# Risk-Adjusted Underwriting Performance Measurement

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**ABSTRACT**

To measure economic profits generated by an insurance policy during its lifetime, we compare the terminal assets of the policy account with certain break-even value. The break-even value is an increasing function of the claims risk and the asset investment risk. It can be calculated with closed-form formulas. We study policies with multiyear loss payments and tax payments. Profits from underwriting and from capital investment are measured separately. Relationships between the cost of capital and the risk-adjusted discount rate of loss are derived. Methods developed in the paper are also useful for fair premium calculation.

**KEYWORDS**

*Risk-adjusted performance measure, risk-adjusted discount rate, policy-year profit, cost of capital, EVA, fair premium*

# 1. Introduction

A typical property-casualty insurance policy covers one accident year, but its claims stay open for many years. The ultimate profitability of the policy cannot be determined until all claims are settled. In the interim, profits may be estimated by projecting future investment gains and claim and expense payments. At the moment the premium is collected, a policy account is established, of which its initial asset is the premium net of acquisition expenses. The value of the asset increases as investment incomes accumulate, and decreases as losses, expenses, and taxes are paid. At the end of the policy life, the terminal asset of the policy account is the ultimate profit generated by the policy. A policy is considered profitable if its terminal asset exceeds a certain break-even value. A main goal of the paper is to calculate this break-even value.

A number of favorable factors allow a policy to generate positive profits. These include higher premiums collected, lower losses and expenses paid, payments made later rather than sooner, and extraordinary investment gains. If all these factors are at their fair or expected values, the resulting terminal asset is the break-even terminal asset. Modern finance teaches that the expected return of an asset investment is in direct proportion to the investment's risk. Likewise, the break-even terminal asset will be shown to be directly related to the claims risk and the asset investment risk.

For investment portfolios, there are many risk-adjusted performance measures, including the well-known Sharpe ratio, the Treynor ratio (ratio-based tests) and Jensen's alpha (a value-based test), all thoroughly discussed in Part 7 of Bodie, Kane, and Marcus (2002). In insurance, the risk-adjusted return on capital (RAROC) and the economic value added (EVA) have become popular. All these measures, however, are designed for testing performances in one time period. It is much harder to construct a performance test for a real insurance policy, whose claim payments span across multiple years. The RAROC and the EVA are usually applied on the calendar-year basis, and thus cannot answer the question whether a policy (or a policy year) is ultimately profitable.[1] The internal rate of return (IRR) of equity flows is a valid policy-year metric. Yet it cannot be called a risk-adjusted measure unless it is explicitly linked to the underwriting and investment risks. Such a link will be discussed in this paper. Further, the RAROC, IRR, and EVA all measure combined profits from both underwriting and investment operations. It is useful for the underwriting managers to know if the underwriting operation alone, for a particular policy or in a particular year, is successful. Our approach will address this issue directly.

Several researchers have studied performance measurement for a multi-year property-casualty (P&C) policy. Bingham (1993) and Bender (1997) divide the balance sheet asset into an insurance product account and a surplus account, and study cash flows from one account to the other or to shareholders. Their insurance product account is similar to our policy account. Schirmacher and Feldblum (2006) provide a detailed study on accounting issues. It examines how profits emerge over time and shows that calendar-year profits depend on the accounting system. Our focus is on the ultimate economic profit, which is independent of the accounting system used. In computing the EVA, Schirmacher and Feldblum (2006) assumes that a cost of capital (COC) is given extraneously. As just mentioned, we will relate it to internal risk metrics.

Profit measurement is intertwined with fair premium determination. The break-even value of profit is derived by assuming that premium is at its fair level, and investment returns and loss and expense payments are all at their expected values. Conversely, the fair premium may be determined by setting to zero the market value of the policy account terminal asset. Our study thus may be found useful for premium calculation.

The paper is organized as follows. Section 2 starts with a discussion of the risk-adjusted loss discount

[1] Some one-year tests have been adapted for use in a multi-year framework and on the policy-year basis, see Goldfarb (2006). But these are only tentative solutions and lack solid theoretical support.

rate. The discount rate is used to quantify a policy's risk. Then, for a single-year model, the break-even value of terminal assets is derived. The break-even value is an increasing function of asset and claims risks. This result is generalized to a multiyear model in Section 3. Here a numerical example is introduced that will be used throughout the paper to illustrate calculations. The main results of the paper are stated in Section 4. Income tax is brought into the model. Closed-form formulas are derived for the fair premium and break-even terminal assets. To obtain tractable results, tax rules are simplified. In Section 5, we give equations linking the break-even terminal assets to the COC. A consequence of the relationship is that the COC is an increasing function of the claims risk, and a decreasing function of the initial capital. In Section 6, we show how to define the EVA for the policy account only, and separately measure profits from underwriting and from capital investment. Section 7 further shows that, since the loss discount rate and the COC both characterize the internal risk of the company (assuming investments are risk-free), each can be derived from the other with simple equations. The fair premium for a policy may be calculated by selecting either a loss discount rate or a target COC. Section 8 concludes the paper.

## 2. Benchmark of underwriting profit

Issuing an insurance policy establishes a mini-bank, which we will call a policy account. If we assume the acquisition expense is paid at the moment the premium is collected, then the starting asset of a policy account is premium minus acquisition expenses. The value of the asset changes over time. Investment gains increase the value, and loss and expense payments decrease it. The remaining assets right after the last payment is made are called the terminal assets. Actuaries who have done analysis on finite reinsurance contracts should be familiar with the calculation of terminal assets. For a policy to be profitable over its lifetime, the terminal assets must be positive and sufficiently high. Intuitively, the more risky the policy, the greater the terminal assets need to be. The purpose of this paper is to derive a benchmark for terminal assets in the framework of modern finance.

### 2.1. Discount rate for liabilities

Pricing actuaries use various "loadings" to quantify risk embedded in claims. A loading may be additive (a dollar amount charge) or multiplicative (a percentage charge), or may be calculated by risk-adjusted discounting (using lower discount rates to create a higher present value). The most convenient way for presenting our results is risk-adjusted discounting.

The following example illustrates how the risk-adjusted discount rate reflects the risk level of a policy. Assume a policy has only one loss, which will be paid in one year, and the expected payment is \$100. The risk-free interest rate is 4 percent. As in Myers and Cohn (1987) and Taylor (1994), we define the fair premium for the policy to be the sum of market values of future claims, expenses, and capital costs. If the claim amount is \$100 certain, its market value is the present value at the risk-free rate 100/1.04 = 96.15. Market value of a claim increases with riskiness of the claim, whereas the risk-adjusted discount rate decreases. Table 1 shows the relationship between the discount rate and the market value, corresponding to different levels of claims risk.

In general, let $r_f$ be the risk-free rate, $L$ the random claim amount, and $r_l$ the claim's risk-adjusted discount rate. Then the risk-free present value is $\mathrm{PV}(L) = E[L]/(1 + r_f)$, and the market value is $\mathrm{MV}(L) = E[L]/(1 + r_l)$. For a risky claim, $\mathrm{MV}(L) > \mathrm{PV}(L)$, the difference being the claim's risk margin.[2] This implies $r_l < r_f$. If risk is low, $r_l$ is close to $r_f$. For a highly risky policy, $r_l$ can be negative.

[2]Although it is intuitively clear that if a claim is risky, its value should contain a positive risk margin, empirical evidence that supports the assertion is sparse. Most authors assume the risk margin is positive (Myers and Cohn 1987; Bingham 2000). But Feldblum (2006) suggests that most P&C liabilities have no systematic risk (i.e., uncorrelated with the market return), thus the risk margin equals zero. The disagreement can only be settled with further empirical research.

**Table 1. Risk-adjusted discount rate**

| Riskiness of Claim (1) | Expected Claim (2) | Risk-free Present Value (3) | Market Value (4) | Risk-Adjusted Discount Rate (5) |
|---|---|---|---|---|
| No Risk | 100.00 | 96.15 | 96.15 | 4% |
| Low Risk | 100.00 | 96.15 | 97.09 | 3% |
| High Risk | 100.00 | 96.15 | 200.00 | –50% |

(5) = (2)/(4) – (1)

## 2.2. Break-even terminal assets

Let $p$ be a policy premium net of acquisition expenses. Then $p$ is the starting balance of the policy account. Assume the balance is invested in a financial security of which its annual return is a random variable $R_a$; the policy losses $L$, paid one year later, has a risk-adjusted discount rate of $r_l$. So the expected return of assets is $r_a = E[R_a]$, and the market value of loss is $\mathrm{MV}(L) = E[L]/(1 + r_l)$. Capital cost is not considered in this section.

If $p$ is the fair premium, then $p = \mathrm{MV}(L) = E[L]/(1 + r_l)$. At the end of the year the assets grow to $\mathrm{MV}(L)(1 + R_a) = E[L](1 + R_a)/(1 + r_l)$. After loss $L$ is paid, the terminal assets[3] are

$$A_1 = \mathrm{MV}(L)(1+R_a) - L.$$

The expected value of terminal assets is

$$\begin{aligned} a_1 = E[A_1] &= \mathrm{MV}(L)(1+r_a) - E[L] \\ &= \mathrm{MV}(L)(r_a - r_l) = \frac{E[L]}{1+r_l}(r_a - r_l), \qquad (2.1) \end{aligned}$$

since $r_a \geq r_f$ and $r_l \leq r_f$, $a_1 \geq 0$. A more risky security has a greater $r_a$, and a more risky liability has a smaller $r_l$. So the spread $r_a - r_l$ is a measure of total risk of the policy account. $a_1$ is in proportion to the total risk.

The term $a_1$ is the risk-adjusted break-even value of terminal assets. If the actual realized terminal assets are greater than $a_1$, the company makes money on the policy. If the terminal assets are only slightly positive, but less than $a_1$, the company appears to make money, but actually does not make enough to compensate for risk. If the claim or the investment is very risky, the terminal assets need to be very high.

Comparing the actual terminal assets with $a_1$ gives us a value-based test for underwriting performance.[4] It is interesting to compare this with the combined ratio, a ratio-based underwriting performance measure. The undiscounted combined ratio is used most often. But it has an obvious shortcoming—it does not reflect the time value of money. Two lines of business may have the same combined ratio, but the longer-tailed line pays out losses more slowly, generates more investment income along the way, and is more profitable. The economic combined ratio (ECR) is introduced to correct this problem. In the calculation of ECR, all underwriting cash flows are discounted at the risk-free rate to the time of policy inception. The ECR is strongly advocated in Swiss Re (2006), which claims that the ECR of 100 percent "truly indicates the watershed between profit and loss" (p. 24). When investments and claims are risky, however, the risk-adjusted break-even ECR is not 100 percent, but something lower. Table 2 gives the break-even ECR and the break-even terminal assets for the three policies in Table 1, with the following additional assumptions: first, fair premium is charged and there is no expense, i.e., $p = \mathrm{MV}(L)$; and second, investment is riskless, i.e., $r_a = r_f = 4\%$.

The ECRs in Table 2 are computed with the fair premium in the denominator. So they are the break-even ECRs. The table shows that if claims risk is high, the break-even ECR is much lower than 100 percent, and $a_1$ is very large compared to $E[L]$.

[3]Terminal assets are profits generated by the policy over its lifetime. I choose to use "assets" rather than "profits" because, for a multiyear model, I will keep track of values of assets from year to year.

[4]Note that if $A_1 < a_1$ ($A_1 > a_1$), then the policy loses (makes) money in this particular year. However, this may be the result of pure randomness and have nothing to do with the quality of the underwriting decision. To assess the quality of underwriting decision, one needs to track profits over a long period or look at the aggregate profit of many policies.

**Table 2. Break-even ECR and break-even terminal assets**

| Riskiness of Claim (1) | $E[L]$ (2) | $r_l$ (3) | $p$ = MV($L$) (4) | PV($L$) (5) | Break-even ECR (6) | $a_1$ (7) |
|---|---|---|---|---|---|---|
| No Risk | 100.00 | 4% | 96.15 | 96.15 | 100.0% | 0.00 |
| Low Risk | 100.00 | 3% | 97.09 | 96.15 | 99.0% | 0.97 |
| High Risk | 100.00 | –50% | 200.00 | 96.15 | 48.1% | 108.00 |

(6) = (5)/(4)

(7) = (4) × (4% – (3))

# 3. Multiyear underwriting profit measure

## 3.1. Break-even terminal assets $a_n$

A typical P&C policy has a multiyear payout pattern. Assume a policy is written at time 0, and the loss payments are random variables $L_1, \ldots, L_n$, $L_i$ paid at time $i$. The nominal total loss is $L = L_1 + \cdots + L_n$. We will derive a break-even value for terminal assets at time $n$, after all losses are paid. Assume a loss discount rate $r_l$, constant throughout the $n$ years, can be found that correctly reflects the risk of the payments $L_i$.[5] Then the market value, at time 0, of the payment stream is

$$\mathrm{MV}_0(L) = \frac{E[L_1]}{1+r_l} + \cdots + \frac{E[L_n]}{(1+r_l)^n}$$
$$= \sum_{i=1}^{n} \frac{E[L_i]}{(1+r_l)^i}. \quad (3.1)$$

The fair premium of the policy, net of expenses, equals this market value: $p = \mathrm{MV}_0(L)$. Let the premium be invested risk free, and $r_f$ be the constant risk-free rate.[6] The following formulas give the net assets at each time $i$, right after loss $L_i$ is paid

$$A_1 = p(1+r_f) - L_1$$
$$A_2 = p(1+r_f)^2 - L_1(1+r_f) - L_2$$
$$\vdots \qquad \vdots$$
$$A_n = p(1+r_f)^n - L_1(1+r_f)^{n-1} - \cdots - L_n.$$

Then the expected terminal asset $a_n = E[A_n]$ is the benchmark value of terminal assets of the policy. Taking the expected value of the last formula and substituting Equation (3.1) for $p$, we get

$$a_n = (1+r_f)^n \sum_{i=1}^{n} E[L_i]\left((1+r_l)^{-i} - (1+r_f)^{-i}\right)$$
$$= (1+r_f)^n \left(\mathrm{MV}_0(L) - \mathrm{PV}_0(L)\right). \quad (3.2)$$

$\mathrm{MV}_0(L) - \mathrm{PV}_0(L)$ is the risk margin of the policy losses, which is greater than 0 if $r_l < r_f$. A less risky policy has a smaller spread $r_f - r_l$, and a relatively smaller $a_n$. A very risky policy can have a negative $r_l$, and a very large $a_n$.

The above $a_n$ is computed from expected values of loss payments, investment returns, and discount rates. In practice, the expected loss payments are given by an actuarial payout pattern, and the expected investment returns are provided by the market (spot rates of bonds with suitable maturities). On the other hand, there is no consensus on the best approach of getting the loss discount rates. Assuming this information can be obtained in the pricing process, we may calculate $a_n$ using (3.2). As the years go by, we trace the actual policy account assets $A_i$ until the year $n$ when all losses are paid.[7] The actual terminal asset

[5]The constancy of $r_l$ is a simplifying assumption. To accurately reflect the riskiness of each $L_i$, the discount rate should be allowed to vary over time. In this more general scenario, similar equations can be derived for $A_i$, but a closed-form solution for $a_n$ would not be possible when tax enters the calculation later in the paper.

[6]For simplicity, results on multiyear models in this paper are stated for a constant risk-free investment return. The same derivation works for random (risky) rates of return, too, but it would not be possible to obtain a closed-form formula for the expected terminal asset.

[7]The actual closing year may be longer or shorter. Here we assume it is exactly $n$ for simplicity.

$A_n$ is then compared with $a_n$ to assess the policy's ultimate profitability. An interim projection of the ultimate profitability is possible, if loss payments and investment returns in the remaining years can be projected. A number of beneficial factors can render a policy more profitable: higher (than expected) premium, smaller losses, slower loss payments or higher investment returns. Among them the premium is what a company has the most control over.

## 3.2. A numerical example

I will use a multiyear numerical example to illustrate the calculations in this paper. The example is borrowed from Schirmacher and Feldblum (2006), so we can compare their methods with ours. Assume a policy is issued on Dec. 31, 20XX, for accident year 20XX+1. The underwriting cash flows are as follows. On Dec. 31, 20XX (time 0), a premium of \$1,000 is collected and acquisition expenses of \$275 paid. General expenses of \$150 are paid six months later (time 0.5). The policyholder has one accident in the year and will receive one payment of \$650 on Dec. 31, 20XX+3 (time 3).

Schirmacher and Feldblum (2006) choose a surplus requirement of 25 percent of the unearned premium reserve plus 15 percent of the loss reserve. The risk-free rate is 8 percent per year compounded semiannually (4 percent per half year). The policy account cash flows and balances are summarized in Table 3.

The nominal losses ($L$) and expenses ($X$) add up to \$1,075, and the premium ($p$) is \$1,000. The underwriting profit is –\$75 and the combined ratio is 1,075/1,000 = 107.5%. The risk-free present value of losses and expenses $PV_0(L + X) = 932.94$, and the ECR equals $PV_0(L + X)/p = 93.29\%$. By the ECR standard (as proposed in Swiss Re 2006) the policy is profitable, but the ECR standard incorrectly ignores risk.

To compute the risk-adjusted break-even value $a_{3.0}$, we need a few more assumptions. Assume $r_l$ = 3% per half year, and the only loss payment of \$650 at time 3 is both the expected and the actual loss. By (3.2), the break-even net assets at time 3 are $(1 + 0.04)^6 \cdot 650 \cdot ((1 + 0.03)^{-6} - (1 + 0.04)^{-6}) = 38.80$. Since the actual terminal assets are \$84.86 (column 6 of Table 3, last entry), greater than \$38.80, the policy is profitable under the risk-adjusted measure.[8]

**Table 3. Policy account assets—no tax**

| Time (1) | Premium (2) | Expense (3) | Loss (4) | Investment Income (5) | Policy Account Assets (6) |
|---|---|---|---|---|---|
| 0.0 | 1000.00 | 275.00 | 0.00 | 0.00 | 725.00 |
| 0.5 | 0.00 | 150.00 | 0.00 | 29.00 | 604.00 |
| 1.0 | 0.00 | 0.00 | 0.00 | 24.16 | 628.16 |
| 1.5 | 0.00 | 0.00 | 0.00 | 25.13 | 653.29 |
| 2.0 | 0.00 | 0.00 | 0.00 | 26.13 | 679.42 |
| 2.5 | 0.00 | 0.00 | 0.00 | 27.18 | 706.59 |
| 3.0 | 0.00 | 0.00 | 650.00 | 28.26 | 84.86 |
| Sum | 1000.00 | 425.00 | 650.00 | 159.86 | |
| PV | 1000.00 | 419.23 | 513.70 | | |

$(6)_{0.0} = (2)_{0.0} - (3)_{0.0} - (4)_{0.0}$

For $i > 0.0$, $(5)_i = (6)_{i-0.5} \times r_f$

For $i > 0.0$, $(6)_i = (6)_{i-0.5} + (2)_i - (3)_i - (4)_i + (5)_i$

The economic value added by the policy is 84.86 – 38.80 = 46.06. However, this assessment overstates the profitability of the policy, because taxes are omitted so far. Taxes are a significant cost, which I will discuss in the following sections.

# 4. After-tax profit measures

Insurance companies have a greater tax burden than non-financial companies. In addition to taxes on profits from underwriting and premium investment, a company has to pay taxes on income from capital investment. This cost is dubbed double taxation—a shareholder's capital investment is taxed twice, first at the corporate level and then at the personal level (when he sells the stock). Since an investor does not pay the corporate tax if he directly buys securities on the market, this tax must be covered by premium to guarantee that an insurance company shareholder does not have a cost disadvantage. This means that all tax payments, including that charged on income from capital investment, need to be deducted from policy

[8]In this illustration, for simplicity, I assume that the actual loss payments, investment returns and discount rates are identical to their expected values. But these two sets of values are usually different.

account assets. Income taxes are generally a fixed percentage of the pre-tax income. But the precise IRS tax codes are complex. I will make simplifying assumptions to obtain closed-form, tractable results.

## 4.1. Single-year model

Let $c$ be the initial capital contributed by shareholders. The capital serves two purposes. First, the company invests the capital to earn income. Second, with the safety margin provided by the capital, the company is able to issue insurance policies. Assume the company issues policies and collects premium $p$ (net of expenses), and invests the total cash $c + p$ in securities. The policy loss $L$ is paid one year later, and the remaining assets are returned to shareholders.

Assume there is a single tax rate, denoted by $t$, for both underwriting and investment profits. The pretax operating income, from both the policy account and the capital investment, is $(p - L) + (p + c)R_a = p(1 + R_a) - L + cR_a$. The total income tax, paid at time 1, is $t(p(1 + R_a) - L) + tcR_a$.[9] The whole tax payment should be deducted from the policy account. The policy account's after-tax net assets are

$$A_1 = p(1+R_a) - L - t\left(p(1+R_a) - L\right) - tcR_a$$
$$= (1-t)\left(p(1+R_a) - L - \frac{tc}{1-t}R_a\right). \qquad (4.1)$$

The policy premium $p$ is fair, as defined in Section 2.1, if it makes the market value of terminal assets zero. In (4.1), we set $\mathrm{MV}(A_1) = 0$. Clearly, $\mathrm{MV}(1 + R_a) = 1$ and $\mathrm{MV}(1) = 1/(1 + r_f)$, which imply $\mathrm{MV}(R_a) = r_f/(1 + r_f)$. These formulas give

$$p = \mathrm{MV}(L) + \frac{tcr_f}{(1-t)(1+r_f)}. \qquad (4.2)$$

The second term is the amount of premium needed to cover taxes on the investment income of capital, which is in direct proportion to $c$. So too much capital hurts the company in price competition. Also, note that the fair premium is not affected by how the premium and capital are invested. To derive the expected terminal assets for the policy, we substitute (4.2/4.1) into (4), and calculate the expected values

$$a_1 = (1-t)\left(\left(\mathrm{MV}(L) + \frac{tcr_f}{(1-t)(1+r_f)}\right)\left(1+E[R_a]\right) - E[L]\right) - tcE[R_a]$$

$$= (1-t)\mathrm{MV}(L)(r_a - r_l) - \frac{tc}{1+r_f}(r_a - r_f)$$

$$= (1-t)\frac{E[L]}{1+r_l}(r_a - r_l) - \frac{tc}{1+r_f}(r_a - r_f). \qquad (4.3)$$

This is the risk-adjusted break-even value for the after-tax terminal assets. In (4.3) the first term is essentially the after-tax version of the break-even value (2.1), and the second term reflects tax on capital. A policy generates a profit if and only if its after-tax terminal assets are greater than $a_1$.

Consider a special case where the investment is risk free, i.e., $r_a = r_f$. Formula (4.3) reduces to

$$a_1 = (1-t)\frac{E[L]}{1+r_l}(r_f - r_l). \qquad (4.4)$$

Remember $r_l \le r_f$, and the riskier the policy, the smaller the $r_l$. So a riskier policy has a greater break-even value (4.4). Note that capital $c$ does not appear in (4.4). This is because, as the investment is risk free, the amount of tax on capital gain is certain, and is exactly covered by the second component of fair premium (4.2).

## 4.2. Multiyear model

The main goal of the paper is to derive the break-even value of terminal assets in the most general setting—multiyear, with income tax, which I will do in the remainder of the paper. To obtain tractable results I will make simplified assumptions on the timing and amount of tax payments. Assume taxes are paid at time 1, 2, . . . . (The interval between $i - 1$ and $i$ need not be one year. In the example in Table 3, each time period is one half year.) Tax paid at time $i$ equals tax rate (a constant) times taxable income earned between times $i - 1$ and $i$. For the purpose of income calculation, I assume loss reserves are discounted at

[9] In practice, when taxable income is negative, the company may not be able to receive the full tax credit in the current year. But we ignore this complication here.

the risk-adjusted rate (also a constant). The simplified tax rules are summarized below.

$$\text{Tax Paid}_i = t \times \begin{pmatrix} \text{Underwriting Gain}_i \\ + \text{Investment Gain}_i \end{pmatrix}$$

$$\text{Investment Gain}_i = r_f \times \text{Investible Assets}_{i-1}$$

$$\text{Underwriting Gain}_1 = p - L_1 - \text{Loss Reserve}_1$$

$$\begin{aligned}\text{Underwriting Gain}_i = & -L_i - \text{Loss Reserve}_i \\ & + \text{Loss Reserve}_{i-1}\end{aligned}$$

$$\text{Loss Reserve}_i = \text{MV}_i\left(\text{Unpaid Loss}_i\right)$$

where Unpaid Loss$_i$ means the future payment stream $L_{i+1}$, $L_{i+2}$, . . . , and MV$_i$(Unpaid Loss$_i$) is the present value of the payment stream discounted at $r_l$. These rules will be used to derive a break-even value for after-tax terminal assets. A test on profitability is to compare actual terminal assets, which follows the IRS tax rules, with this break-even value. Deviation of the simplified tax rules from the IRS rules would create some distortion, which I hope is not material.

Derivation of results in a multiyear model is inevitably complicated. I will state the main results here, and present their proofs in appendices. Again, use $A_i$, $i = 0, 1, \ldots, n-1$, to denote the (random) assets of the policy account at time $i$, after loss $L_i$ is paid. (Note that $A_0 = p$.) Let $c_i$, $i = 0, 1, \ldots, n-1$, be the amount of capital held, so that $A_i + c_i$ is the total investable assets of the company at time $i$. The term $c_i$ might be an amount required by regulators (as assumed in Schirmacher and Feldblum 2006), or desired by the company management. Again we assume assets are invested risk free, and the risk-free rate $r_f$ is constant for all years.

The fair premium (net of expenses) $p$ satisfies the equation MV$_0(A_n) = 0$. The following theorem gives a concise, closed-form formula for the fair premium.

**Theorem 1.** The fair premium $p$ is given by

$$\begin{aligned}p = \text{MV}_0\left(L\right) + \frac{tr_f}{\left(1-t\right)\left(1+r_f\right)}\Bigg(c_0 + \frac{c_1}{1+\left(1-t\right)r_f} \\ + \cdots + \frac{c_{n-1}}{\left(1+\left(1-t\right)r_f\right)^{n-1}}\Bigg). \qquad (4.5)\end{aligned}$$

The first term in the formula is the fair premium when there is no tax. The second term can be considered a present value of the tax payments on capital investments, where the discount rate is the after-tax interest rate $(1 - t)r_f$. Premium calculation will be discussed in depth in Section 7. Next, to derive the break-even value for the after-tax terminal assets, we start from time 0 with premium (4.5), and successively compute underwriting, investment and tax cash flows, and the net policy account assets $A_i$. The result is also a simple closed-form formula.

**Theorem 2.** The break-even value for the after-tax terminal assets in the policy account is given by

$$\begin{aligned}a_n = & \frac{\left(1-t\right)\left(r_f - r_l\right)\left(1+\left(1-t\right)r_f\right)^n}{\left(1-t\right)r_f - r_l} \\ & \sum_{i=1}^{n} E\left[L_i\right]\left(\frac{1}{\left(1+r_l\right)^i} - \frac{1}{\left(1+\left(1-t\right)r_f\right)^i}\right) \\ = & \frac{\left(1-t\right)\left(r_f - r_l\right)\left(1+\left(1-t\right)r_f\right)^n}{\left(1-t\right)r_f - r_l} \\ & \left(\text{MV}_0\left(L\right) - \text{PV}_0^{\text{tax}}\left(L\right)\right), \qquad (4.6)\end{aligned}$$

where $\text{PV}_0^{\text{tax}}(L)$ stands for the present value discounted at the after-tax interest rate $(1 - t)r_f$. Recall that $r_l$ is the discount rate in MV$_0(L)$ and $r_l$ usually is less than $r_f$. Obviously, $\text{MV}_0(L) - \text{PV}_0^{\text{tax}}(L)$ and $(1 - t) r_f - r_l$ always have the same sign, which means (4.6) is always positive. Equation (4.6) reduces to (3.2) when there is no tax.

The fair premium $p$ carries a charge for tax on the investment income of capital (second term in (4.5)), so $p$ is a function of the capital amounts $c_i$. Interestingly, calculation of the break-even value $a_n$, Formula (4.6), does not involve $c_i$. This is because the tax charge exactly covers all those taxes, under the condition that investment returns are deterministic.[10] As in the case of single-year model (Equation (4.3)), if investments are risky, the break-even terminal assets will depend on $c_i$.

[10]For example, if capitals are larger, then (1) income taxes on capital investments are greater; (2) the fair premium is higher based on (4.5). The additional tax expense is exactly funded by the additional premium, and the expected terminal assets remain the same.

**Table 4. Policy account assets—after tax**

| Time (1) | Premium (2) | Expense (3) | Loss (4) | Tax (5) | Investment Income (6) | Policy Account Assets (7) |
|---|---|---|---|---|---|---|
| 0.0 | 1,000.00 | 275.00 | 0.00 | –26.25 | 0.00 | 751.25 |
| 0.5 | 0.00 | 150.00 | 0.00 | 32.45 | 30.05 | 598.86 |
| 1.0 | 0.00 | 0.00 | 0.00 | 29.39 | 23.95 | 593.42 |
| 1.5 | 0.00 | 0.00 | 0.00 | 8.13 | 23.74 | 609.03 |
| 2.0 | 0.00 | 0.00 | 0.00 | 7.97 | 24.36 | 625.43 |
| 2.5 | 0.00 | 0.00 | 0.00 | –3.57 | 25.02 | 654.01 |
| 3.0 | 0.00 | 0.00 | 650.00 | –3.38 | 26.16 | 33.55 |
| Sum | 1,000.00 | 425.00 | 650.00 | 44.73 | 153.28 | |
| PV | 1,000.00 | 419.23 | 513.70 | 40.55 | | |

Column (5) from Table 2 in Schirmacher and Feldblum (2006).

$(7)_{0.0} = (2)_{0.0} - (3)_{0.0} - (4)_{0.0} - (5)_{0.0}$

For $i > 0.0$, $(6)_i = (7)_{i-0.05} \times r_f$

For $i > 0.0$, $(7)_i = (7)_{i-0.5} + (2)_i - (3)_i - (4)_i - (5)_i + (6)_i$

These formulas are easy to apply. In Table 4, a tax column is added to Table 3, and the policy account assets at each time are recalculated by deducting taxes. (Note that the tax column contains all taxes, including that on capital investments.) The last entry of column 7 gives the policy's after-tax terminal assets of \$33.55. Given the rates $r_f = 4\%$, $r_l = 3\%$ and $t = 35\%$, we compute $\mathrm{MV}_0(L) = 650/(1 + 0.03)^6 = 544.36$, and $\mathrm{PV}_0^{\mathrm{tax}}(L) = 650/(1 + (1 - 0.35) \times 0.04)^6 = 557.22$. Substituting these figures into (4.6), we get the break-even terminal assets $a_{3.0} = 24.37$. Since this is less than the actual terminal assets of \$33.55, the policy is profitable. The value added by the policy is $33.55 - 24.37 = 9.18$.

## 5. Linking break-even terminal assets to cost of capital

The cost of capital (COC) is the rate of return on capital required by shareholders. The rate of return is a random variable, and the COC is the expected value of this rate. According to the Capital Asset Pricing Model (see Bodie, Kane, and Marcus 2002), the COC is in direct proportion to the systematic risk. In more general settings, it is customary to expect that the higher the risk (however it is defined), the higher the COC. For the insurance models under consideration, risk of the shareholder return comes from two sources—the volatile investment returns and the volatile claim payments. In the preceding sections, I have shown that the expected values of claims and investment gains determine the expected value of terminal assets in the policy account. The total return to shareholders is the sum of policy account terminal assets and the investment return on capital. Therefore, the COC is a simple function of the break-even terminal assets and the expected investment return on capital. This relationship may also be used in the reverse manner: if the COC is obtained through stock analysis, a required level of terminal assets can be deduced, from which a fair premium may be calculated.

### 5.1. Single-year model

The expected value of policy account terminal assets is given in Equation (4.3). The expected value of capital investment at time 1 is $c(1 + r_a)$. Therefore, the total expected after-tax net assets at time 1 are

$$(1-t)\mathrm{MV}(L)(r_a - r_l) - \frac{tc}{1+r_f}(r_a - r_f) + c(1+r_a). \tag{5.1}$$

The expected rate of return on capital is

$$\mathrm{COC}=\frac{(1-t)\mathrm{MV}(L)}{\mathrm{c}}(r_a-r_l)-\frac{t}{1+r_f}(r_a-r_f)+r_a. \tag{5.2}$$

By (5.2), the COC is the sum of three components: the investment rate of return $r_a$; the after-tax spread $(1-t)(r_a-r_l)$ times the "leverage ratio" $\mathrm{MV}(L)/c$; and a quantity related to taxes on capital investment, which vanishes if the investment is risk free. The following factors would cause the COC to increase (i.e., shareholders to require a greater return): riskier investments (greater $r_a$), more volatile claims (smaller $r_l$), or a higher leverage ratio. Increasing the amount of capital would reduce the leverage ratio, hence reduce the COC.

In Appendix B, I will explain that Formula (5.2) is consistent with the Capital Asset Pricing Model (CAPM). In the CAPM world, an asset's expected return is in direct proportion to its β, which is a measure of the asset's systematic risk. I will use the given asset rate $r_a$ and liability rate $r_l$ to determine β of the shareholder return, and show the expected value of the return—the COC—is exactly given by (5.2).

## 5.2. Multiyear model

In a multiyear model, shareholders contribute an initial capital $c_0$ and establish a capital account. The capital account then earns investment income and pays out dividends (releases capital). Let $c_i$ be the amount of capital held at time $i$. Then the total assets at time $i$ are $A_i + c_i$.[11] Assume dividends prior to the ending year are entirely deducted from the capital account; the policy account only releases its profit at time $n$.[12] Dividends can be determined if the $c_i$ are known: the dividend at time $i$ is $c_{i-1}$ plus its investment income in the year minus $c_i$. If each $c_i$ is invested at the constant risk-free rate $r_f$, then the dividend flows (out of the capital account) are $-c_0$, $c_0(1+r_f)-c_1$, $c_1(1+r_f)-c_2, \ldots, c_{n-1}(1+r_f)$. Obviously, the internal rate of return (IRR) of these flows is $r_f$.

The expected terminal assets $a_n$ of the policy account is given in (4.6). Thus the expected total dividend flows, from the company to its shareholders, are $-c_0$, $c_0(1+r_f)-c_1$, $c_1(1+r_f)-c_2, \ldots, c_{n-1}(1+r_f)+a_n$. The IRR of the total dividend flows is given by the equation

$$c_0=\frac{c_0(1+r_f)-c_1}{1+IRR}+\frac{c_1(1+r_f)-c_2}{(1+IRR)^2}+\cdots+\frac{c_{n-1}(1+r_f)+a_n}{(1+IRR)^n}. \tag{5.3}$$

This IRR is the average—over $n$ years—cost of capital of the company. After a policy has run its course, we can compute the IRR of the actual capital flows. If the IRR is greater than (less than) the average COC given by (5.3), the company's overall operation is profitable (unprofitable). It is worth noting that, since $a_n > 0$, the COC must be greater than the expected rate of return of asset investment. On the other hand, a greater claims risk implies a greater spread $r_f - r_l$, thus a greater $a_n$ and a greater COC.

Back to the example of Table 4. In their paper, Schirmacher and Feldblum (2006) assume the required capital is 25 percent of the unearned premium reserve plus 15 percent of the loss reserve. They then compute the required assets at each time $i$ and the corresponding dividend flows. Table 5 shows the dividend flows out of the capital account (column 6), the total dividend flows (column 7), and the break-even flows (column 8). These three columns only differ in their last entry. The IRR for column 6 equals the asset rate of return of 4 percent, as expected. The IRR for column 7 is 6.18% (obtained also in Schirmacher and Feldblum 2006). The IRR for column 8, 5.62%, is the COC. Since the IRR of the actual total dividend flows is greater than the COC, the company adds value for shareholders.

[11]The asset in this paper corresponds to the income-producing asset in Schirmacher and Feldblum (2006). Non-income-producing assets are not considered.

[12]This distinction between the policy account and the capital account does not affect profit measurement of the company as a whole. But it is important for measuring the policy account profit separately from the capital account.

**Table 5. Dividend flows**

| Time (1) | Policy Account Assets (2) | Capital Held (3) | Total Assets (4) | Investment Income on Capital (5) | Capital Account Dividend Flow (6) | Total Dividend Flow (7) | Break-even Dividend Flow (8) |
|---|---|---|---|---|---|---|---|
| 0.0 | 751.25 | 428.75 | 1,180.00 | 0.00 | –428.75 | –428.75 | –428.75 |
| 0.5 | 598.86 | 362.62 | 961.48 | 17.15 | 83.28 | 83.28 | 83.28 |
| 1.0 | 593.42 | 149.53 | 742.95 | 14.50 | 227.60 | 227.60 | 227.60 |
| 1.5 | 609.03 | 122.54 | 731.58 | 5.98 | 32.97 | 32.97 | 32.97 |
| 2.0 | 625.43 | 94.77 | 720.20 | 4.90 | 32.67 | 32.67 | 32.67 |
| 2.5 | 654.01 | 79.84 | 733.85 | 3.79 | 18.73 | 18.73 | 18.73 |
| 3.0 | 33.55 | 0.00 | 33.55 | 3.19 | 83.03 | 116.58 | 107.40 |
| IRR | | | | | 4.00% | 6.18% | 5.62% |

Column (2) is column (7) in Table 4.

Column (4) from Table 7 in Schirmacher and Feldblum (2006).

For $i > 0.0$, $(5)_i = (3)_{i-0.5} \times r_f$

For $i > 0.0$, $(6)_i = (2)_{i-0.5} + (5)_i - (2)_i$

For $i < 3.0$, $(7)_i = (6)_i$; $(7)_{3.0} = (6)_{3.0} + (7)_{3.0}$ in Table 4

For $i < 3.0$, $(8)_i = (6)_i$; $(8)_{3.0} = (6)_{3.0} + a_{3.0}$ ($a_{3.0}$ calculated in Section 4.2)

## 6. Decomposing the EVA

Shareholders expect to earn a rate of return equal to the cost of capital. If they earn more than (less than) the COC, then the investment adds (destroys) value. The economic value added is defined as (see, e.g., Schirmacher and Feldblum 2006)

$$\text{EVA} = \text{After-tax Net Income} - \text{COC} \times \text{Capital Held}.$$

The second term in the formula, COC $\times$ Capital Held, is the break-even value of the after-tax income. So this approach of measuring profits is similar to what we described in Sections 2 to 4. The difference is that the EVA measures the total profits, while our method addresses profitability of the policy account. However, it is straightforward to decompose the EVA into one measure for the policy account and another for the capital account.

### 6.1. Single-year model

For the single-year model, the COC is given in Formula (5.2), which can be split into two parts. The last term, $r_a$, is the hurdle rate for the capital account. If the actual return on capital is greater than $r_a$, then the investment operation adds value. By Section 4.1, the first two terms of (5.2) give the hurdle rate of the policy account. This prompts us to define the EVA for the two accounts separately:

$$\text{EVA}_c = c \times \left(\text{Actual Investment Rate} - r_a\right), \text{ and} \quad (6.1)$$

$$\text{EVA}_p = \text{Actual After-tax Terminal Assets} - (1-t)\text{MV}(L)(r_a - r_l) + \frac{tc}{1+r_f}(r_a - r_f). \quad (6.2)$$

In (6.2) the actual assets in the policy account are after all taxes, including those on capital gains. Readers familiar with investment portfolio analysis may recognize that the rate spread, Actual Investment Rate $- r_a$, in (6.1) is Jensen's alpha for the asset portfolio. Obviously, $\text{EVA} = \text{EVA}_c + \text{EVA}_p$. If an actuary wishes to review the past underwriting performance and determine price changes, $\text{EVA}_p$ would provide more accurate information than the EVA.

Other methods have been developed to separately measure underwriting and investment activities. Bingham (2004) proposes to allocate capital between underwriting and asset investment, find a cost of capital for each of the functions, and separately calculate their value creation. In practice, some companies build models to calculate the underwriting ROE, the investment ROE, and ROEs at various policy group or investment portfolio levels. Our method has some unique features. First, it emphasizes that income tax on capital investment should be covered by policy account profits. Second, it treats the capital account as an ordinary investment portfolio, and tests it with the established Jensen's alpha. Third, it is consistent with the CAPM (see Appendix B), so is theoretically solid.

The policy account itself consists of two activities, underwriting (collecting premiums and paying losses) and investment of premium. These two activities are intertwined and cannot be measured separately. For example, an increase in premium is an achievement of the underwriting department. The resulting gain in profit should be credited entirely to underwriting, not to investment. But the additional premium generates an additional investment income, which cannot be cleanly attributed to either underwriting or investment. Also, the policy account covers income tax on capital investments. It is not clear whether this tax should be covered by underwriting profits or by investment income. Even in our method, performances of the capital account and the policy account are not completely independent. If capital investment generates a higher return, the corresponding income tax increases, which reduces $EVA_p$.

## 6.2. Multiyear model

In a general, multiyear setting, the EVA is calculated annually based on each year's income. A policy with an $n$-year payout pattern has a stream of $n$ EVAs whose value depend on the valuation of loss reserves under selected accounting rules. Schirmacher and Feldblum (2006) compute the EVA stream in two accounting systems, the net present value (NPV) and the IRR. I will not deal with accounting here, but only discuss the measurement of ultimate economic profits at the end of the policy life.

The IRR of the total dividend flows, denoted by $IRR_{tot}$, is a standard profit measure of shareholders' investment. The cost of capital is the break-even value of $IRR_{tot}$. As shown in Section 5.2, the total dividend flows are the sum of two component flows: a single flow at time $n$—the terminal assets—from the policy account, and a stream of dividends from the capital account. Methods of evaluating the two component flows have essentially been derived in previous sections, which are summarized below.

For the policy account, we define $EVA_p$ thus:

$$EVA_p = \text{Actual After-tax Terminal Assets} - a_n. \tag{6.3}$$

$EVA_p$ is the ultimate cash value added by the policy. For the example in Table 5, $EVA_p = 33.55 - 24.37 = 9.18$. For the capital account, we use $IRR_c$ to denote the IRR of the dividends that flow out of the capital account. The break-even value of $IRR_c$ is the expected investment return $r_a$ (or $r_f$, if capitals are invested risk free). Clearly, if $EVA_p > 0$ and $IRR_c > r_a$, then $IRR_{tot} > COC$; conversely, if $EVA_p < 0$ and $IRR_c < r_a$, then $IRR_{tot} < COC$. Note that $EVA_p$, $IRR_c$, and $IRR_{tot}$ are all independent of the accounting system. $EVA_p$ provides more useful information to an underwriting manager than $IRR_{tot}$ does.

To illustrate the importance of measuring the underwriting performance separately from capital investment, we give an example in which a profitable company has a very unprofitable underwriting result. An unexpected large loss exhausts policy account assets before claims are settled. That is, $A_i \leq 0$ for some $i < n$. The policy account assets thus stay negative for all later years. The negative $EVA_p$ correctly indicates that the policy is unprofitable. On the other hand, the capital investment happens to generate large returns, which produces an $IRR_{tot}$ exceeding the COC, indicating a profitable overall operation. $IRR_{tot}$ here is no indication of the underwriting performance.

## 7. Comparing direct and indirect pricing methods

Insurance pricing methods are broadly divided into two types. The *direct* methods are exemplified by Myers and Cohn (1987), Equation (3.4). In these methods, risk of future claims is quantified by a risk loading—in this paper, the risk-adjusted discount rate. In an *indirect* method, a target return on capital is first chosen, which reflects the total risk of claims and investments. Then the premium is back-solved to achieve this target. If both methods correctly capture risk, they should produce identical fair premiums. The formulas derived in Sections 4 and 5 give us a mathematical relationship between the two methods.

Formula (4.5) is a direct method for computing $p$. The key parameter that captures the risk of claims is $r_l$.[13] A corresponding indirect method works through the following steps: Choose a COC, substitute it into (5.3) for IRR to get $a_n$, solve (4.6) (using a numerical method like Goal Seek or Solver in Excel) for $r_l$, and then compute $p$ with (4.5). This also shows how $r_l$ and the COC, the two parameters characterizing risk, uniquely determine each other.

I will demonstrate these calculations using the same familiar example. In Section 4.2, we selected $r_l = 3\%$ and calculated $\mathrm{MV}_0(L) = 544.36$. Substituting this $\mathrm{MV}_0(L)$ and the $c_i$ in column 3 of Table 5 into (4.5), yields $p = 569.08$. Loading in the present value of expenses, \$419.23, we get the full policy premium of \$988.31. (This break-even value is less than the actual charge of \$1,000. Thus the policy is a good deal to begin with.) To illustrate the steps leading from the COC to $r_l$ and $p$, we adopt the assumption in Schirmacher and Feldblum (2006) that the COC is 5 percent. Plugging this IRR into the denominators of (5.3), and the $c_i$ in column 3 of Table 5 into the numerators, we obtain $a_n = \$14.76$. Now use Equation (4.6). Since there is only one loss payment of \$650 at time 3, $\mathrm{MV}_0(L) = 650/(1 + r_l)^6$, where $r_l$ is an unknown variable. Also $\mathrm{PV}_0^{\mathrm{tax}}(L) = 557.22$. Plugging these and all known parameters into (4.6) and solving for $r_l$, we have $r_l = 3.39\%$. Using this $r_l$ in (4.5) gives $p = 556.98$. Adding in \$419.23 for expenses, we obtain the total policy premium of \$976.21. To sum up, $r_l = 3\%$ corresponds to a COC of 5.62% (column 8 of Table 5), and $r_l = 3.39\%$ corresponds to a COC of 5 percent; the first scenario is more risky, and has a higher premium of \$988.31 (vs. \$976.21 for the second scenario).

To use these approaches in a pricing project, it is imperative to determine the capitals (required or desired) $c_0, c_1, \ldots, c_{n-1}$. In a multiline company, the company-wide capital needs to be allocated at each time $i$. There are numerous research papers on capital allocation.

[13]Although we have been addressing calculating $\mathrm{MV}_0(L)$ with the risk-adjusted discount rate, Formula (4.5) would still apply if $\mathrm{MV}_0(L)$ can be obtained with another method.

## 8. Conclusions

Risk-adjusted performance measures are studied in this paper. The focus is on the year-by-year change of assets in the policy account. A break-even value for the policy account terminal assets is found, which, when compared with the actual terminal assets, determines whether the policy has added value over its life time. The main results of the paper are the two theorems in Section 4.2. In contrast to most existing risk-adjusted performance measures, like the RAROC and the EVA, our approach addresses the underwriting profits separately from capital investment results. Relationships between the break-even terminal assets, the cost of capital, and the EVA are also discussed.

A key input for calculating the break-even terminal assets and the fair premium, using Formulas (4.5) and (4.6), is the loss discount rate $r_l$. It characterizes the claims risk of the policy. There is a straightforward link between $r_l$ and the COC, the latter reflecting the combined risk of claims and investments. If one is given, the other can be derived. How to select either $r_l$ or the COC to correctly reflect risk is a challenge. The COC is computed relative to a sequence of capitals $c_i$. Determination of the capitals also needs further research.

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# Appendices

## A. Proof of the theorems

### A.1. Proof of Theorem 1

Formula (4.5) can be written as

$$p = \sum_{i=1}^{n} \left( \mathrm{MV}_0(L_i) + \frac{tr_f c_{i-1}}{(1-t)(1+r_f)(1+(1-t)r_f)^{i-1}} \right). \tag{A.1}$$

We only need to prove the theorem for each $i$, that is, for a policy with a single loss payment $L_i$ at time $i$, and with a single nonzero capital $c_{i-1}$ at time $i-1$ (considered beginning of year $i$). The fair premium is given by

$$p = \mathrm{MV}_0(L_i) + \frac{tr_f c_{i-1}}{(1-t)(1+r_f)(1+(1-t)r_f)^{i-1}}. \tag{A.2}$$

If (A.2) holds for all $i$, then (A.1) is true simply by additivity.

If time $i$ is the only time losses and capital account taxes are paid, then at all other times $j$, $j = 1, \ldots i-1, i+1, \ldots n$, the only payments are taxes on policy account profits. These profits need to be carefully calculated according to the rules stated in Section 4.2.

Let $\mathrm{MV}_j(L_i)$ be the market value of $L_i$ at time $j < i$. The value of $\mathrm{MV}_j(L_i)$ will not be known until time $j$. Therefore, viewed at time 0, $\mathrm{MV}_j(L_i)$ is a random variable.[14] To simplify notations, let $\mathrm{MV}_j(L_i)$ be denoted by $V_j$. The loss reserve at time $j < i$ is $V_j$, and loss reserves are zero after time $i$. In the following derivation, I start by calculating the net assets at time 1, move forward in time, and end up with terminal assets at time $n$. I then set the market value of the terminal assets to zero, and solve for the fair premium $p$.

At time 1, the underwriting gain is $p - V_1$ and the investment gain is $pr_f$. Then the tax is $t(p(1 + r_f) - V_1)$. The net assets at time 1 are

$$\begin{aligned} A_1 &= p(1+r_f) - t(p(1+r_f) - V_1) \\ &= p(1+r_f)(1-t) + tV_1. \end{aligned}$$

At time 2, the underwriting gain is $V_1 - V_2$ and the investment gain is $A_1 r_f$. Then the tax is $t(V_1 - V_2 + A_1 r_f)$. The net assets at time 2 are

$$\begin{aligned} A_2 &= A_1(1+r_f) - t(V_1 - V_2 + A_1 r_f) \\ &= p(1+r_f)(1-t)(1+(1-t)r_f) + tV_2 + r_f t(1-t)V_1. \end{aligned}$$

In general, the tax at any time $j < i$ is $t(V_{j-1} - V_j + A_{j-1} r_f)$, and it is not hard to prove by induction that the net assets at $j$ are given by the formula

$$\begin{aligned} A_j &= p(1+r_f)(1-t)(1+(1-t)r_f)^{j-1} \\ &\quad + tV_j + r_f t(1-t)\big(V_{j-1} + (1+(1-t)r_f)V_{j-2} \\ &\quad + \cdots + (1+(1-t)r_f)^{j-2} V_1\big). \end{aligned} \tag{A.3}$$

Now this formula holds for $A_{i-1}$. At time $i$, there are two additional payments, loss $L_i$ and tax on capital

[14] Rigorously, the market values $\mathrm{MV}_0(L_i)$, $\mathrm{MV}_1(L_i)$, . . . , $\mathrm{MV}_{i-1}(L_i)$, $L_i$, are a stochastic process adapted to a filtration indexed by time $j$.

investment $tr_f c_{i-1}$, and no further loss reserves. So the total tax is $t(V_{i-1} - L_i + A_{i-1}r_f) + tr_f c_{i-1}$, and the net assets are

$$
\begin{aligned}
A_i &= A_{i-1}(1+r_f) - t(V_{i-1} - L_i + A_{i-1}r_f) - tr_f c_{i-1} \\
&= p(1+r_f)(1-t)(1+(1-t)r_f)^{i-1} - (1-t)L_i \\
&\quad + r_f t(1-t)\Big(V_{i-1} + (1+(1-t)r_f)V_{i-2} \\
&\quad + \cdots + (1+(1-t)r_f)^{i-2}V_1\Big) - r_f tc_{i-1}. \qquad \text{(A.4)}
\end{aligned}
$$

After time $i$, every year the assets $A_i$ are reinvested and taxes on the investment gains paid. The after-tax investment rate of return is $(1 - t)r_f$. So, the terminal assets at time $n$ are

$$A_n = A_i(1+(1-t)r_f)^{n-1}. \qquad \text{(A.5)}$$

The fair premium $p$ is so defined as to make the market value of $A_n$ zero. By (A.5), $\mathrm{MV}_0(A_n) = 0$ if and only if $\mathrm{MV}_0(A_i) = 0$. So we need to calculate the market value of each term in (A.4).

The first and the last term in (A.4) are nonrandom constants. The market value, at time 0, of a constant is the constant divided by $(1 + r_f)^i$. The market values of other terms in (A.4) are obtained using the formula

$$\mathrm{MV}_0(V_j) = \mathrm{MV}_0(L_i)\big/(1+r_f)^{i-j}. \qquad \text{(A.6)}$$

I will use this formula now to complete the proof. The formula itself will be proved later in the section.

$$
\begin{aligned}
&\mathrm{MV}_0(A_i) \\
&\quad = \mathrm{MV}_0\left(p(1+r_f)(1-t)(1+(1-t)r_f)^{i-1}\right) \\
&\qquad - (1-t)\mathrm{MV}_0(L_i) + r_f t(1-t)\Big(\mathrm{MV}_0(V_{i-1}) \\
&\qquad + (1+(1-t)r_f)\mathrm{MV}_0(V_{i-2}) + \cdots \\
&\qquad + (1+(1-t)r_f)^{i-2}\mathrm{MV}_0(V_1)\Big) - \mathrm{MV}_0(r_f tc_{i-1})
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{p(1-t)(1+(1-t)r_f)^{i-1}}{(1+r_f)^{i-1}} - (1-t)\mathrm{MV}_0(L_i) \\
&\quad + \frac{r_f t(1-t)}{1+r_f}\mathrm{MV}_0(L_i)\left(1 + \frac{1+(1-t)r_f}{1+r_f}\right. \\
&\quad \left. + \cdots + \frac{(1+(1-t)r_f)^{i-2}}{(1+r_f)^{i-2}}\right) - \frac{r_f tc_{i-1}}{(1+r_f)^i} \\
&= \frac{p(1-t)(1+(1-t)r_f)^{i-1}}{(1+r_f)^{i-1}} - (1-t)\mathrm{MV}_0(L_i) \\
&\quad + \frac{r_f t(1-t)}{1+r_f}\mathrm{MV}_0(L_i)\left(1 - \frac{(1+(1-t)r_f)^{i-1}}{(1+r_f)^{i-1}}\right) \\
&\quad \div \left(1 - \frac{1+(1-t)r_f}{1+r_f}\right) - \frac{r_f tc_{i-1}}{(1+r_f)^i} \\
&= \frac{p(1-t)(1+(1-t)r_f)^{i-1}}{(1+r_f)^{i-1}} \\
&\quad - \frac{(1-t)(1+(1-t)r_f)^{i-1}}{(1+r_f)^{i-1}}\mathrm{MV}_0(L_i) - \frac{r_f tc_{i-1}}{(1+r_f)^i}
\end{aligned}
$$

Setting $\mathrm{MV}_0(A_i) = 0$ and solving for $p$, we get the formula (A.2). This proves Theorem 1.

Note that in the derivation of (8) we do not need the assumption that there is a constant risk-adjusted discount rate $r_l$. Therefore, (8) can be used to calculate the fair premium whenever the market value $\mathrm{MV}_0(L)$ can be reasonably estimated.

## A.2. Proof of formula (A.6)

In formula (A.4), $A_i$ is a random variable conditioned on all information up to time $i$. This conditioning statement is important when computing the market value of $V_j$. $V_j = \mathrm{MV}_j(L_i)$ is a random variable viewed at any point $j' < j$, but is nonrandom at any $j' > j$. So it is easy to first discount $V_j$ to time $j$,

$$\mathrm{MV}_j(V_j) = V_j\big/(1+r_f)^{i-j}.$$

Then, further discounting the above to time 0, we get

$$\mathrm{MV}_0(V_j) = \mathrm{MV}_0(\mathrm{MV}_j(L_i)) / (1+r_f)^{i-j}$$
$$= \mathrm{MV}_0(L_i) / (1+r_f)^{i-j},$$

which proves (A.6).[15]

## A.3. Proof of Theorem 2

Theorem 2 says if a policy charges premium (4.5), then its expected terminal assets at time $n$ have the form (4.6). I will again prove the theorem by splitting it into $n$ simpler components. For any $i < n$, assume that a subpolicy $i$ has premium (A.2), makes only one loss payment $L_i$ at time $i$, and is supported by one nonzero capital $c_{i-1}$ at time $i-1$. I will prove that the expected terminal assets of the subpolicy, at time $n$, are given by the formula

$$a_{n,i} = \frac{(1-t)(r_f - r_l)(1+(1-t)r_f)^n}{(1-t)r_f - r_l} E[L_i]\left(\frac{1}{(1+r_l)^i} - \frac{1}{(1+(1-t)r_f)^i}\right). \quad \text{(A.7)}$$

Obviously, the expected terminal assets of the original policy is the sum of these $a_{n,i}$. This will prove Theorem 2.

Substituting (A.2) into the right-hand side of (A.4), we have

$$\begin{aligned} A_i = {} & \left(V_0 + \frac{t r_f c_{i-1}}{(1-t)(1+r_f)(1+(1-t)r_f)^{i-1}}\right) \\ & \times (1+r_f)(1-t)(1+(1-t)r_f)^{i-1} - (1-t)L_i \\ & + r_f t(1-t)\left(V_{i-1} + (1+(1-t)r_f)V_{i-2}\right. \\ & \left. + \cdots + (1+(1-t)r_f)^{i-2} V_1\right) - r_f t c_{i-1}. \end{aligned}$$

The assumption that there is a constant loss discount rate $r_l$ implies that $E[V_j] = E[L_i]/(1 + r_l)^{i-j}$. Noting that the $c_{i-1}$ terms cancel out, we have

$$\begin{aligned} E[A_i] = {} & (1+r_f)(1-t)(1+(1-t)r_f)^{i-1}\frac{E[L_i]}{(1+r_l)^i} \\ & - (1-t)E[L_i] + \frac{r_f t(1-t)}{1+r_l}E[L_i] \\ & \left(1 + \frac{1+(1-t)r_f}{1+r_l} + \cdots + \frac{(1+(1-t)r_f)^{i-2}}{(1+r_l)^{i-2}}\right) \\ = {} & (1+r_f)(1-t)(1+(1-t)r_f)^{i-1}\frac{E[L_i]}{(1+r_l)^i} \\ & - (1-t)E[L_i] + \frac{r_f t(1-t)}{1+r_l}E[L_i] \\ & \left(1 - \frac{(1+(1-t)r_f)^{i-1}}{(1+r_l)^{i-1}}\right) \div \left(1 - \frac{1+(1-t)r_f}{1+r_l}\right) \\ = {} & (1+r_f)(1-t)(1+(1-t)r_f)^{i-1}\frac{E[L_i]}{(1+r_l)^i} \\ & - (1-t)E[L_i] + \frac{r_f t(1-t)}{r_l - (1-t)r_f}E[L_i] \\ & \left(1 - \frac{(1+(1-t)r_f)^{i-1}}{(1+r_l)^{i-1}}\right) \\ = {} & \left((1+r_f)(1-t)(1+(1-t)r_f)^{i-1}\frac{E[L_i]}{(1+r_l)^i}\right. \\ & \left. - \frac{r_f t(1-t)}{r_l - (1-t)r_f}E[L_i]\frac{(1+(1-t)r_f)^{i-1}}{(1+r_l)^{i-1}}\right) \\ & + \left(-(1-t)E[L_i] + \frac{r_f t(1-t)}{r_l - (1-t)r_f}E[L_i]\right) \\ = {} & \frac{(1-t)(r_f - r_l)(1+(1-t)r_f)^i}{(1-t)r_f - r_l}\frac{E[L_i]}{(1+r_l)^i} \\ & - \frac{(1-t)(r_f - r_l)}{(1-t)r_f - r_l}E[L_i]. \end{aligned}$$

$A_i$ is the net asset at time $i$. After time $i$, the asset grows at the after-tax investment yield $(1-t)r_f$. So the

[15] A rigorous proof of the formula may be stated with stochastic discount factors. The technique is standard in asset pricing theory.

terminal net assets at time $n$ is $A_{n,i} = A_i(1 + (1 - t)r_f)^{n-i}$. Therefore,

$$a_{n,i} = E[A_{n,i}] = E[A_i](1+(1-t)r_f)^{n-i}$$
$$= \frac{(1-t)(r_f - r_l)(1+(1-t)r_f)^n}{(1-t)r_f - r_l}$$
$$E[L_i]\left(\frac{1}{(1+r_l)^i} - \frac{1}{(1+(1-t)r_f)^i}\right).$$

This proves (A.7), and thus completes the proof of Theorem 2.

## B. Formula (5.2) is consistent with the CAPM

Assume our insurance company exists in a CAPM world. That is, the invested assets, the claims and the shareholders' capital all satisfy the CAPM. For the invested assets we have

$$E[R_a] = r_a, \qquad r_a - r_f = \beta_a m, \tag{B.1}$$

where $m$ is the market risk premium. For a policy with premium $p$ (net of expenses) and future claim $L$, we define its "return" to be $R_l = (L - p)/p$, and assume that

$$E[R_l] = r_l, \qquad r_l - r_f = \beta_l m. \tag{B.2}$$

This liability CAPM is a natural extension of the standard (investment) CAPM, and has been proposed by many authors; see, for example, Sherris (2003). $r_l \le r_f$ implies $\beta_l \le 0$. An implicit assumption in this CAPM framework is that all assets are traded at the market value, and all policies are charged the fair premium, which equals the market value of claims.

Formula (4.2) provides a policy's fair premium $p$. Substituting it into formula (4.1) gives the after-tax policy account net assets at time 1. Adding in the value of the capital amount we have the following total net assets:

$$C = (1-t)\mathrm{MV}(L)(R_a - R_l) - \frac{tc}{1+r_f}(R_a - r_f)$$
$$+ c(1+R_a). \tag{B.3}$$

The return on capital, $R_c = (C - c)/c$, is

$$R_c = \frac{(1-t)\mathrm{MV}(L)}{c}(R_a - R_l) - \frac{t}{1+r_f}(R_a - r_f) + R_a$$
$$= \left(1 + \frac{(1-t)\mathrm{MV}(L)}{c} - \frac{t}{1+r_f}\right)R_a$$
$$- \frac{(1-t)\mathrm{MV}(L)}{c}R_l + \frac{t}{1+r_f}r_f. \tag{B.4}$$

Based on formula (B.4), the shareholder investment $C$ can be replicated by the following three investments: short-selling an asset, whose return is $R_l$, and receiving cash $(1 - t)\mathrm{MV}(L)$; lending an amount $tc/(1 + r_f)$ at the risk-free rate $r_f$; and buying an asset, whose return is $R_a$, with the net cash $c + (1 - t)\mathrm{MV}(L) - tc/(1 + r_f)$. Thus, the $\beta$ of the investment $C$ is given by the weighted average of $\beta$'s of the three investments

$$\beta_c = \left(1 + \frac{(1-t)\mathrm{MV}(L)}{c} - \frac{t}{1+r_f}\right)\beta_a$$
$$- \frac{(1-t)\mathrm{MV}(L)}{c}\beta_l. \tag{B.5}$$

If $C$ satisfies the CAPM, then $E[R_c] - r_f = \beta_c m$. Using (B.1) and (B.2), we obtain

$$E[R_c] = \left(1 + \frac{(1-t)\mathrm{MV}(L)}{c} - \frac{t}{1+r_f}\right)r_a$$
$$- \frac{(1-t)\mathrm{MV}(L)}{c}r_l + \frac{t}{1+r_f}r_f$$
$$= r_a + \frac{(1-t)\mathrm{MV}(L)}{c}(r_a - r_l) - \frac{t}{1+r_f}(r_a - r_f). \tag{B.6}$$

This is exactly the COC in Formula (5.2).