General Iteration Algorithm for Classification Ratemaking

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ABSTRACT

In this study, we propose a flexible and comprehensive iteration algorithm called "general iteration algorithm" (GIA) to model insurance ratemaking data. The iteration algorithm is a generalization of a decades-old iteration approach known as "minimum bias models." We will demonstrate how to use GIA to solve all the multiplicative minimum bias models published to date and the commonly used multiplicative generalized linear models (GLMs), such as gamma, Poisson, normal, and inverse Gaussian models. In addition, we will demonstrate how to apply GIA to solve the broad range of GLM models, mixed additive and multiplicative models, and constraint-optimization problems that pricing actuaries often deal with in their practical work.

KEYWORDS

GIA, GLM, classification ratemaking, weighted average

1. Introduction

Insurance rating for property and casualty lines of business went through a great expansion after World War II. The expansion laid down the foundation for modern rating plans, which are fairly complex in that they typically consist of a wide range of rating factors. However, it also created a significant challenge for the insurance industry in how to determine the optimal values for each rating variable in the plan. For example, a typical personal automobile rating plan contains garage territory, driver age, driver gender, driver marital status, vehicle usage, driving distance, vehicle model year, vehicle symbols, driver history (accidents and violations), and a number of special credits and debits such as multi-car discounts, driving school discounts, and good student discounts.

To respond to the challenge of rating plan expansion, Bailey and Simon [2] and Bailey [1] proposed a "heuristic" iteration approach called "minimum bias models," which utilizes an iterative procedure in determining simultaneously the "optimal" values for the rating variables. During iteration, the procedure will minimize a target "bias" function. Compared to the traditional one-way or two-way analysis, such "multivariate procedures" can reduce estimation errors. Until recent interest in generalized linear models (GLMs), the minimum bias approach was the major technique used by property and casualty pricing actuaries in determining the rate relativities for a class plan with multiple rating variables.

We will illustrate how the minimum bias approach can be used to derive indicated class plan factors. Because multiplicative models are more popular than additive ones, we will focus first and primarily on multiplicative models. Later, we will also illustrate how to generalize the approach by developing additive and mixed additive-multiplicative models.

Assume that we are conducting a two-variable (X and Y) rating plan analysis based on loss cost. Variable X has a total of m categories of values, variable Y has a total of n categories of values, and the categories are represented by the subscript of i (from $1,2,\ldots,m$) and j (from $1,2,\ldots,n$). Define $r_{i,j}$ as the observed loss cost relativity, and $w_{i,j}$ as the earned exposures or weight for the classification i and j for variables X and Y, respectively, and let x_i and y_j be the relativities for classification i and classification j, respectively. The multiplicative rating plan proposed by Bailey [1] is:

$$\mathrm{E}(r_{i,j}) = x_i y_j, \qquad \text{where} \quad i = 1,2,\ldots,m \quad \text{and} \quad j = 1,2,\ldots,n.$$

With the above multiplicative formula, one type of "error" proposed by Bailey is to measure the difference between the "estimated cost" and the "observed cost." The errors across the variable Y are $\sum_{j=1}^{n} w_{i,j}(r_{i,j} - x_i y_j)$ for $i = 1,2,\ldots,m$. The errors across the variable X are $\sum_{i=1}^{m} w_{i,j} \cdot (r_{i,j} - x_i y_j)$ for $j = 1,2,\ldots,n$.

When the above errors are set to "zero" for every X and Y, it can be shown that the estimated relativities, $\hat{x}_i$, $\hat{y}_j$, can be derived iteratively as follows:

Algorithm 1:

$$\hat{x}_i = \frac{\sum_j w_{i,j} r_{i,j}}{\sum_j w_{i,j} y_j} \qquad \hat{y}_j = \frac{\sum_i w_{i,j} r_{i,j}}{\sum_i w_{i,j} x_i}. \tag{1.1}$$

Strictly speaking, it is somewhat misleading to describe Bailey's approach as "minimum bias models." First, what Bailey proposed is essentially an iteration algorithm, not a set of statistical models. The iterative procedure is a "fixed point iteration technique" commonly employed in numeric analysis for root finding. Second, the "error" function above is not consistent with the bias concept in statistics. Bias generally refers to the difference between the mean of an estimator and the true value of the parameter being estimated. For example, suppose we are trying to estimate relativity x_i using an estimator $\hat{x}_i$ (which is some function of observed data). Then the bias is defined as $E(\hat{x}_i) - x_i$. If $E(\hat{x}_i) - x_i = 0$, $\hat{x}_i$ is called an unbiased estimator of x_i. Although Algorithm 1 does not measure the bias of the mean estimated relativity from the true value, the approach has long been recognized as the "minimum bias" method by actuaries. In fact, it is essentially a cross-classification estimation algorithm. In this paper, we describe our new and generalized approach, the "general iteration algorithm" (GIA), which has greater statistical rigor.

Brown [3] was the first one to introduce statistical models and link Bailey and Simon's minimum bias approach to the maximum likelihood estimations of the statistical theories:

$$L_{i,j} = Br_{i,j} = Bx_i y_j + \varepsilon_{i,j},$$

where $L_{i,j}$ is the observed loss cost, B is the base, $\varepsilon_{i,j}$ is a random error, and $L_{i,j}$ follows a statistical distribution. Returning to Algorithm 1, it can be proven that Algorithm 1 is equivalent to applying the maximum likelihood (ML) method with an assumption that $L_{i,j}$ follows a Poisson distribution. Therefore, the results from Algorithm 1 are the same as those from the "ML Poisson model."

With the introduction of statistical theories and statistical models to the minimum bias approach, Brown further expanded the approach with four more minimum bias algorithms (three multiplicative and one additive) by assuming different distributions for $L_{i,j}$ (or $r_{i,j}$):

Algorithm 2:

$$\hat{x}_i = \frac{1}{n}\sum_j \frac{r_{i,j}}{y_j}. \tag{1.2}$$

Algorithm 2 assumes that $L_{i,j}$ follows an exponential distribution.

Algorithm 3:

$$\hat{x}_i = \frac{\sum_j w_{i,j}^2 r_{i,j} y_j}{\sum_j w_{i,j}^2 y_j^2}. \tag{1.3}$$

Algorithm 3 is equivalent to an ML normal model.

Algorithm 4:

$$\hat{x}_i = \frac{\sum_j w_{i,j} r_{i,j} y_j}{\sum_j w_{i,j} y_j^2}. \tag{1.4}$$

Algorithm 4 results from the least squares model.

Another minimum bias algorithm proposed by Bailey and Simon [2] has a complicated format:

Algorithm 5:

$$\hat{x}_i = \left(\frac{\sum_j w_{i,j} r_{i,j}^2 y_j^{-1}}{\sum_j w_{i,j} y_j}\right)^{1/2}. \tag{1.5}$$

Feldblum and Brosius [5] summarized these minimum bias algorithms into four categories: "balance principle," "maximum likelihood," "least squares," and "χ-squared."

- Algorithm 1 could be derived from the so-called "balance principle," that is, *"the sum of the indicated relativity = the sum of observed relativity."* Such a balance relationship can be formulated as:

$$\sum_j w_{i,j} r_{i,j} = \sum_j w_{i,j} x_i y_j.$$

- Algorithms 1, 2, and 3 can be derived from the associated log likelihood functions of observed pure premium relativities.
- Algorithm 4 can be derived by minimizing the sum of the squared errors:

$$\min_{x,y} \sum_{i,j} w_{i,j}(r_{i,j} - x_i y_j)^2.$$

- Algorithm 5 can be derived by minimizing the "χ-squared" error, the squared error divided by the indicated relativity:

$$\min_{x,y} \sum_{i,j} w_{i,j} \frac{(r_{i,j} - x_i y_j)^2}{x_i y_j}.$$

In his milestone paper, Mildenhall [9] further demonstrated that classification rates determined by various linear bias functions are essentially the same as those from GLMs. One main advantage of using statistical models such as GLM is that the characteristics of the models, such as the parameters' confidence intervals and hypothesis testing, can be thoroughly studied and determined by statistical theories. Also, the contribution and significance of the variables in the models can be statistically evaluated. Another advantage is that GLMs are more efficient because they do not require actuaries to program the iterative process in determining the parameters.[1] However, this advantage can be discounted somewhat due to the powerful calculation capability of modern computers. Due to these advantages, GLMs have become more popular in recent years. Of course, actuaries need to acquire the necessary statistical knowledge in understanding and applying the GLMs and rely on specific statistical modeling tools or software.

On the other hand, we believe that the formats and the procedures for the minimum bias types of iteration algorithms are simple and straightforward. The approach is based on a target or error function along with an iterative procedure to minimize the function without distribution assumptions. Actuaries have been using the approach for many decades. So, compared to GLM, some advantages of the iteration approach are that it is easy to understand; easy to use; easy to program using many different software tools (for example, an Excel spreadsheet); and does not require advanced statistical knowledge, such as maximum likelihood estimations and deviance functions of GLM.

One issue associated with most previous work on the minimum bias approach and GLM is the model-selection limitation. GLMs assume the underlying distributions are from the exponential family. Also, commonly used statistical software typically provides limited selection of GLM distributions, such as Poisson, Gamma, normal, negative binomial, and inverse Gaussian. On the other hand, only five types of multiplicative models and four types of additive models are available from previous minimum bias work.[2] These limitations, we believe, may reduce estimation accuracy in practice since insurance and actuarial data are rarely perfect and may not fit well the exponential family of distributions or existing bias models.

In addition, there are two other common and practical issues that actuaries have to deal with in their daily pricing exercises. First, many real-world rating plans are essentially a mixed additive and multiplicative model. For example, for personal auto pricing, the primary class plan factor for age, gender, marital status, and vehicle use is often added with the secondary class plan factor for past accident and violation points, and then the result is multiplied with other factors. The commonly used GLM software, to our knowledge, does not provide options that can solve such mixed models because the identity link function implies an additive model, while the log link function implies a multiplicative model.

Second, it is possible that using either a GLM or a previous minimum bias iteration approach will result in parameters for some variables which are questionable or unacceptable by the marketplace. One practical way to deal with this issue is to select or constrain the factors for the variables based on business and competitive reasons while leaving other factors to be determined by multivariate modeling techniques. Since all the variables are connected in the multivariate analysis, any "constrained" factors should flow through the analysis, and the constraint will impact the results for the other "unconstrained" factors. We can call this issue a "constraint optimization" problem.

[1]GLMs may also involve an iterative approach. The most commonly used numerical method to solve the GLM is the "iterative reweighted least squares" algorithm.

[2]Feldblum and Brosius [5] list six multiplicative minimum bias models in their summary table. However, the balance principle model is the same as the maximum likelihood Poisson model.

In this study, we propose a more flexible and comprehensive approach within the minimum bias framework, called a "general iteration algorithm" (GIA). The key features of GIA are:

- It will significantly broaden the assumptions for distributions in use, and, to a certain degree, it totally relaxes any specific form for the distributions. Therefore, GIA will be able to provide a much wider array of models from which actuaries may choose. This will increase the model-selection flexibility.
- Its flexibility will improve the accuracy and the goodness of fit of classification rates. We will demonstrate this result through a case study later.
- Similar to past minimum bias approaches, it is easy to understand and does not require advanced statistical knowledge. For practical purposes, GIA users only need to select the target functions and the iteration procedure because the approach is distribution free.
- While GIA still requires the iterative process in determining the parameters, we believe that the effort is not significant with today's powerful computers.
- It can solve the mixed additive-multiplicative models and the constraint optimization problems.

In the following sections, we will first prove that all five existing multiplicative minimum bias algorithms are special cases of GIA. We will also propose several more multiplicative algorithms that actuaries may consider for ratemaking based on insurance data. Then, we will demonstrate how to apply GIAs to solve the mixed models and constraint optimization problems.

The numerical analysis of multiplicative and additive models given later is based on severity data for private passenger auto collision in Mildenhall [9] and McCullagh and Nelder [8]. The results from selected algorithms will be compared to those from the GLM models. Following Bailey and Simon [2], the weighted absolute bias and the Pearson chi-square statistic are used to measure the goodness of fit. We also calculate the weighted absolute percentage bias, which indicates the magnitude of the errors relative to the predicted values.

The remainder of this paper is organized as follows:

- Section 2 discusses the details of 2-parameter multiplicative, 3-parameter multiplicative, constraint, additive, and mixed GIA.
- Section 3 addresses the residual diagnosis of GIA.
- Section 4 investigates the calculation efficiency of GIA. It shows that GIA could converge rapidly and is not necessarily inefficient in numerical calculations.
- Section 5 reviews numerical results for two case studies using multiplicative and mixed models.
- Section 6 outlines our conclusions.
- The appendix reports the numerical results for the examples discussed in Section 5 with several selected multiplicative GIAs. It also shows the iterative convergences of selected multiplicative, additive, and mixed GIAs.

2. General iteration algorithm (GIA)

2.1. Two-parameter GIAs

Following the notation used previously, in the multiplicative framework for two rating factors, the expected relativity for cell (i,j) should be equal to the product of x_i and y_j:

$$\mathrm{E}(r_{i,j}) = \mu_{i,j} = x_i y_j. \tag{2.1}$$

By (2.1), there are a total of n alternative estimates for x_i and a total of m estimates for y_j:

$$\begin{aligned} \hat{x}_{i,j} &= r_{i,j}/y_j, \qquad j = 1,2,\ldots,n \\ \hat{y}_{j,i} &= r_{i,j}/x_i, \qquad i = 1,2,\ldots,m. \end{aligned} \tag{2.2}$$

Following actuarial convention, the final estimates of x_i and y_j could be calculated by the weighted average of $\hat{x}_{i,j}$ and $\hat{y}_{j,i}$. If we use the straight average to estimate the relativity:

$$\hat{x}_i = \sum_j \frac{1}{n}\hat{x}_{i,j} = \frac{1}{n}\sum_j \frac{r_{i,j}}{y_j}. \tag{2.3}$$

Similarly, $\hat{y}_j = \sum_i (1/m)\hat{y}_{j,i} = (1/m)\sum_i (r_{i,j}/x_i)$. This is Algorithm 2, the ML exponential model introduced by Brown [3].

If the relativity-adjusted exposure, $w_{i,j}\mu_{i,j}$, is used as the weight in determining the estimates:

$$\hat{x}_i = \sum_j \frac{w_{i,j}\mu_{i,j}}{\sum_j w_{i,j}\mu_{i,j}}\hat{x}_{i,j} = \sum_j \frac{w_{i,j}y_j}{\sum_j w_{i,j}y_j}\frac{r_{i,j}}{y_j} = \frac{\sum_j w_{i,j}r_{i,j}}{\sum_j w_{i,j}y_j}. \tag{2.4}$$

Similarly,

$$\hat{y}_j = \sum_i \frac{w_{i,j}\mu_{i,j}}{\sum_i w_{i,j}\mu_{i,j}}\hat{y}_{i,j} = \sum_i \frac{w_{i,j}x_i}{\sum_i w_{i,j}x_i}\frac{r_{i,j}}{x_i} = \frac{\sum_i w_{i,j}r_{i,j}}{\sum_i w_{i,j}x_i}.$$

The resulting model is the same as Algorithm 1, the "balance principle" or ML Poisson model.

If the square of the relativity-adjusted exposure, $w_{i,j}^2\mu_{i,j}^2$, is used as the weight:

$$\hat{x}_i = \sum_j \frac{w_{i,j}^2\mu_{i,j}^2}{\sum_j w_{i,j}^2\mu_{i,j}^2}\hat{x}_{i,j} = \frac{\sum_j w_{i,j}^2 r_{i,j}y_j}{\sum_j w_{i,j}^2 y_j^2}. \tag{2.5}$$

The resulting model is the same as Algorithm 3, the ML normal model.

If the exposure adjusted by the square of relativity, $w_{i,j}\mu_{i,j}^2$, is used as the weight:

$$\hat{x}_i = \sum_j \frac{w_{i,j}\mu_{i,j}^2}{\sum_j w_{i,j}\mu_{i,j}^2}\hat{x}_{i,j} = \frac{\sum_j w_{i,j}r_{i,j}y_j}{\sum_j w_{i,j}y_j^2}. \tag{2.6}$$

The resulting model is the same as Algorithm 4, the least-squares model.

From the above results, we propose the 2-parameter GIA approach by using $w_{i,j}^p\mu_{i,j}^q$ as the weights for the bias function:

2-Parameter GIA:

$$\hat{x}_i = \sum_j \frac{w_{i,j}^p\mu_{i,j}^q}{\sum_j w_{i,j}^p\mu_{i,j}^q}\hat{x}_{i,j} = \frac{\sum_j w_{i,j}^p r_{i,j}y_j^{q-1}}{\sum_j w_{i,j}^p y_j^q}. \tag{2.7}$$

When

- $p = q = 0$, it is the ML exponential model, Algorithm 2;
- $p = q = 1$, it is the ML Poisson model, Algorithm 1;
- $p = q = 2$, it is the ML normal model, Algorithm 3
- $p = 1$ and $q = 2$, it is the least-squares model, Algorithm 4.

In addition, there are two more models that correspond to GLM with the exponential family of gamma and inverse Gaussian distributions.[3] When the exposure is used as the weights, that is, $p = 1$ and $q = 0$, the GIA will lead to a GLM gamma model and becomes:

Algorithm 6:

$$\hat{x}_i = \sum_j \frac{w_{i,j}}{\sum_j w_{i,j}}\hat{x}_{i,j} = \frac{\sum_j w_{i,j}r_{i,j}y_j^{-1}}{\sum_j w_{i,j}}. \tag{2.8}$$

When $p = 1$ and $q = -1$, the GIA leads to a GLM inverse Gaussian model and becomes:

Algorithm 7:

$$\hat{x}_i = \sum_j \frac{w_{i,j}y_j^{-1}}{\sum_j w_{i,j}y_j^{-1}}\hat{x}_{i,j} = \frac{\sum_j w_{i,j}r_{i,j}/y_j^2}{\sum_j w_{i,j}/y_j}. \tag{2.9}$$

Equation (2.7) suggests that in theory there is no limitation for the values of p and q that can be used and they can take on any real values. It is with this feature that GIA should greatly

[3]For detailed information, please refer to Section 7 of Mildenhall [9].

enhance the flexibility for actuaries when they apply the algorithm to fit their data. Of course, in reality we do not expect that extreme values for p and q will be found useful. In ratemaking applications, earned premium could be used if exposure is not available. Normalized premium (premium divided by relativity) is a reasonable option for the weight. This suggests that q could be negative. In general, p should be positive: the more exposure/claims/premium, the more weight assigned.

2.2. Three-parameter GIAs

So far, we have used the 2-parameter GIA in Equation (2.7) to represent several commonly used models, Algorithms 1 to 4, but not Algorithm 5, the "χ-squared" multiplicative model. In order to represent Algorithm 5, we further expand the 2-parameter GIA to a 3-parameter GIA using the link function concept from GLM.

One generalization of GLMs as compared to a more basic linear model is done by introducing a link function to link the linear predictor to the response variable. Similarly, we introduce a relativity link function to link the GIA estimate to the relativity. The proposed relativity link function is different in several aspects from the link function in GLMs. In GLMs, the link function determines the type of model: log link implies a multiplicative model and identity link implies an additive model. This is not the case for GIA. Multiplicative GIA, for example, could have a log, power, or exponential relativity link function.

For a 3-parameter GIA, instead of using (2.2), we estimate the relativity link functions of $f(\hat{x}_i)$ and $f(\hat{y}_j)$ from $f(\hat{x}_{i,j})$ and $f(\hat{y}_{j,i})$ first; and then calculate $\hat{x}_i$ and $\hat{y}_j$ by inverting the relativity link function, $f^{-1}(f(\hat{x}_i))$ and $f^{-1}(f(\hat{y}_j))$. The functions $f(\hat{x}_{i,j})$ and $f(\hat{y}_{j,i})$ can be estimated by:

$$f(\hat{x}_{i,j}) = f(r_{i,j}/y_j), \qquad j = 1,2,\ldots,n$$
$$f(\hat{y}_{j,i}) = f(r_{i,j}/x_i), \qquad i = 1,2,\ldots,m. \tag{2.10}$$

Taking the weighted average using parameters p and q:

$$f(\hat{x}_i) = \sum_j \frac{w_{i,j}^p \mu_{i,j}^q}{\sum_j w_{i,j}^p \mu_{i,j}^q} f(\hat{x}_{i,j}) = \frac{\sum_j w_{i,j}^p y_j^q f\left(\frac{r_{i,j}}{y_j}\right)}{\sum_j w_{i,j}^p y_j^q} \tag{2.11}$$

$$f(\hat{y}_j) = \sum_i \frac{w_{i,j}^p \mu_{i,j}^q}{\sum_i w_{i,j}^p \mu_{i,j}^q} f(\hat{y}_{j,i}) = \frac{\sum_i w_{i,j}^p x_i^q f\left(\frac{r_{i,j}}{x_i}\right)}{\sum_i w_{i,j}^p x_i^q}.$$

Thus,

$$\hat{x}_i = f^{-1}\left(\frac{\sum_j w_{i,j}^p y_j^q f\left(\frac{r_{i,j}}{y_j}\right)}{\sum_j w_{i,j}^p y_j^q}\right) \tag{2.12}$$

$$\hat{y}_j = f^{-1}\left(\frac{\sum_i w_{i,j}^p x_i^q f\left(\frac{r_{i,j}}{x_i}\right)}{\sum_i w_{i,j}^p x_i^q}\right).$$

One possible selection of the relativity link function is the power function, $f(\hat{x}_i) = \hat{x}_i^k$ and $f(\hat{y}_j) = \hat{y}_j^k$. In this case, equation (2.12) leads to a 3-parameter GIA:

$$\hat{x}_i = \left(\frac{\sum_j w_{i,j}^p r_{i,j}^k y_j^{q-k}}{\sum_j w_{i,j}^p y_j^q}\right)^{1/k}. \tag{2.13}$$ [4]

When $k = 2$, $p = 1$, and $q = 1$, Equation (2.13) is equivalent to:

$$\hat{x}_i = \left(\frac{\sum_j w_{i,j} r_{i,j}^2 y_j^{-1}}{\sum_j w_{i,j} y_j}\right)^{1/2}, \tag{2.14}$$

and this is Algorithm 5, the "χ-squared" multiplicative model.

Another example of a new iterative algorithm occurs when $k = 1/2$, $p = 1$, and $q = 1$:

Algorithm 8:

$$\hat{x}_i = \left(\frac{\sum_j w_{i,j} r_{i,j}^{1/2} y_j^{1/2}}{\sum_j w_{i,j} y_j}\right)^2. \tag{2.15}$$

[4] There is no unique solution for these equations. For one group of solutions, we can divide each x by a factor and multiply each y by the same factor to obtain another group of solutions. To guarantee a unique solution, we can add a constraint to force the average of the x's to be one.

Mildenhall [10] indicated that the 3-parameter GIA is equivalent to a GLM with the parameters x^k and y^k, the weight w^p, and the response variable r^k following a distribution with variance function $\mathrm{Var}(\mu) = \mu^{2-q/k}$. When $k = 1$ and $p = 1$, we can conclude that:

- when $q = 2$, the normal GLM model is the same as the GIA Algorithm 4 in Equation (1.4);
- when $q = 1$, the Poisson GLM model is the same as the GIA Algorithm 1 in Equation (1.1);
- when $q = 0$, the gamma GLM model is the same as the GIA Algorithm 6 in Equation (2.8);
- when $q = -1$, the inverse Gaussian GLM model is the same as the GIA Algorithm 7 in Equation (2.9).

Also, for "χ-squared" minimum bias model with $k = 2$, $p = 1$, and $q = 1$, the GIA theory indicates that r^2 follows a Tweedie distribution with a variance function $\mathrm{Var}(\mu) = \mu^{1.5}$.

In actuarial exercises, we often exclude the extremely high and low values from the weighted average to yield more robust results. In the case of several rating variables, there may be thousands of alternative estimates. Actuaries have the flexibility to use the weighted average within selected ranges (e.g., the average without the highest and the lowest 1% percentile). This is similar to the concept of "trimmed" regression used with GLMs whereby observations with undue influence on a fitted value are removed.

Finally, we would like to extend GIA to reserve applications. Mack [7] discussed the connection between ratemaking models of auto insurance and IBNR reserve calculation because reserves can be estimated by a ratemaking model with two "rating" variables, accident year and development year. He showed that the minimum bias method produces the same result as the chain ladder loss development method. Recently, actuaries have applied GLMs to estimate reserves using the incremental loss as the response variable.

Let $P_{i,j}$ be the incremental paid loss in accident year i and development year j, that is, $P_{i,j}$ is the cell (i,j) of the incremental payment triangle. England and Verrall [4] used the following GLM with log link function and Poisson distribution to estimate the expected values of future payments:

$$E(P_{i,j}) = m_{i,j} \quad \text{and} \quad \mathrm{Var}(P_{i,j}) = \phi m_{i,j},$$
$$\log(m_{i,j}) = C + \alpha_i + \beta_j$$
$$\alpha_1 = \beta_1 = 0.$$

Several other models were also proposed for reserve estimates. For example, Renshaw and Verrall [11] applied the GLM with a gamma distribution. The only difference between the gamma and Poisson models is that the gamma model's variance function is $\mathrm{Var}(P_{i,j}) = \phi m_{i,j}^2$.

Let $\mu_{i,j} = (m_{i,j}/m_{1,1})$, $x_i = e^{\alpha_i}$, and $y_j = e^{\beta_j}$; then the above GLM reserve models can be similarly transferred to the GIA multiplicative algorithm by setting $\mu_{i,j} = x_i y_j$. So GIA can also be used to estimate reserves based on the triangles of incremental paid loss. When $k = 1$, $p = 1$ and $q = 1$, GIA yields the same result as a Poisson GLM reserve model; when $k = 1$, $p = 1$, and $q = 0$, GIA produces the same result as a gamma GLM model.

2.3. Constraint GIA

In real-world ratemaking applications, some factors need to be selected or capped within a certain range for business or competitive reasons. Since in a multivariate analysis, all the variables are related, other factors should be adjusted to reflect the impact of the subjective selections. When this issue arises, the standard GLM or other approaches may have limitations if the selected factors are outside of the fitted confidence interval.

For example, the multi-car discount used for private passenger auto pricing is typically between 5% and 25%. Any factor outside this range is not likely to be accepted by the market, no matter what the fitted value is for the "indicated" discount. In the following we will demonstrate how to apply GIA to solve the issue.

For example, let x_1 and x_2 be the single and multi-car factors, respectively, and we will cap the multi-car discount to be between 5% and 25%. The constraint can be represented by $0.75x_1 \le x_2 \le 0.95x_1$. Adding this constraint to (2.13), we can solve the problem by:

$$\hat{x}_1 = \left(\frac{\sum_j w_{1,j}^p r_{1,j}^k y_j^{q-k}}{\sum_j w_{1,j}^p y_j^q}\right)^{1/k},$$

$$\hat{x}_2 = \max\left(0.75\hat{x}_1, \min\left(0.95\hat{x}_1, \left(\frac{\sum_j w_{2,j}^p r_{2,j}^k y_j^{q-k}}{\sum_j w_{2,j}^p y_j^q}\right)^{1/k}\right)\right). \quad (2.16)$$

With the constraint, we can continue the iteration process until the values for all other rating factors converge. This flexibility[5] associated with GIA will provide actuaries another benefit in dealing with their practical problems.

2.4. Additive GIA

Following the same notations as above, the expected cost for classification cell (i,j) with an additive model should be equal to the sum of x_i and y_j:

$$E(r_{i,j}) = \mu_{i,j} = x_i + y_j. \quad (2.17)$$

Thus,

$$\hat{x}_{i,j} = r_{i,j} - y_j, \qquad j = 1,2,\ldots,n$$
$$\hat{y}_{j,i} = r_{i,j} - x_i, \qquad i = 1,2,\ldots,m. \quad (2.18)$$

In the multiplicative models, we use the relativity-adjusted exposure, $w_{i,j}^p \mu_{i,j}^q$, as the weighting function and introduce the power relativity link function. However, the weighting functions and the relativity link functions cannot be applied in an additive process.

For the additive GIA, we are limited to the following one-parameter model using $w_{i,j}^p$ as the weight:

$$\hat{x}_i = \sum_j \frac{w_{i,j}^p}{\sum_j w_{i,j}^p}\hat{x}_{i,j} = \frac{\sum_j w_{i,j}^p (r_{i,j} - y_j)}{\sum_j w_{i,j}^p}. \quad (2.19)$$

When $p = 1$, it leads to the model introduced by Bailey [1] or the "Balance Principle" model in Feldblum and Brosius [5]. Mildenhall [9] also proved that it is equivalent to an additive normal GLM model. When $p = 2$, it leads to the ML additive normal model introduced by Brown [3]. When $p = 0$, it leads to the least squares model by Feldblum and Brosius [5]. There is no further generalization for the additive GIAs with additional parameters or link functions.

Except for the exponential family of distributions, the lognormal distribution is probably the most widely used distribution in actuarial ractice. If $r_{i,j}$ follows a lognormal distribution, $\log(r_{i,j})$ will follow a normal distribution and the multiplicative rating plan can be transformed to $\log(r_{i,j}) = \log(x_i) + \log(y_j) + \varepsilon_{i,j}$. The additive GIA algorithms can be used to derive the parameters for the lognormal distribution assumption.[6]

2.5. Mixed additive and multiplicative GIAs

A simplified mixed additive and multiplicative model[7] can be illustrated as follows:

$$r_{i,j,h} = (x_i + y_j) \times z_h + \varepsilon_{i,j,h}, \quad (2.20)$$

where $i = 1,2,\ldots,m$; $j = 1,2,\ldots,n$; and $h = 1,2,\ldots,l$. There are $n \times l$ alternative estimates for x_i:

$$\hat{x}_{i,j,h} = \frac{r_{i,j,h}}{z_h} - y_j.$$

[5] Using the offset term, GLMs can solve fixed factor constraints, such as $x_2 = 0.8$. GIA is more flexible in its capability of solving almost all formats of constraints.

[6] $E(r_{i,j}) = x_i \times y_j \times \exp(0.5 \times \mathrm{Var}(\varepsilon_{i,j}))$.

[7] The models with more complex additive-multiplicative structures can be derived similarly.

There are $m \times l$ alternative estimates for y_j:

$$\hat{y}_{i,j,h} = \frac{r_{i,j,h}}{z_h} - x_i.$$

There are $m \times n$ alternative estimates for z_h:

$$\hat{z}_{i,j,h} = \frac{r_{i,j,h}}{x_i + y_j}.$$

Using $w^p_{i,j,h}$ as the weight:

$$\begin{aligned}
\hat{z}_h &= \frac{\sum_i \sum_j w^p_{i,j,h} \times \hat{z}_{i,j,h}}{\sum_i \sum_j w^p_{i,j,h}} \\
&= \frac{\sum_i \sum_j w^p_{i,j,h} \times \left(\frac{r_{i,j,h}}{x_i + y_j}\right)}{\sum_i \sum_j w^p_{i,j,h}}, \\
\hat{x}_i &= \frac{\sum_j \sum_h w^p_{i,j,h} \times \hat{x}_{i,j,h}}{\sum_j \sum_h w^p_{i,j,h}} \\
&= \frac{\sum_j \sum_h w^p_{i,j,h} \times \left(\frac{r_{i,j,h}}{z_h} - y_j\right)}{\sum_j \sum_h w^p_{i,j,h}}; \\
\hat{y}_j &= \frac{\sum_i \sum_h w^p_{i,j,h} \times \hat{y}_{i,j,h}}{\sum_i \sum_h w^p_{i,j,h}} \\
&= \frac{\sum_i \sum_h w^p_{i,j,h} \times \left(\frac{r_{i,j,h}}{z_h} - x_i\right)}{\sum_i \sum_h w^p_{i,j,h}}.
\end{aligned} \tag{2.21}$$

There is no unique solution for (2.21). For example, if a, b, and c are a solution to estimate the factors x, y, and z, then $2a$, $2b$, and $0.5c$ are another possible solution. In order to facilitate the iteration convergence, we need to add some constraints in the procedure. If we use the sample mean as the base, the weighted average of multiplicative factors from one rating variable should be close to one. So in each iteration we can adjust all the z's proportionally so that the average is reset to one. Mathematically, the constraint is:

$$\frac{\sum_i \sum_j \sum_h w^p_{i,j,h} \hat{z}_h}{\sum_i \sum_j \sum_h w^p_{i,j,h}} = 1. \tag{2.22}$$

3. Residual diagnosis

For a statistical data-fitting exercise, it is important to conduct a diagnostic test to validate the distribution assumption in use. Such diagnostic tests typically consist of a residual plot in which the residuals are the difference between the fitted values and the actual values. In this section, we will describe how to conduct such residual analysis for GIA, and the residual plot results for the case study are given in the next section.

We have discussed that a 3-parameter GIA is equivalent to a GLM, assuming the response variable r^k follows a distribution with variance function $\mathrm{Var}(\mu) = \mu^{2-q/k}$. The raw residuals $(r^k_{i,j} - \hat{x}^k_i \hat{y}^k_j)$ from GIA do not asymptotically follow an independent and identical normal distribution because the variances of residuals are positively correlated to the predicted values.[8] As in GLM, we define the scaled Pearson residual of GIA as

$$e_{i,j} = \frac{r^k_{i,j} - \hat{r}^k_i}{\sqrt{\mathrm{Var}(\mu)}} = \frac{r^k_{i,j} - \hat{x}^k_i \hat{y}^k_j}{\sqrt{(\hat{x}^k_i \hat{y}^k_j)^{2-q/k}}} = \frac{r^k_{i,j} - \hat{x}^k_i \hat{y}^k_j}{\sqrt{\hat{x}_i^{2k-q} \hat{y}_j^{2k-q}}}, \tag{3.1}$$

where $e_{i,j}$ is approximately independent and identically distributed since

$$\begin{aligned}
\mathrm{Var}(e_{i,j}) &= \mathrm{Var}\left(\frac{r^k_{i,j} - \hat{x}^k_i \hat{y}^k_j}{\sqrt{(\hat{x}^k_i \hat{y}^k_j)^{2-q/k}}}\right) \\
&= \frac{\mathrm{Var}(r^k_{i,j})}{(\hat{x}^k_i \hat{y}^k_j)^{2-q/k}} = 1.
\end{aligned} \tag{3.2}$$

We can use the scaled Pearson residuals to conduct the residual diagnosis for GIA, such as developing a scattered residuals plot and a quantile-to-quantile (Q-Q) plot. If the GIA algorithms fit the data well, scaled Pearson residuals are randomly scattered and the Q-Q plot is close to a straight line.

[8] The additive models are equivalent to GLM normal models, so that the raw residuals from additive and mixed models can be used directly for diagnosis tests.

4. Calculation efficiency

One issue associated with GIA is the calculation efficiency. Mildenhall [9] discussed that one advantage of GLMs compared to the minimum bias models is the calculation efficiency because GLMs do not require an iterative process in estimating the parameters. He showed that the additive minimum bias model by Bailey [1], or GIA with $p = 1$, does not converge even after 50 iterations using the well-investigated data given by McCullagh and Nelder [8].

However, with several adjustments to the iteration methodology, we can show that GIA can converge very quickly. Using the same data, the additive GIA can complete the convergence in five iterations. One adjustment is to include as much updated information as possible—that is, the latest y's should be used to estimate the next x's and vice versa.

In GLMs and previous minimum bias models, a specific class is usually selected as the base (e.g., age 60+ and pleasure). For GIA, we suggest using the average as the base, because, when using a specific class as the base, the numerical value of the base will vary from one iteration to next, requiring additional iterations to force the factor for the base class to be one.

Another well-known issue for the iteration procedure concerns how to set the starting point for the first iteration. The closer the starting point to the final results, the faster the convergence. Using average frequency/severity/pure premium as the base, the average factor of a rating variable is one for multiplicative models and the average discount is zero for the additive models. Therefore, in this study, we chose the starting values of $x_{i,0}$ and $y_{j,0}$ to be 1 for the multiplicative models and 0 for the additive models.

5. Numerical analysis

The numerical analysis of testing various multiplicative GIAs is based on the severity data for private passenger auto collision given in Mildenhall [9] and McCullagh and Nelder [8]. Using this well-researched data will help us to compare the empirical results of this paper with previous studies. The data includes 32 severity observations for two classification variables: eight age groups and four types of vehicle use. In this severity case study, the weight $w_{i,j}$ is the number of claims. Table 1 in the Appendix lists the data.

In order to test mixed additive and multiplicative GIAs, we need at least three variables in the data. The data in Mildenhall [9] and McCullagh and Nelder [8] contain only two variables. Therefore, we will use another collision pure premium dataset to demonstrate the mixed algorithm. In addition to age and vehicle use, this data includes credit score as a third variable, with four classifications from low to high. In this pure premium case study, the weight $w_{i,j}$ is the earned exposure. Table 2 in the Appendix displays the data.

Four criteria are used to evaluate the performance of these GIAs: the absolute bias, the absolute percentage bias, the Pearson chi-squared statistic, and the combination of absolute bias and the chi-squared statistic:

- The weighted absolute bias (wab) criterion is proposed by Bailey and Simon [2]. It is the weighted average of absolute dollar difference between the observations and fitted values:

$$wab = \frac{\sum w_{i,j}|Br_{i,j} - Bx_i y_j|}{\sum w_{i,j}}.$$

- The second one, weighted absolute percentage bias ($wapb$), measures the absolute bias relative to the predicted values:

$$wapb = \frac{\sum w_{i,j}\dfrac{|Br_{i,j} - Bx_i y_j|}{Bx_i y_j}}{\sum w_{i,j}}.$$

- The weighted Pearson chi-squared ($wChi$) statistic is also proposed by Bailey and Simon [2] and it is appropriate to test if "differences between the raw data and the estimated relativities should be small enough to be caused by chance":

$$wChi = \frac{\sum w_{i,j} \frac{(Br_{i,j} - Bx_i y_j)^2}{Bx_i y_j}}{\sum w_{i,j}}.$$

- Lastly, we combine the absolute bias and Pearson chi-squared statistic, $\sqrt{wab \times wChi}$, to be the fourth criterion for the model selection.

Table 3 lists the relativities for Algorithms 1–8 and Table 4 displays the four performance statistics of those models, *wab*, *wapb*, *wChi*, and $\sqrt{wab \times wChi}$.[9] In all the cases, class "age 60+" and "pleasure" are used as the base.

To illustrate the residual diagnosis of GIA, we show the residual plots for GIA with $k = 1$, $p = 1$, and $q = -0.5$. Figure 1 in the Appendix reports the scattered residuals by observations; Figures 2 and 3 show the scattered residuals by age and by vehicle use, respectively; Figure 4 is the Q-Q plot. It is clear that the classification of age 17–20 and business use is an outlier.[10] This is not surprising because of the small sample size in the cell (five claims). A practical way to solve the problem is to cap the severity.

As stated before, we find that GLMs with common exponential family distribution assumptions are special cases of GIA ($k = 1$ and $p = 1$). Comparing the GIA factors in Table 3 with those from GLMs with normal, Poisson, gamma, and inverse Gaussian distributions, we confirm:

- when $k = 1$, $p = 1$, and $q = 2$, the "least squares" GIA has the same results as GLM with a normal distribution;[11]
- when $k = 1$, $p = 1$, and $q = 1$, GIA is the same as a Poisson GLM;
- when $k = 1$, $p = 1$, and $q = 0$, GIA is the same as a gamma GLM; and
- when $k = 1$, $p = 1$ and $q = -1$, GIA is the same as a GLM with inverse Gaussian distribution.

As discussed in Section 2, a GIA with $k = 1$ and $p = 1$

$$\left(\hat{x}_i = \frac{\sum_j w_{i,j} r_{i,j} y_j^{q-1}}{\sum_j w_{i,j} y_j^q} \right)$$

is equivalent to the multiplicative GLMs with the variance function of $\mathrm{Var}(\mu) = \mu^{2-q}$ for an assumed exponential family distribution. It is well known that insurance and actuarial data is generally positively skewed. The skewness for the symmetric normal distribution is zero, and is increasingly positive from Poisson, to gamma, and to inverse Gaussian. For the multiplicative GIA algorithms, the skewness can be represented by q. When $q = 2$, the GIA is the same as a normal GLM. When $q = 1$, it is the same as a Poisson GLM. It is the same as a gamma when $q = 0$ and the same as inverse Gaussian when $q = -1$. Thus, smaller q values should be selected when the GIA is applied to more skewed data.

The authors also attempted to find the "global minimum error" points.[12] In this case study, if *wab* is used to measure the model performance, when $k = 1.95$, $p = 3.15$, and $q = -14.06$, the weighted absolute error is minimized with $wab = 10.0765$. If *wapb* is used to measure the model performance, when $k = 1.98$, $p = 3.15$, and $q = -14.04$, the weighted absolute percentage error is minimized with $wapb = 3.461\%$. The result suggests that the best-fit model, in this example, does not occur with any of the commonly

[9] We tested hundreds of 3-parameter algorithms. For the detailed reports on all the tested models, please refer to Fu and Wu [6].

[10] The severity of age 17–20 and business use does not fit any tested GLMs and GIAs well.

[11] The underlying assumption of "least squares" regression is that the residuals follow a normal distribution. So the "least squares" method is the same as a normal GLM.

[12] Resolving such global minimum error issues requires additional in-depth research and is beyond the scope of this paper. Since the error measures are easy to calculate explicitly in a spreadsheet, an Excel built-in tool like Solver can be used to find the optimization solutions. If the data is larger than the spreadsheet's capacity, interested readers can apply Newton's method to obtain the minimums using SAS, Splus, or Matlab. The first and second derivatives can be estimated using a finite difference method.

Figure 1. Scattered residual plot of GIA with $k = 1$, $p = 1$, and $q = -0.5$.

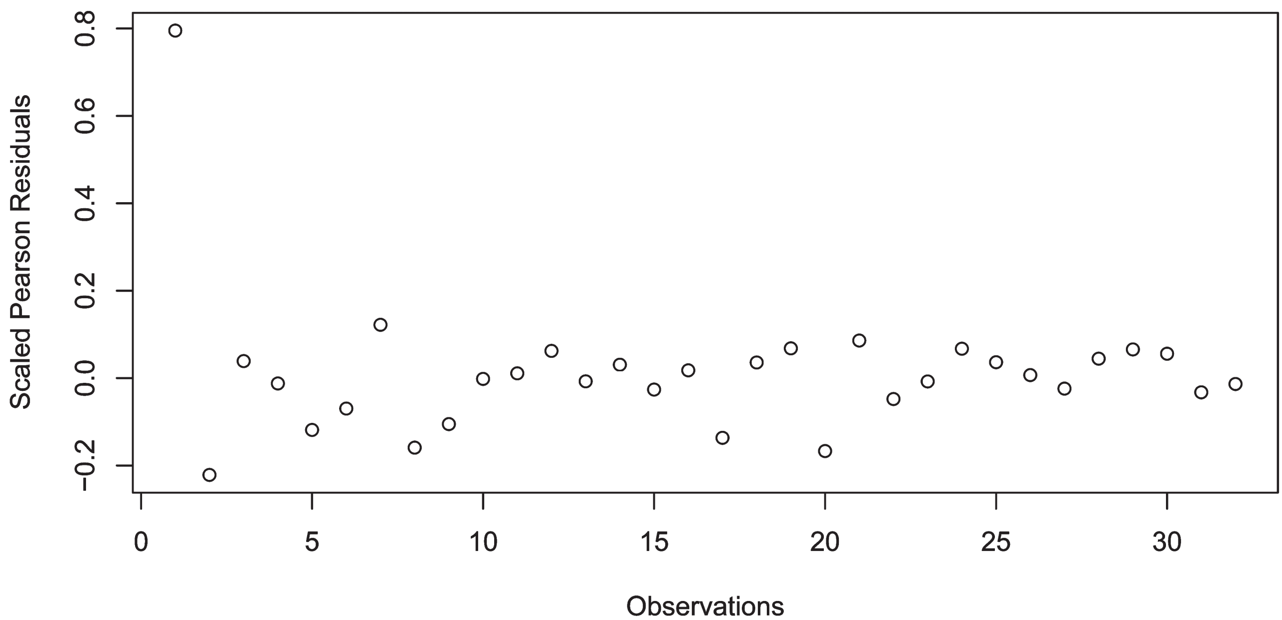
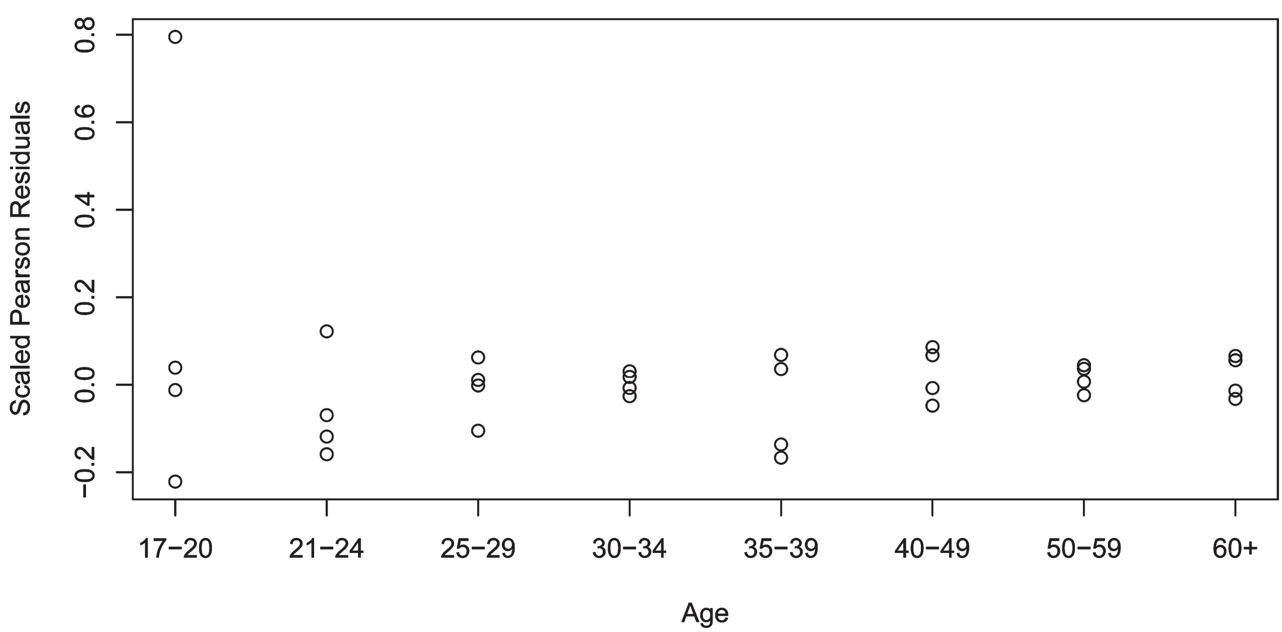


Figure 2. Scattered residual plot by age of GIA with $k = 1$, $p = 1$, and $q = -0.5$.



used minimum bias models and generalized linear models. It clearly demonstrates the fact that insurance data may not be perfect for predetermined distributions.

On the other hand, if $wChi$ is used, the "χ-squared" model ($k = 2$, $p = 1$, and $q = 1$) provides the best solution. This is expected because the "χ-squared" model is calculated by minimizing the Pearson chi-squared statistic.

If we use the criterion of $\sqrt{wab \times wChi}$ to select models, when $k = 2.45$, $p = 1.16$, and $q = -0.06$, the combined error is minimized with

Figure 3. Scattered residual plot by vehicle use of GIA with $k = 1$, $p = 1$, and $q = -0.5$.

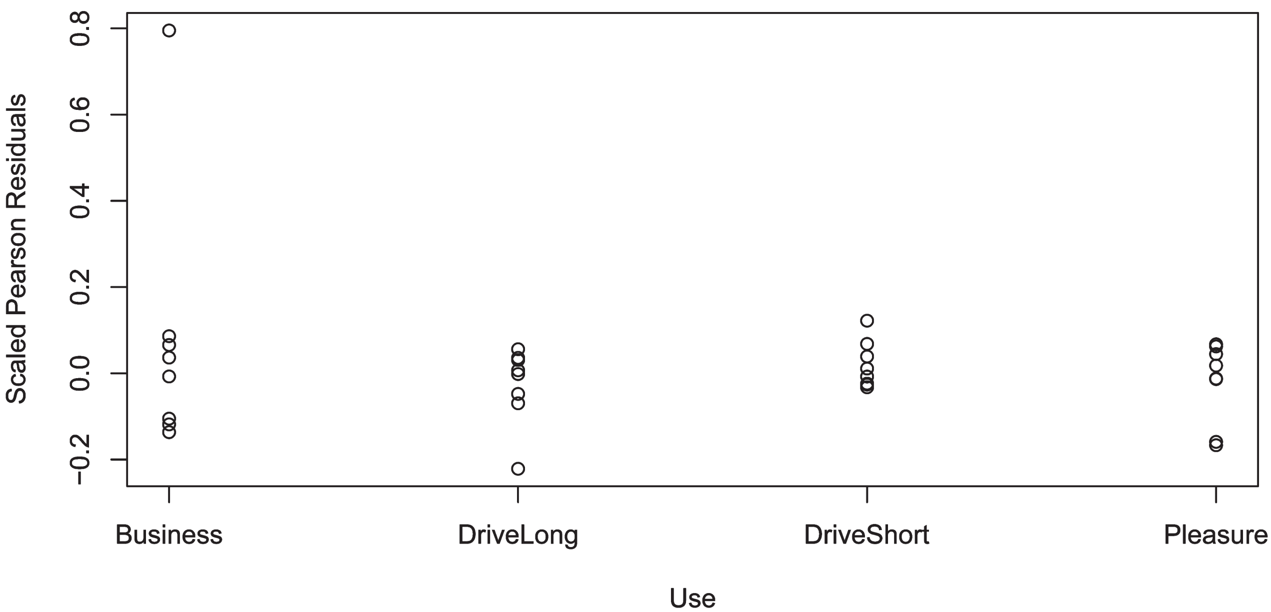
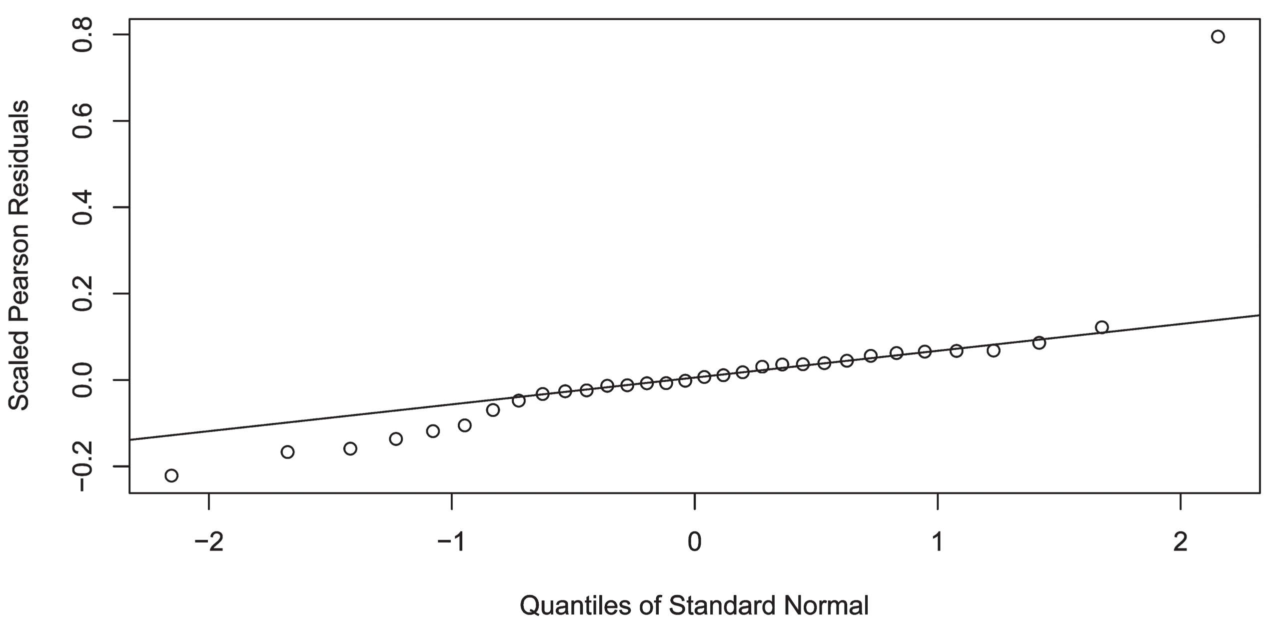


Figure 4. Q-Q plot of GIA with $k = 1$, $p = 1$, and $q = -0.5$.



$\sqrt{wab \times wChi} = 3.3061$. Again, the five commonly used minimum bias algorithms are not the best solution when absolute bias and the chi-squared statistic are considered simultaneously.

Based on the results of this research and our experience, we suggest for actuarial applications the following ranges of values for k, p, q:

- $1 \leq k \leq 3$.
- $p \geq q$, $0.5 \leq p \leq 4$, and $q \leq 1$.
- The higher the skewness of the data, the smaller the value of q should be.

Finally, we use another collision pure premium dataset to demonstrate the results for the mixed algorithm. Table 11 reports the final factors of the model. For the purpose of illustration, we will only calculate the model with $p = 1$.

To show that GIAs can converge rapidly, in the Appendix we report the iteration processes of selected GIAs:

- Table 5 shows the multiplicative factors for the gamma GIA using average severity as the base.
- Table 6 translates those factors using the classification age 60+ and pleasure as the base.
- Table 7 reports the iterative process for the coefficients of a GLM with the gamma distribution and log link.
- Table 8 translates those coefficients to the multiplicative factors of a gamma GLM.
- Table 9 lists the additive factors for the GIA with $p = 1$.
- Table 10 shows the additive dollar values for the GIA with $p = 1$ and uses the classification age 60+ and pleasure as the base.
- Table 11 reports the convergence process of the mixed model with $p = 1$.

From Tables 5–8, the multiplicative gamma GIA converges in four iterations. This is as fast as the corresponding GLM model. As expected, the numerical solutions between the two models are identical, and the solutions are also identical to the previous results of Algorithm 6 given in Table 3 for $k = 1$, $p = 1$, and $q = 0$.

Tables 9 and 10 report the iterative process for the GIA additive algorithm with $p = 1$. Mildenhall [9] used this model as an example to show that the minimum bias approach is not efficient. He showed that the minimum bias model converges slowly to the GLM results, and that the dollar values at the 50th iteration are about two cents different from those by GLM. However, using our numerical algorithm, the GIA calculation converges completely in five iterations with solutions identical to GLM results.

Table 11 shows the iterative process for the GIA mixed model. Even though the algorithm is more complicated than the multiplicative and additive models, the convergence takes only six iterations.

The above example illustrates an optimization case with two and three variables. However, in typical rating plans, we need to optimize more than two variables. Our experience indicates that the improved numerical approach for GIA will converge fairly quickly for typical actuarial rating exercises with five to 15 variables.

4. Conclusions

In this research, we propose a general iteration algorithm by including different weighting functions and relativity link functions in the approach. As indicated by the severity example given previously, insurance and actuarial data are rarely perfect, so we expect that the best fitted results typically will not be based on a predetermined distribution, such as those in the exponential family of distributions. Therefore, GIA can provide actuaries a great deal of flexibility in data fitting and model selection. The case studies given in the paper indicate that the "best" fitted results occur when the underlying distribution assumptions are not commonly used distributions.

In theory, the parameters in GIA can take on any real values and there is no limitation on the relativity link functions when GIA is applied to a dataset. Therefore, GIA will provide actuaries many more options than previous minimum bias algorithms or GLMs. However, due to the fact that insurance and actuarial data is positively skewed in nature, we do not expect that a very wide range of weighting or relativity link functions needs to be used in practice.

For the severity example used in the study, we searched and identified the best models with the minimum fitted errors. One issue may exist: GIA uses an iterative process in determining the parameters, so when it further incorporates multiple distribution assumptions in the searching process, the approach may become even more time-consuming and inefficient. However, we do not believe this issue is significant because of the powerful computational capability of modern computers.

Mildenhall [10] indicates, in his comments on our prior work, that GLM can be extended to replicate the comprehensive GIA proposed in this study. However, since commonly used GLM software has limited selections for the statistical distribution assumptions, it is difficult to perform Mildenhall's extension. In addition, we demonstrate how to extend GIA to solve mixed additive-multiplicative models and constraint optimization problems. To our knowledge, at this stage, there is no solution provided by GLM users to deal with these issues.

With the fast development of information technology, actuaries can analyze data in ways they could not imagine a decade ago. Currently there is a strong interest in data mining and predictive modeling in the insurance industry, and this calls for more powerful data analytical tools for actuaries. While some new tools, such as GLM, neural networks, decision trees, and MARS, have emerged recently and have received a great deal of attention, we believe that the decades-old minimum bias algorithms still have several advantages over other techniques, including being easy to understand and easy to use. We hope that our work in improving the flexibility and comprehensiveness of the minimum bias iteration approach is a timely effort and that this approach will continue to be a useful tool for actuaries in the future.

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Appendix. Data and numerical results

Table 1. PPA collision severity data for multiplicative and additive algorithms

Age	VUSE	Severity	Claim
17–20	Pleasure	250.48	21
17–20	DriveShort	274.78	40
17–20	DriveLong	244.52	23
17–20	Business	797.80	5
21–24	Pleasure	213.71	63
21–24	DriveShort	298.60	171
21–24	DriveLong	298.13	92
21–24	Business	362.23	44
25–29	Pleasure	250.57	140
25–29	DriveShort	248.56	343
25–29	DriveLong	297.90	318
25–29	Business	342.31	129
30–34	Pleasure	229.09	123
30–34	DriveShort	228.48	448
30–34	DriveLong	293.87	361
30–34	Business	367.46	169
35–39	Pleasure	153.62	151
35–39	DriveShort	201.67	479
35–39	DriveLong	238.21	381
35–39	Business	256.21	166
40–49	Pleasure	208.59	245
40–49	DriveShort	202.80	970
40–49	DriveLong	236.06	719
40–49	Business	352.49	304
50–59	Pleasure	207.57	266
50–59	DriveShort	202.67	859
50–59	DriveLong	253.63	504
50–59	Business	340.56	162
60+	Pleasure	192.00	260
60+	DriveShort	196.33	578
60+	DriveLong	259.79	312
60+	Business	342.58	96

Table 2. PPA collision pure premium data for the mixed algorithm

Age	VUSE	Credit	Exposure	Loss	Pure Prem
17–20	Business	1	5.2	0.0	0.00
17–20	Business	2	3.3	0.0	0.00
17–20	Business	3	7.3	0.0	0.00
17–20	Business	4	6.2	0.0	0.00
17–20	DriveLong	1	66.5	9,513.6	143.06
17–20	DriveLong	2	48.8	19,380.4	397.14
17–20	DriveLong	3	116.3	31,301.1	269.21
17–20	DriveLong	4	59.7	10,038.2	168.28
17–20	DriveShort	1	1,010.9	350,529.8	346.76
17–20	DriveShort	2	781.4	255,723.2	327.28
17–20	DriveShort	3	2,294.3	612,357.7	266.90
17–20	DriveShort	4	1,258.5	331,804.0	263.65
17–20	Pleasure	1	752.9	204,925.3	272.18
17–20	Pleasure	2	689.2	253,729.9	368.14
17–20	Pleasure	3	2,376.6	599,740.7	252.35

Table 2. (Continued)

Age	VUSE	Credit	Exposure	Loss	Pure Prem
17–20	Pleasure	4	1,285.9	237,747.6	184.89
21–24	Business	1	3.7	7,148.7	1,954.97
21–24	Business	2	3.3	0.0	0.00
21–24	Business	3	8.2	1,885.7	229.92
21–24	Business	4	2.2	140.0	63.49
21–24	DriveLong	1	126.0	28,433.6	225.61
21–24	DriveLong	2	145.6	43,135.2	296.32
21–24	DriveLong	3	187.7	82,429.8	439.07
21–24	DriveLong	4	80.7	12,261.0	151.88
21–24	DriveShort	1	1,427.9	277,123.8	194.07
21–24	DriveShort	2	1,771.4	427,339.5	241.25
21–24	DriveShort	3	2,831.4	509,032.4	179.78
21–24	DriveShort	4	1,170.3	123,744.3	105.74
21–24	Pleasure	1	643.4	153,109.7	237.95
21–24	Pleasure	2	792.4	214,037.3	270.10
21–24	Pleasure	3	1,811.2	380,801.1	210.25
21–24	Pleasure	4	955.0	156,535.5	163.92
25–29	Business	1	7.9	10,008.0	1,267.24
25–29	Business	2	14.8	8,806.8	595.42
25–29	Business	3	24.3	4,569.5	187.94
25–29	Business	4	3.8	0.0	0.00
25–29	DriveLong	1	242.3	64,343.7	265.52
25–29	DriveLong	2	280.6	66,854.4	238.27
25–29	DriveLong	3	508.1	91,732.4	180.54
25–29	DriveLong	4	70.8	9,346.9	132.07
25–29	DriveShort	1	2,685.0	474,584.1	176.75
25–29	DriveShort	2	2,918.1	484,317.1	165.97
25–29	DriveShort	3	4,908.4	725,874.1	147.88
25–29	DriveShort	4	813.0	121,589.9	149.55
25–29	Pleasure	1	1,140.1	252,874.9	221.79
25–29	Pleasure	2	1,173.3	143,197.2	122.05
25–29	Pleasure	3	1,984.2	261,112.7	131.59
25–29	Pleasure	4	465.5	52,280.0	112.31
30–34	Business	1	12.7	2,447.0	192.76
30–34	Business	2	20.1	11,168.7	555.47
30–34	Business	3	41.6	6,039.5	145.08
30–34	Business	4	2.5	0.0	0.00
30–34	DriveLong	1	351.4	44,128.5	125.57
30–34	DriveLong	2	280.1	32,023.1	114.34
30–34	DriveLong	3	752.4	76,489.0	101.66
30–34	DriveLong	4	141.4	28,522.0	201.66
30–34	DriveShort	1	3,125.5	529,866.3	169.53
30–34	DriveShort	2	2,726.8	339,734.4	124.59
30–34	DriveShort	3	6,534.6	837,467.0	128.16
30–34	DriveShort	4	1,142.2	111,598.4	97.70
30–34	Pleasure	1	1,668.3	223,172.5	133.77
30–34	Pleasure	2	1,566.8	217,866.0	139.05
30–34	Pleasure	3	3,713.6	272,824.1	73.47
30–34	Pleasure	4	704.2	101,294.5	143.85
35–39	Business	1	24.8	0.0	0.00
35–39	Business	2	26.3	0.0	0.00
35–39	Business	3	93.9	16,303.5	173.64
35–39	Business	4	21.0	6,283.5	299.78
35–39	DriveLong	1	381.5	55,915.9	146.57
35–39	DriveLong	2	349.1	57,144.1	163.68
35–39	DriveLong	3	1,026.2	83,512.8	81.38
35–39	DriveLong	4	284.5	18,426.5	64.77

Table 2. (Continued)

Age	VUSE	Credit	Exposure	Loss	Pure Prem
35–39	DriveShort	1	2,916.6	462,268.3	158.50
35–39	DriveShort	2	2,671.6	406,378.3	152.11
35–39	DriveShort	3	7,354.7	791,419.1	107.61
35–39	DriveShort	4	2,051.5	177,397.5	86.47
35–39	Pleasure	1	1,759.2	304,840.8	173.28
35–39	Pleasure	2	1,895.8	253,239.5	133.58
35–39	Pleasure	3	5,284.4	634,395.0	120.05
35–39	Pleasure	4	1,719.3	158,092.6	91.95
40–49	Business	1	58.6	6,144.3	104.86
40–49	Business	2	71.6	2,904.0	40.54
40–49	Business	3	241.5	20,189.9	83.59
40–49	Business	4	119.4	8,570.0	71.79
40–49	DriveLong	1	740.4	108,353.8	146.35
40–49	DriveLong	2	796.7	116,826.4	146.63
40–49	DriveLong	3	2,345.8	272,806.7	116.30
40–49	DriveLong	4	1,071.7	90,416.8	84.36
40–49	DriveShort	1	6,005.9	909,030.1	151.36
40–49	DriveShort	2	5,920.7	872,424.1	147.35
40–49	DriveShort	3	17,811.1	1,922,925.0	107.96
40–49	DriveShort	4	8,117.4	768,209.6	94.64
40–49	Pleasure	1	4,141.6	750,012.4	181.09
40–49	Pleasure	2	4,655.8	654,742.2	140.63
40–49	Pleasure	3	15,053.2	1,842,087.0	122.37
40–49	Pleasure	4	7,641.1	727,944.0	95.27
50–59	Business	1	47.5	13,664.7	287.88
50–59	Business	2	80.6	9,389.2	116.54
50–59	Business	3	274.9	81,673.0	297.08
50–59	Business	4	153.2	17,521.7	114.37
50–59	DriveLong	1	531.5	39,548.2	74.41
50–59	DriveLong	2	617.7	62,526.3	101.22
50–59	DriveLong	3	1,977.2	166,025.0	83.97
50–59	DriveLong	4	1,290.2	88,343.5	68.47
50–59	DriveShort	1	4,367.9	598,852.8	137.10
50–59	DriveShort	2	4,635.3	615,743.5	132.84
50–59	DriveShort	3	15,020.7	1,512,889.7	100.72
50–59	DriveShort	4	9,795.8	725,559.5	74.07
50–59	Pleasure	1	4,128.7	682,331.4	165.27
50–59	Pleasure	2	4,719.2	608,792.4	129.00
50–59	Pleasure	3	15,841.4	1,653,298.5	104.37
50–59	Pleasure	4	11,439.1	923,608.0	80.74
60+	Business	1	18.7	0.0	0.00
60+	Business	2	36.2	1,331.6	36.76
60+	Business	3	134.5	23,698.8	176.21
60+	Business	4	133.1	16,844.7	126.56
60+	DriveLong	1	174.2	25,849.2	148.41
60+	DriveLong	2	203.7	31,320.5	153.79
60+	DriveLong	3	776.2	55,812.1	71.90
60+	DriveLong	4	705.7	38,051.9	53.92
60+	DriveShort	1	1,400.5	175,722.8	125.47
60+	DriveShort	2	1,648.6	157,108.6	95.30
60+	DriveShort	3	6,334.3	556,852.1	87.91
60+	DriveShort	4	6,236.2	553,343.2	88.73
60+	Pleasure	1	5,237.7	650,696.3	124.23
60+	Pleasure	2	4,725.1	567,174.2	120.03
60+	Pleasure	3	22,656.9	2,129,405.7	93.99
60+	Pleasure	4	31,601.4	2,601,434.7	82.32

Table 3. The age and vehicle-use relativities for Algorithms 1–8

Algorithm	17–20	21–24	25–29	30–34	35–39	40–49	50–59	60+	Business	DTW Long	DTW Short	Pleasure
1	1.319	1.280	1.190	1.151	0.919	1.005	1.019	1.000	1.642	1.262	1.042	1.000
2	1.483	1.204	1.178	1.140	0.872	1.012	1.020	1.000	1.801	1.260	1.087	1.000
3	1.276	1.351	1.205	1.161	0.953	1.002	1.020	1.000	1.646	1.239	1.020	1.000
4	1.343	1.256	1.171	1.145	0.905	1.003	1.015	1.000	1.641	1.260	1.042	1.000
5	1.371	1.289	1.190	1.150	0.922	1.005	1.018	1.000	1.647	1.261	1.040	1.000
6	1.307	1.301	1.206	1.156	0.931	1.007	1.022	1.000	1.644	1.264	1.042	1.000
7	1.303	1.318	1.220	1.159	0.939	1.010	1.026	1.000	1.647	1.266	1.042	1.000
8	1.298	1.276	1.190	1.152	0.918	1.004	1.019	1.000	1.639	1.263	1.043	1.000

Table 4. The performance measurements for Algorithms 1–8

Algorithm	wab	$wapb$	$wChi$	$\sqrt{wab * wChi}$
1	11.190	4.45%	1.022	3.3815
2	14.588	5.96%	1.426	4.5612
3	10.577	4.01%	1.096	3.4051
4	11.664	4.70%	1.032	3.4696
5	11.192	4.42%	1.015	3.3705
6	10.826	4.26%	1.029	3.3376
7	10.669	4.15%	1.043	3.3358
8	11.208	4.47%	1.029	3.3967

Table 5. Numerical iterations for multiplicative gamma GIA factors using average severity as the base

Iteration	Base	17–20	21–24	25–29	30–34	35–39	40–49	50–59	60+	Business	DTW Long	DTW Short	Pleasure
1	241.46	1.20355	1.20764	1.15438	1.12367	0.89052	0.97098	0.95341	0.92183	1.39343	1.07369	0.88745	0.85417
2	241.46	1.23983	1.23441	1.14473	1.09725	0.88339	0.95575	0.96996	0.94864	1.39890	1.07549	0.88654	0.85098
3	241.46	1.24057	1.23475	1.14465	1.09689	0.88323	0.95554	0.97017	0.94908	1.39898	1.07551	0.88653	0.85093
4	241.46	1.24059	1.23476	1.14465	1.09688	0.88323	0.95554	0.97017	0.94909	1.39898	1.07551	0.88653	0.85093

Table 6. Numerical iterations for multiplicative gamma GIA factors using 60+ and pleasure as the base

Iteration	Base	17–20	21–24	25–29	30–34	35–39	40–49	50–59	60+	Business	DTW Long	DTW Short	Pleasure
1	190.126	1.30561	1.31004	1.25228	1.21896	0.96604	1.05332	1.03426	1.00000	1.63132	1.25700	1.03896	1.00000
2	194.924	1.30696	1.30124	1.20671	1.15666	0.93122	1.00749	1.02247	1.00000	1.64387	1.26382	1.04178	1.00000
3	195.003	1.30713	1.30100	1.20606	1.15574	0.93062	1.00681	1.02222	1.00000	1.64406	1.26393	1.04183	1.00000
4	195.004	1.30714	1.30100	1.20605	1.15573	0.93061	1.00680	1.02221	1.00000	1.64406	1.26393	1.04183	1.00000

Table 7. Numerical iterations for multiplicative gamma GLM coefficients using 60+ and pleasure as the base

Iteration	Base	17–20	21–24	25–29	30–34	35–39	40–49	50–59	60+	Business	DTW Long	DTW Short	Pleasure
1	5.271	0.24473	0.25635	0.18704	0.14541	−0.07528	0.00653	0.02240	0.00000	0.49442	0.23602	0.04304	0.00000
2	5.273	0.26833	0.26300	0.18728	0.14470	−0.07206	0.00682	0.02197	0.00000	0.49730	0.23429	0.04118	0.00000
3	5.273	0.26783	0.26312	0.18735	0.14473	−0.07192	0.00677	0.02197	0.00000	0.49717	0.23423	0.04099	0.00000
4	5.273	0.26784	0.26313	0.18735	0.14473	−0.07192	0.00677	0.02197	0.00000	0.49717	0.23423	0.04098	0.00000

Table 8. Numerical iterations for multiplicative gamma GLM factors using 60+ and pleasure as the base

Iteration	Base	17–20	21–24	25–29	30–34	35–39	40–49	50–59	60+	Business	DTW Long	DTW Short	Pleasure
1	194.612	1.27728	1.29221	1.20567	1.15651	0.92749	1.00655	1.02265	1.00000	1.63954	1.26620	1.04398	1.00000
2	194.986	1.30778	1.30082	1.20596	1.15570	0.93047	1.00684	1.02222	1.00000	1.64427	1.26401	1.04204	1.00000
3	195.004	1.30713	1.30099	1.20605	1.15573	0.93061	1.00680	1.02221	1.00000	1.64406	1.26394	1.04184	1.00000
4	195.004	1.30714	1.30100	1.20605	1.15573	0.93061	1.00680	1.02221	1.00000	1.64406	1.26393	1.04183	1.00000

Table 9. Numerical iterations for additive factors of additive GIA with $p = 1$ using average severity as the base

Iteration	Base	17–20	21–24	25–29	30–34	35–39	40–49	50–59	60+	Business	DTW Long	DTW Short	Pleasure
1	241.46	1.20355	1.20764	1.15438	1.12367	0.89052	0.97098	0.95341	0.92183	0.39376	0.07209	−0.11175	−0.14513
2	241.46	1.24727	1.21924	1.13818	1.10128	0.87575	0.95865	0.97266	0.95556	0.39840	0.07410	−0.11308	−0.14928
3	241.46	1.24807	1.21951	1.13796	1.10091	0.87552	0.95841	0.97293	0.95619	0.39848	0.07413	−0.11310	−0.14936
4	241.46	1.24808	1.21952	1.13796	1.10090	0.87552	0.95840	0.97293	0.95620	0.39848	0.07413	−0.11310	−0.14936
5	241.46	1.24808	1.21952	1.13796	1.10090	0.87552	0.95840	0.97293	0.95620	0.39848	0.07413	−0.11310	−0.14936

Table 10. Numerical iterations for dollar values of additive GIA with $p = 1$ using age 60+ and pleasure as the base

Iteration	Base	17–20	21–24	25–29	30–34	35–39	40–49	50–59	60+	Business	DTW Long	DTW Short	Pleasure
1	187.541	68.024	69.011	56.153	48.736	−7.559	11.867	7.626	0.000	130.121	52.452	8.060	0.000
2	194.684	70.435	63.668	44.094	35.184	−19.272	0.746	4.128	0.000	132.244	53.937	8.743	0.000
3	194.816	70.477	63.583	43.892	34.945	−19.478	0.537	4.043	0.000	132.281	53.964	8.756	0.000
4	194.818	70.478	63.581	43.889	34.941	−19.481	0.533	4.041	0.000	132.282	53.964	8.756	0.000
5	194.818	70.478	63.581	43.889	34.941	−19.481	0.533	4.041	0.000	132.282	53.964	8.756	0.000

Table 11. Convergence process of mixed GIA with $p = 1$

Iter-ation	Busi-ness	DTW Long	DTW Short	Pleasure	17–20	21–24	25–29	30–34	35–39	40–49	50–59	60+	Credit1	Credit2	Credit3	Credit4
1	1.3493	0.9731	1.0532	0.9539	2.2395	1.6630	1.3075	1.0138	1.0127	1.0042	0.8634	0.7888	0.3006	0.1852	−0.0468	−0.1722
2	1.4916	0.9169	0.9752	1.0245	2.2467	1.6483	1.2706	0.9787	0.9894	1.0028	0.8745	0.8154	0.3188	0.2046	−0.0446	−0.1947
3	1.4887	0.9221	0.9800	1.0199	2.2433	1.6412	1.2586	0.9691	0.9835	1.0002	0.8753	0.8239	0.3200	0.2057	−0.0441	−0.1966
4	1.4886	0.9246	0.9823	1.0177	2.2423	1.6397	1.2563	0.9674	0.9824	0.9996	0.8752	0.8253	0.3200	0.2056	−0.0440	−0.1967
5	1.4886	0.9251	0.9827	1.0173	2.2421	1.6395	1.2560	0.9672	0.9823	0.9995	0.8752	0.8255	0.3200	0.2056	−0.0440	−0.1966
6	1.4886	0.9251	0.9828	1.0172	2.2421	1.6394	1.2560	0.9672	0.9823	0.9995	0.8752	0.8255	0.3200	0.2056	−0.0440	−0.1966