $j = 2$, only the most recent accident year is included in the total, so the parameter risk of the total is equal to the parameter risk of the most recent year: $\Delta^2(X_2) = \Delta^2(f_1) \cdot C_{I,1}^2$. For $j = 3, 4, \ldots$,

$$\Delta^2\left(X_j\right) = \left(M_{j-1} + C_{I-j+2,\,j-1}\right)^2\Delta^2\left(f_{j-1}\right) + f_{j-1}^2\Delta^2\left(X_{j-1}\right) + \Delta^2\left(f_{j-1}\right)\Delta^2\left(X_{j-1}\right). \quad (12)$$

3.2.2.2. Process risk: Variance of X_j

Model (1) assumes all accident years are independent. Therefore the process variance of the sum of the future values as of a given age is the sum of the process variances:

$$\Gamma^2\left(X_j\right) = \sum_{i=I-j+2}^{I}\Gamma^2\left(C_{i,j}\right). \quad (13)$$

4. An example

We consider the triangle of RAA data analyzed in Mack (1993), Barnett and Zehnwirth (2000) and elsewhere in the literature and illustrate spreadsheet calculations of process risk and parameter risk within the CLFM framework. We selected simple and volume-weighted average link ratios for a few ages, and "judgmental" selections for other periods to demonstrate the concepts. Losses, link ratios, simple and volume-weighted averages and the selections are shown in Table 2.

The mean and standard error estimates based on this triangle D, the selected factors, and the CLFM formulas are summarized in Table 3. We will illustrate the CLFM calculations for a few representative entries.

4.1. Expected value calculations

Table 4 shows the projected chain-ladder values based on the latest diagonal and the selected factors.

[13] See Bardis, Majidi, and Murphy (2008) for more details.

Table 2. RAA data

Losses

AY/Age	1	2	3	4	5	6	7	8	9	10
1	5,012	8,269	10,907	11,805	13,539	16,181	18,009	18,608	18,662	18,834
2	106	4,285	5,396	10,666	13,782	15,599	15,496	16,169	16,704	
3	3,410	8,992	13,873	16,141	18,735	22,214	22,863	23,466		
4	5,655	11,555	15,766	21,266	23,425	26,083	27,067			
5	1,092	9,565	15,836	22,169	25,955	26,180				
6	1,513	6,445	11,702	12,935	15,852					
7	557	4,020	10,946	12,314						
8	1,351	6,947	13,112							
9	3,133	5,395								
10	2,063									

Link Ratios

AY/Dev. Period	1 to 2	2 to 3	3 to 4	4 to 5	5 to 6	6 to 7	7 to 8	8 to 9	9 to 10	
1	1.650	1.319	1.082	1.147	1.195	1.113	1.033	1.003	1.009	
2	40.425	1.259	1.977	1.292	1.132	0.993	1.043	1.033		
3	2.637	1.543	1.163	1.161	1.186	1.029	1.026			
4	2.043	1.364	1.349	1.102	1.113	1.038				
5	8.759	1.656	1.400	1.171	1.009					
6	4.260	1.816	1.105	1.226						
7	7.217	2.723	1.125							
8	5.142	1.887								
9	1.722									
Simple average	8.206	1.696	1.315	1.183	1.127	1.043	1.034	1.018	1.009	
Volume-weighted average	2.999	1.624	1.271	1.172	1.113	1.042	1.033	1.017	1.009	
Selected	8.206	1.624	1.275	1.175	1.115	1.042	1.035	1.018	1.009	1.000

Table 3. CLFM calculations for representative entries

AY/Age	Estimated Ultimate	Current Diagonal	Estimated Unpaid	Total Risk	CV
1	18,834	18,834	—	—	—
2	16,858	16,704	154	9	6.0%
3	24,109	23,466	643	620	96.4%
4	28,781	27,067	1,714	798	46.6%
5	29,006	26,180	2,826	1,500	53.1%
6	19,583	15,852	3,731	1,979	53.0%
7	17,874	12,314	5,560	2,180	39.2%
8	24,266	13,112	11,154	5,606	50.3%
9	16,210	5,395	10,815	6,433	59.5%
10	50,866	2,063	48,803	81,878	167.8%
All	246,387	160,987	85,400	82,838	97.0%

Table 4. Projected loss by accident year and age

AY\ Age	1	2	3	4	5	6	7	8	9	10=Ultimate
1										
2										16,858
3									23,888	24,109
4								28,014	28,519	28,781
5							27,278	28,233	28,741	29,006
6						17,675	18,416	19,061	19,404	19,583
7					14,469	16,133	16,809	17,398	17,711	17,874
8				16,718	19,643	21,902	22,821	23,620	24,045	24,266
9			8,759	11,168	13,122	14,631	15,245	15,778	16,062	16,210
10		16,929	27,485	35,043	41,176	45,911	47,836	49,511	50,402	50,866
All (*X*)		16,929	36,244	62,929	88,410	116,252	148,405	181,614	208,771	227,553

For example, for accident year 10 the projected value in the first future diagonal is the product of the diagonal value and the 1-2 selected factor ($2{,}063 \cdot 8.206 = 16{,}929$). For the next diagonal the projected value is the product of the age 2 projection and the 2-3 selected factor ($16{,}929 \cdot 1.624 = 27{,}485$). The values in the bottom row ("All") are the sums of the values in their respective columns.

4.2. Variability calculations

4.2.1. Selection-consistent alphas

The simple average was selected for development period 1-2 and the volume-weighted average for periods 2-3 and 6-7. Accordingly, the respective selection-consistent alphas are 2 and 1 by convention. For the remaining selections the selection-consistent alphas are the solutions of Equation (6), which we solved in Excel with a Newton-Raphson technique.[14] The values of α shown in Table 5 thus identify selection-consistent members of the CLFM family.

4.2.2. σ^2

We chose the OLS approach to illustrate how to carry out the CLFM calculations in Excel. For example, for period 3-4, $\alpha = 1.158$ (Table 5), the data for the transformed model (7) are given in Table 6, and the LINEST estimate for σ is 13.03.

The 9 to 10 development period has only one observation, insufficient for regression; we used Mack's suggested heuristic [10, p. 363] $\sigma^2_{n-1} = \min(\sigma^4_{n-2}/\sigma^2_{n-3}, \min(\sigma^2_{n-3}, \sigma^2_{n-2}))$. Table 7 summarizes the σ^2 estimates for all development periods.

4.2.3. $\Delta^2(f_j)$

For the standard error of the selected link ratio, denoted in our paper as $\Delta^2(f_j)$, either refer to the output of the software employed—LINEST[15] in our case—or use the formula [(Mack 1999 p. 363); see footnote 10] $\Delta^2(f_j) = \frac{\hat{\sigma}^2_j}{\sum_{i=1}^{n-j} c_{i,j}^{\alpha_j}}$ which we did for the problematic 9-10 development period. Table 8 summarizes these estimates.

4.2.4. Parameter risk (Δ) for projected loss

Parameter risk is estimated recursively in an analogous fashion to the expected value. Table 9 displays the parameter risk estimates by accident year as of each future evaluation and for all accident years combined.

[14] For development years 9-to-10, where we do not have sufficient data to perform a regression, we selected a selection-consistent alpha equal to the one calculated for the 8-to-9 development years.

[15] LINEST labels the estimate of σ as "se_y" and the standard error of the slope parameter as "se_1."

Table 5. Selection-Consistent alpha

1 to 2	2 to 3	3 to 4	4 to 5	5 to 6	6 to 7	7 to 8	8 to 9	9 to 10
2.000	1.000	1.158	1.305	1.117	1.000	2.565	2.005	2.005

Table 6. Transformed data for OLS regression

AY/Age	3	4
1	50.072	54.195
2	37.235	73.600
3	55.408	64.466
4	58.473	78.871
5	58.582	82.009
6	51.577	57.012
7	50.148	56.415

4.2.4.1. $\Delta^2(C)$ for an individual accident year

To illustrate how we calculate these parameter risk estimates for an individual accident year, let's work with accident year 10. For the first period after the current diagonal ($i = 10$ and $j = 2$) we use Formula (8), the actual loss in Table 2, and the link ratio uncertainty estimate from Table 8:

$$\Delta^2(C_{10,2}) = C_{10,1}^2 \cdot \Delta^2(f_1) = 2{,}063^2 \cdot 16.921$$
$$= 72{,}014{,}303$$

For the next development period we use Formula (9), the estimated projected loss $\mu_{10,2}$ from Table 4, the selected link ratio in Table 2, Table 8 and the result of the previous calculation:

$$\Delta^2(C_{10,3}) = \mu_{10,2}^2 \Delta^2(f_2) + f_2^2 \Delta^2(C_{10,2}) + \Delta^2(f_2)\Delta^2(C_{10,2})$$
$$= 16{,}929^2 \cdot 0.018 + 1.624^2 \cdot 72{,}014{,}303$$
$$+ 0.018 \cdot 72{,}014{,}303$$
$$= 196{,}434{,}086.$$

Estimates for the remaining ages are iterated in a similar fashion.

4.2.4.2. Parameter risk: $\Delta^2(X)$ for all accident years combined

For all accident years combined, the parameter risk for age 2 is identical with the parameter risk for accident year 10 alone: $\Delta^2(X_2) = 72{,}014{,}303$. For age 3, we use Formula (12):

$$\Delta^2(X_3) = (M_2 + C_{9,2})^2 \Delta^2(f_2)$$
$$+ \hat{f}_2^2 \Delta^2(X_2) + \Delta^2(f_2)\Delta^2(X_2)$$
$$= (16{,}929 + 5{,}395)^2 \cdot 0.018$$
$$+ 1.624^2 \cdot 72{,}014{,}303 + 0.018 \cdot 72{,}014{,}303$$
$$= 200{,}341{,}585.$$

The value for $M_2 = E(X_2)$ comes from Table 4, the actual diagonal value $C_{9,2}$ from Table 3 and the value of $\Delta^2(X_2)$ from the previous recursion step. Estimates for the remaining ages are iterated in a similar fashion.

4.2.5. Process risk (Γ) for projected loss

Table 10 summarizes the process risk estimates by accident year and for all accident years combined. The process risk estimates for all accident years combined is the sum of the process risk estimates for the individual accident years. The process risk estimates for individual accident years are calculated recursively. We illustrate with accident year 10.

Table 7. σ^2 estimates

1 to 2	2 to 3	3 to 4	4 to 5	5 to 6	6 to 7	7 to 8	8 to 9	9 to 10
152.287	1,108.526	169.856	3.327	37.370	40.820	0.00000029	0.00044	0.00000029

Table 8. $\Delta^2(f_j)$

1 to 2	2 to 3	3 to 4	4 to 5	5 to 6	6 to 7	7 to 8	8 to 9	9 to 10
16.921	0.018	0.009	0.001	0.001	0.001	0.000025	0.00023	0.00000000000000079

Table 9. Parameter risk estimates—$\Delta^2(C_{i,j})$

AY\ Age	2	3	4	5	6	7	8	9	10=Ultimate
1									
2									0
3								125,438	127,761
4							17,980	197,415	201,070
5						349,384	392,514	588,435	599,331
6					311,530	497,615	541,359	643,908	655,832
7				102,711	387,361	553,402	599,728	690,599	703,388
8			1,537,823	2,313,511	3,357,455	3,891,196	4,180,951	4,460,856	4,543,463
9		537,048	1,564,071	2,244,932	3,007,204	3,375,358	3,621,318	3,810,392	3,880,954
10	72,014,303	196,434,086	327,842,268	453,681,119	566,692,078	616,580,023	660,524,087	685,225,577	697,914,670
All	72,014,303	200,341,585	349,261,694	486,270,855	618,623,671	682,251,827	731,569,874	767,890,482	782,110,374

Table 10. Process risk estimates—$\Gamma^2(C_{i,j})$

AY\ Age	2	3	4	5	6	7	8	9	10=Ultimate
1									
2									84
3								251,221	256,045
4							67,040	427,841	436,009
5						1,068,664	1,213,302	1,621,884	1,652,168
6					1,828,861	2,708,217	2,926,316	3,199,551	3,258,915
7				727,566	2,556,825	3,435,687	3,700,424	3,974,441	4,048,136
8			9,974,886	14,867,845	20,818,156	23,495,904	25,215,193	26,397,237	26,886,246
9		5,980,499	16,050,554	22,825,400	29,880,893	33,037,943	35,408,793	36,824,611	37,506,622
10	648,128,730	1,727,121,088	2,839,654,629	3,925,360,699	4,886,849,026	5,307,176,777	5,686,523,927	5,896,827,944	6,006,028,710
All	648,128,730	1,733,101,587	2,865,680,069	3,963,781,510	4,941,933,761	5,370,923,191	5,755,054,995	5,969,524,731	6,080,072,937

For the first period after the current diagonal (i = 10 and j = 2), we use Formula (10), the actual loss in Table 2, and the scale parameter estimate from Table 7:

$$\Gamma^2\left(C_{10,2}\right) = C_{10,1}^{\alpha_1}\hat{\sigma}_1^2 = 2{,}063^2 \cdot 152.287 = 648{,}128{,}730.$$

For the next development period (j = 3) we use Formula (11):

$$\begin{aligned}\Gamma^2\left(C_{10,3}\right) &= \mathrm{E}\left(C_{10,2}\middle|D\right)^{\alpha_2} \cdot \Psi\left(\alpha_2, \frac{\Gamma\left(C_{10,2}\right)}{\mathrm{E}\left(C_{10,2}\right)}\right) \cdot \hat{\sigma}_2^2 \\ &\quad + \hat{f}_2^2\Gamma^2\left(C_{10,2}\right) \\ &= 16{,}929^{1.000} \cdot 1 \cdot 1{,}108.526 \\ &\quad + 1.624^2 \cdot 648{,}128{,}730 \\ &= 1{,}727{,}121{,}088.\end{aligned}$$

because $\Psi(\alpha, \kappa) \equiv 1$ when $\alpha = 1$. For the process risk at age $j = 4$ where $\alpha_3 = 1.158$ we linearly interpolate between $\Psi(1, \kappa) = 1$ and $\Psi(2, \kappa) = 1 + \kappa^2$ where $\kappa = \sqrt{1{,}727{,}121{,}088 \big/ 27{,}485} = 1.512$ and get $\Psi(1.158, 1.51) = 1 + (1.158 - 1)(1.512)^2 = 1.362$. So

$$\begin{aligned}\Gamma^2\left(C_{10,4}\right) &= \mathrm{E}\left(C_{10,3}\middle|D\right)^{\alpha_3} \cdot \Psi\left(\alpha_3, \frac{\Gamma\left(C_{10,3}\right)}{\mathrm{E}\left(C_{10,3}\right)}\right) \cdot \hat{\sigma}_3^2 \\ &\quad + f_3^2\Gamma^2\left(C_{10,3}\right) \\ &= 27{,}485^{1.158} \cdot 1.362 \cdot 169.856 \\ &\quad + 1.275^2 \cdot 1{,}727{,}121{,}088 \\ &= 2{,}839{,}654{,}629.\end{aligned}$$

Estimates for the remaining ages are iterated in a similar fashion.

4.3. Comparison of the CLFM vs. the Mack method

The question of how the CLFM and Mack results compare often arises.[16] As we understand the popular practice of the method of Mack (1993), the Mack method CV assuming weighted average link ratios and all years in the triangle would be applied to the point estimate based on a different set of factors. The Mack method CV from the RAA data is 51.6%.[17] This is about half the CLFM CV in Table 3. Thus, the CLFM risk estimate would be about twice the value of the risk estimate from the Mack method as we understand its common implementation in practice.

5. Summary

This paper presents a family of models that is consistent with the implementation of the chain-ladder method as used in practice. Our approach is different from the methods of Mack (1993, 1999) and Murphy (1994) because, whereas their models assume that the selected chain-ladder link ratio is a volume-weighted or simple average, our model accepts an actuary's judgmentally selected factor as a fundamental input. By enlarging the domain of the exponent of the chain ladder method's "explanatory variable" (the value of loss at the beginning of the development period) in its influence on modeling loss development variability, our approach allows for many more selected link ratios than just the usual averages to be considered best linear unbiased estimates within a chain-ladder-consistent stochastic model. As a result, point estimates and risk estimates of unpaid claim liabilities can be calculated simultaneously. This avoids the need to *scale* chain ladder point estimates based on one model (selected factors) with CVs based on a different model (e.g., volume-weighted or simple averages) or with CVs based on a different methodology entirely (e.g., bootstrapping). Our approach can be implemented in a spreadsheet, thus avoiding the need for more sophisticated statistical software.

The theory of our approach and illustrated in the example suggests that scaling a chain-ladder point estimate with a Mack method CV based on the all-year volume-weighted average will understate the

[16]Most recently by a reviewer of the paper.

[17]This cv can be produced by the formula in Mack (1993) or by the approach herein, where unity α is selected for all development periods.

standard error of the projections; the greater the difference between the actuary's selections and the volume-weighted averages, the greater the understatement.

It goes without saying that to model loss development within the CLFM family does not eliminate model risk, an inescapable side effect of any statistical model by definition. The authors also caution that it is not necessarily possible to identify a CLFM family member that is consistent with every potential link ratio selection. Refer to the constraints outlined in the paper.

Various reviewers have suggested that the alpha index that identifies a member of a CLFM family can be considered a "parameter" rather than an "index" and therefore some component of the model risk might possibly be quantified by an estimate of that parameter's estimation risk. The authors had indeed investigated that work stream within a maximum likelihood context. Although the mathematics was interesting, that research thread was abandoned because there was no guarantee that the likelihood maximizing value of alpha would index the CLFM member consistent with the actuary's selection. Others may find this work stream more fruitful, but our primary goal was to identify selection-consistent models that cater to the needs of practitioners who select development factors based on judgment on a daily basis.

For diagnostics regarding the selections relative to potential trends in the triangle, we refer the reader to our first paper (Bardis, Majidi, and Murphy 2008).

The authors also wish to point out the CLFM framework assumes that the only available data that might shed light on link ratio uncertainty is the triangle alone. When exogenous data help determine factor selection, unpaid claim estimate uncertainty will undoubtedly be improved by incorporating additional sources of pertinent quantifiable information within a broader model that is not limited to the triangle alone. We anticipate much research in that area in the future.

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Appendix A

Proof of Lemma 1 (Link Ratio Function)

1. We first note that for arbitrary α we have

$$\sum_{i=1}^{I-j} w_{i,j}^{\alpha} = 1. \qquad \text{(A)}$$

Without loss of generality we can assume $C_{aymin_j,j} < C_{i,j}$ for $i \le I - j$. It is now sufficient to prove that $w^{\alpha}_{aymin_j,j} \to 1$ as $\alpha \to \infty$. This can be proven by rewriting the weight as

$$w^{\alpha}_{aymin_j,j} = C^{2-\alpha}_{aymin_j,j} \Big/ \sum_{k=1}^{I-j} C^{2-\alpha}_{k,j} = C^{2}_{aymin_j,j} \Big/ \sum_{k=1}^{I-j} C^{2}_{k,j} \cdot \left(C_{aymin_j,j} / C_{k,j}\right)^{\alpha}.$$

Obviously $(C_{aymin_j}/C_{k,j}) < 1$ for all $k \neq aymin_j$. Thus all terms converge to 0 except for $k = aymin_j$, so that $\sum_{k=1}^{I-j} C^2_{k,j} \cdot (C_{aymin_j}/C_{k,j})^{\alpha} \to C^2_{aymin_j,j}$ as $\alpha \to \infty$. That proves $w^{\alpha}_{aymin_j} \to 1$ as $\alpha \to \infty$ and subsequently $w^{\alpha}_{ij,j} \to 0$ as $\alpha \to \infty$ for all $i \neq aymin_j$ based on (A). The proposition is then obvious: $\lim_{\alpha\to\infty} \mathrm{LR}_j(\alpha) = F_{aymin_j,j}$.

2. The proof that $\lim_{\alpha\to-\infty} \mathrm{LR}_j(\alpha) = F_{aymax_j,j}$ is similar to 1.

The generalization to the case where the accident years having the minimum/maximum beginning values of loss are not unique is obvious, as the limits of the corresponding weights are 1 as well.

Proof of the Parameter Risk Formulas—single accident year

For the first period after the current diagonal, $\hat{C}_{i,k+1} = \hat{f}_k C_{i,k}$, so $\Delta^2(C_{i,k+1}) = C^2_{i,k}\Delta^2(f^2_k)$ because $C^2_{i,k}$ is a constant. For s > 1 periods after the current diagonal, $\hat{C}_{i,k+s} = \hat{f}_{k+s-1}\hat{C}_{i,k+s-1}$, so based on the "law of total variance":

$$\begin{aligned}
\Delta^2(C_{i,k+s}) &= \mathrm{E}\left(\mathrm{Var}\left(\hat{C}_{i,k+s} \middle| \hat{C}_{i,k+s-1}\right)\right) \\
&\quad + \mathrm{Var}\left(\mathrm{E}\left(\hat{C}_{i,k+s} \middle| \hat{C}_{i,k+s-1}\right)\right) \\
&= \mathrm{E}\left(\hat{C}^2_{i,k+s-1}\mathrm{Var}\left(\hat{f}_{k+s-1}\right)\right) \\
&\quad + \mathrm{Var}\left(\hat{C}_{i,k+s-1}\,\mathrm{E}\left(\hat{f}_{k+s-1}\right)\right) \\
&= \mathrm{Var}\left(\hat{f}_{k+s-1}\right)\mathrm{E}\left(\hat{C}^2_{i,k+s-1}\right) \\
&\quad + \mathrm{Var}\left(\hat{C}_{i,k+s-1} f_{k+s-1}\right) \\
&= \mathrm{Var}\left(\hat{f}_{k+s-1}\right)\left(\mathrm{Var}\left(\hat{C}_{i,k+s-1}\right) + \mathrm{E}^2\left(\hat{C}_{i,k+s-1}\right)\right) \\
&\quad + f^2_{k+s-1}\mathrm{Var}\left(\hat{C}_{i,k+s-1}\right) \\
&= \mu^2_{i,k+s-1}\Delta^2\left(f_{k+s-1}\right) + f^2_{k+s-1}\Delta^2\left(C_{i,k+s-1}\right) \\
&\quad + \Delta^2\left(f_{k+s-1}\right)\Delta^2\left(C_{i,k+s-1}\right).
\end{aligned}$$

Proof of the Process Risk Formulas—single accident year

For the first period after the current diagonal, $\Gamma(C_{i,k+1}) = C^{\alpha_k}_{i,k}\sigma^2_k$. For $s > 1$ periods after the current diagonal, process risk can be calculated recursively according to the formula

$$\Gamma^2\left(C_{i,k+s}\right) = f^2_{k+s-1} \cdot \Gamma^2\left(C_{i,k+s-1}\right) + \mathrm{E}\left(C^{\alpha_{k+s-1}}_{i,k+s-1} \middle| D\right)\sigma^2_{k+s-1}.$$

Proof:

For the first period after its current age ($s = 1$) the process risk for $C_{i,k+1}$ is a direct result of assumption (1):

$$\Gamma^2\left(C_{i,k+1}\right) = C_{i,k}^{\alpha_k}\sigma_k^2$$

because $C_{i,k}^{\alpha_k}$ is a known constant.

For $s > 1$ we again rely on the "law of total variance":

$$\begin{aligned}\Gamma^2\left(C_{i,k+s}\right) &= \mathrm{E}\left(\mathrm{Var}\left(C_{i,k+s}\middle|D\right)\right) + \mathrm{Var}\left(\mathrm{E}\left(C_{i,k+s}\middle|D\right)\right)\\ &= \mathrm{E}\left(C_{i,k+s-1}^{\alpha_{k+s-1}}\middle|D\right) + \mathrm{Var}\left(\mathrm{E}\left(f_{k+s-1}C_{i,k+s-1}\middle|D\right)\right)\\ &= \mathrm{E}\left(C_{i,k+s-1}^{\alpha_{k+s-1}}\middle|D\right)\sigma_{k+s-1}^2 + f_{k+s-1}^2\Gamma^2\left(C_{i,k+s-1}\right)\end{aligned}$$

As explained in the text, in practice we favor approximating $\mathrm{E}\left(C_{i,k+s-1}^{\alpha_{k+s-1}}\middle|D\right)$ with $\left(\mathrm{E}\left(C_{i,k+s-1}\middle|D\right)\right)^{\alpha_{k+s-1}} \cdot \Psi$, where factor Ψ is a function of α and the coefficient of variation κ.

For estimates of Γ^2 we replace all unknown quantities by their best estimates: f_k by $\hat{f}_k$, σ_k^2 by $\hat{\sigma}_k^2$, etc. Again we note here that $\hat{\sigma}_k^2$ and $\hat{f}_k^2$ both depend on $\hat{\alpha}_k$. However, we drop the functional notation $\hat{\sigma}_k^2(\hat{\alpha}_k)$ and $\hat{f}_k^2(\hat{\alpha}_k)$ for convenience of presentation.

Proof of the Parameter Risk Formulas—all accident years combined

For $j = 3, 4, \ldots, \hat{X}_j = \hat{f}_{j-1} \cdot (\hat{X}_{j-1} + C_{I-j+2,j-1})$, where $I - j + 2$ is the only accident year that has matured as of age $j - 1$. By employing the "law of total variance" mentioned above, we have:

$$\begin{aligned}\Delta^2\left(X_j\right) &= \mathrm{E}\left(\mathrm{Var}\left(\hat{X}_j\middle|\hat{X}_{j-1}\right)\right) + \mathrm{Var}\left(\mathrm{E}\left(\hat{X}_j\middle|\hat{X}_{j-1}\right)\right)\\ &= \mathrm{E}\left(\mathrm{Var}\left(\hat{f}_{j-1}\left(\hat{X}_{j-1} + C_{I-j+2,j-1}\right)\middle|\hat{X}_{j-1}\right)\right)\\ &\quad + \mathrm{Var}\left(\mathrm{E}\left(\hat{f}_{j-1}\left(\hat{X}_{j-1} + C_{I-j+2,j-1}\right)\middle|\hat{X}_{j-1}\right)\right)\\ &= \mathrm{E}\left(\left(\hat{X}_{j-1} + C_{I-j+2,j-1}\right)^2\mathrm{Var}\left(\hat{f}_{j-1}\middle|\hat{X}_{j-1}\right)\right)\\ &\quad + \mathrm{Var}\left(\left(\hat{X}_{j-1} + C_{I-j+2,j-1}\right)\mathrm{E}\left(\hat{f}_{j-1}\middle|\hat{X}_{j-1}\right)\right)\\ &= \Delta^2\left(f_{j-1}\right)\mathrm{E}\left(\left(\hat{X}_{j-1} + C_{I-j+2,j-1}\right)^2\right)\\ &\quad + \mathrm{Var}\left(f_{j-1}\left(\hat{X}_{j-1} + C_{I-j+2,j-1}\right)\right)\\ &= \Delta^2\left(f_{j-1}\right)\left\{\mathrm{Var}\left(\hat{X}_{j-1}\right) + \mathrm{E}^2\left(\hat{X}_{j-1} + C_{I-j+2,j-1}\right)\right\}\\ &\quad + f_{j-1}^2\mathrm{Var}\left(\hat{X}_{j-1}\right)\\ &= \left(M_{j-1} + C_{I-j+2,j-1}\right)^2\Delta^2\left(f_{j-1}\right) + f_{j-1}^2\Delta^2\left(X_{j-1}\right)\\ &\quad + \Delta^2\left(f_{j-1}\right)\Delta^2\left(X_{j-1}\right)\end{aligned}$$

because $C_{I-j+2,j-1}$ is a constant.

Proof of the Process Risk Formulas—all accident years combined

The formula for process risk is straightforward since all accident years are assumed to be independent and the process variance of the sum of the losses for all accident years is the sum of the process variance of each accident year.

Appendix B

The Mack (1999) model is based on the assumptions that $E(F_j|C_j) = f_j$ and $\mathrm{Var}(F_j|C_j) = \frac{\sigma_j^2}{C_j^\alpha}$ where, for simplicity, we omit his accident year index i and assume that all weights are equal to 1. Mack (1999) calculates standard error recursively as follows:

$$\text{s.e.}^2(\hat{C}_{k+1}) = \hat{C}_k^2\left(\text{s.e.}^2(F_k) + \text{s.e.}^2(\hat{f}_k)\right) + \hat{f}_k^2\ \text{s.e.}^2(\hat{C}_k).$$

Case 1: Volume-weighted average link ratios

In the Mack framework the volume-weighted average case is achieved for $\alpha = 1$. Thus

$$\text{s.e.}^2(\hat{C}_{k+1}) = \hat{C}_k^2\left(\frac{\hat{\sigma}_k^2}{\hat{C}_k} + \text{s.e.}^2(\hat{f}_k)\right) + \hat{f}_k^2\ \text{s.e.}^2(\hat{C}_k) \Leftrightarrow$$

$$\text{s.e.}^2(\hat{C}_{k+1}) = \hat{C}_k\sigma_k^2 + \hat{C}_k^2\text{s.e.}^2(\hat{f}_k) + \hat{f}_k^2\ \text{s.e.}^2(\hat{C}_k). \qquad \text{(I)}$$

Within the CLFM framework the volume-weighted average case is also achieved for $\alpha = 1$. The CLFM formula for mean square error (mse) in Mack's notation (s.e.^{2}) is

$$\text{s.e.}^2(\hat{C}_{k+1}) = \Delta^2(C_{k+1}) + \Gamma^2(C_{k+1}) \quad \text{(from (9) and (11), respectively)}$$

$$= \left\{\begin{matrix}\hat{C}_k^2\Delta^2(f_k) + \hat{f}_k^2\Delta^2(C_k) \\ +\Delta^2(f_k)\Delta^2(C_k)\end{matrix}\right\} + \left\{E(\hat{C}_k)\sigma_k^2 + \Gamma^2(C_k)\hat{f}_k^2\right\}$$

$$= E(\hat{C}_k)\sigma_k^2 + \hat{C}_k^2\Delta^2(f_k) + \hat{f}_k^2\left[\Delta^2(C_k) + \Gamma^2(C_k)\right] + \Delta^2(f_k)\Delta^2(C_k)$$

$$= E(\hat{C}_k)\sigma_k^2 + \hat{C}_k^2\Delta^2(f_k) + \hat{f}_k^2\ \text{s.e.}^2(\hat{C}_k) + \Delta^2(f_k)\Delta^2(C_k) \Leftrightarrow$$

$$\text{s.e.}^2(\hat{C}_{k+1}) = \hat{C}_k\sigma_k^2 + \hat{C}_k^2\Delta^2(f_k) + \hat{f}_k^2\ \text{s.e.}^2(\hat{C}_k) + \Delta^2(f_k)\Delta^2(C_k). \qquad \text{(II)}$$

The last "cross-variance" term in (II), i.e., $\Delta^2(f_k)\,\Delta^2(C_k)$, is not included in the Mack's volume-weighted average formula (I). This is a well-known result.[18]

Case 2: Simple average link ratios

In the Mack framework the simple average case is achieved for $\alpha = 0$. Thus

$$\text{s.e.}^2(\hat{C}_{k+1}) = \hat{C}_k^2\left(\sigma_k^2 + \text{s.e.}^2(\hat{f}_k)\right) + \hat{f}_k^2\ \text{s.e.}^2(\hat{C}_k) \quad \Leftrightarrow$$

$$\text{s.e.}^2(\hat{C}_{k+1}) = \hat{C}_k^2\sigma_k^2 + \hat{C}_k^2\ \text{s.e.}^2(\hat{f}_k) + \hat{f}_k^2\ \text{s.e.}^2(\hat{C}_k). \qquad \text{(III)}$$

Within the CLFM framework the simple average case is achieved for $\alpha = 2$. Again using Mack's notation, the CLFM mse formula is

$$\text{s.e.}^2(\hat{C}_{k+1}) = \Delta^2(C_{k+1}) + \Gamma^2(C_{k+1})$$

$$= \left\{\hat{C}_k^2\Delta^2(f_k) + \hat{f}_k^2\Delta^2(C_k) + \Delta^2(f_k)\Delta^2(C_k)\right\} + \left\{E(\hat{C}_k^2)\sigma_k^2 + \Gamma^2(C_k)\hat{f}_k^2\right\}$$

[18] See, for example, Buchwalder et al. (2006).

$$= \mathrm{E}\left(\hat{C}_k^2\right)\sigma_k^2 + \hat{C}_k^2\Delta^2\left(f_k\right)$$

$$+ \hat{f}_k^2\left[\Delta^2\left(C_k\right) + \Gamma^2\left(C_k\right)\right] + \Delta^2\left(f_k\right)\Delta^2\left(C_k\right)$$

$$= \left[\mathrm{E}\left(\hat{C}_k\right)^2 + \Gamma^2\left(C_k\right)\right]\sigma_k^2 + \hat{C}_k^2\Delta^2\left(f_k\right)$$

$$+ \hat{f}_k^2 \text{ s.e.}^2\left(\hat{C}_k\right) + \Delta^2\left(f_k\right)\Delta^2\left(C_k\right) \quad \Leftrightarrow$$

$$\text{s.e.}^2\left(\hat{C}_{k+1}\right) = \hat{C}_k^2\sigma_k^2 + \hat{C}_k^2\Delta^2\left(f_k\right) + \hat{f}_k^2\text{s.e.}^2\left(\hat{C}_k\right)$$

$$+ \Delta^2\left(f_k\right)\Delta^2\left(C_k\right) + \Gamma^2\left(C_k\right)\sigma_k^2. \qquad \text{(IV)}$$

So the difference between the CLFM and Mack formulas for mean square error in the simple average case is comprised of the last two "cross-variance terms" in (IV), i.e., $\Delta^2(f_k)\ \Delta^2(C_k) + \Gamma^2(C_k)\sigma_k^2$. As far as the authors can tell, this comparison is a new result.

In both cases, mse estimates based on Mack's formulas will be smaller than those based on CLFM formulas by a magnitude equal to the "additional terms." For most relatively stable triangles, the cross-variance terms will have relatively little impact. But when there is considerable volatility in the empirical loss ratios, the magnitude of the cross-variance terms can be significant. (This was demonstrated for the volume-weighted case in the example). In the straight average case the other term $\Gamma^2(C_k)\sigma_k^2$ not included in Mack's formula can overshadow the cross-variance term, as it does with the Example data (analysis omitted above). When the judgmentally selected link ratio is not one of these two cases, the differences between the CLFM and Mack mse estimators will depend on the proximity of the selection to the straight average and volume-weighted average cases.