Tail Factor Convergence in Sherman's Inverse Power Curve Loss Development Factor Model

Jon Evans

ABSTRACT

The infinite product of the age-to-age development factors in Sherman's inverse power curve model is proven to converge to a finite number when the power parameter is less than −1, and alternatively to diverge to infinity when the power parameter is −1 or greater. For the convergent parameter values, a simple formula is derived, in terms of any finite product of age-to-age factors, for the endpoints of an interval containing the limit of the infinite product. These endpoints converge to the limit as the finite time cutoff point increases. For any finite time cutoff, the product of age-to-age factors lies below the interval, and thus the lower endpoint of the interval is always a better estimate of the limit than the finite product itself. Several numerical examples are included for illustration. The convergence condition and the interval formula are applicable to the selection of a finite cutoff age, review of the reasonability of the convergence rate, and actual numerical calculations of the tail factor.

KEYWORDS

Tail factor, inverse power curve.

1. Background and introduction

Sherman (1984) found that an inverse power curve of the form $1 + a\,(t + c)^b$ fit empirical age-to-age loss development factors better than several other basic functional forms he tested. Lowe and Mohrman (1985) expressed concern about the convergence of the product of the age-to-age factors. Boor (2006, p. 373), and the CAS Tail Factor Working Party (2013, p. 52) noted that there has been no known closed-form expression that approximates the tail generated by the inverse power curve.

In practice, the age-to-age development factors produced by the curve are multiplied together out to some finite age cutoff, such as $t = 80$, to produce a cumulative development factor. The impact of factors beyond that age to ultimate, or the tail factor beyond the cutoff, in this case $t = 81 \ldots$, is assumed to be negligible. Alternatively, if the impact of the tail factor is not negligible, then some other modeling consideration must inform the selection of the cutoff time.

The potential danger in the assumption of negligible tail factor impact is illustrated in Table 1 and Figure 1. Two different sets of parameters share the same initial age-to-age factor of 1.01 at $t = 1$ and the same cumulative factor of 1.30 from $t = 1$ to 100. However, while the cumulative factor for Example 1, using power parameter $b = -4.0$, grows only a little past $t = 100$,

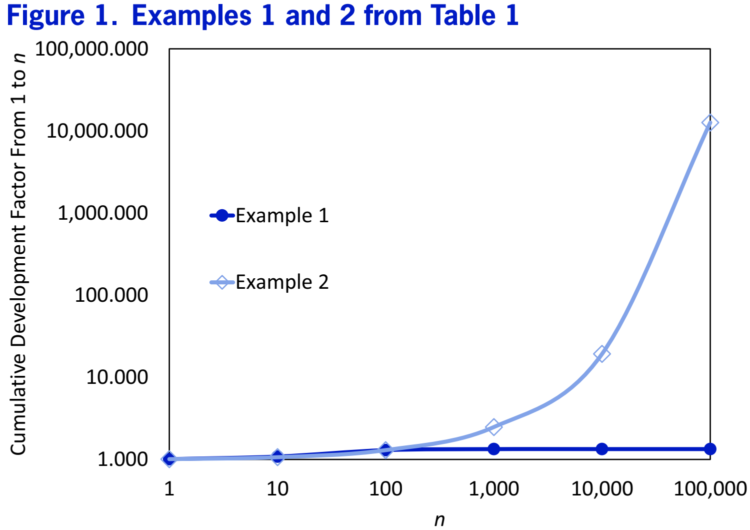
Table 1. Examples of apparently convergent and divergent tail factors for the inverse power curve model

	Parameter Values	
Parameters	Example 1	Example 2
a	545540.243359093	0.0150014750112457
b	–4.0	–0.5
c	84.9422458022239	1.25044252421429
	Cumulative Development Factors From 1 to *n*	
n	Example 1	Example 2
1	1.010	1.010
10	1.085	1.065
100	1.300	1.300
1,000	1.337	2.482
10,000	1.338	19.293
100,000	1.338	1.27E+04
1,000,000	1.338	1.03E+13
10,000,000	1.338	1.54E+41

Example 2, using $b = -0.5$, appears to zoom toward infinity in the tail.

This paper uses basic real analysis (Rudin 1976 being a standard textbook reference) to prove that the infinite product of the age-to-age factors converges to a finite number when the power parameter b is less than -1, and diverges to $+\infty$ when $b \geq -1$. Note, in this paper we refer to a sequence that increases without any upper bound as diverging to $+\infty$, or having a

Figure 1. Examples 1 and 2 from Table 1



limit of $+\infty$. Furthermore, when $b < -1$, for any finite product of the age-to-age factors up to a specific age *n,* there is a simple formula for an interval containing the limit of the infinite product. As *n* increases, the interval becomes tighter and the endpoints each converge to the limit of the infinite product. The lower endpoint of this interval is always a better estimate of the infinite product than the finite product of the age-to-age factors, which is always less than the lower endpoint.

It is worth noting again that tail divergence does not necessarily mean the model is invalid, but simply that any specific finite cutoff point should be otherwise justified. For a convergent tail, either a cutoff point must still be justified by some other consideration or care must be taken that the tail factor past the cutoff is reasonably close to 1. The interval estimate derived in this paper can help answer the latter question.

The proof of convergence/divergence is laid out in Section 2.1, with the proof of several useful lemmas in Appendix A. The interval estimate is derived in Section 2.2. Numerical examples of the progressive convergence/divergence of the finite product and the interval estimate of the infinite product for several sets of parameters are shown in Section 2.3.

2. Convergence theorem and limit estimation

Following the notational conventions of the recent CAS Tail Factor Working Party (2013), in the remainder of this paper, *d,* instead of *t,* is used for age or time.

2.1. Statement and proof of primary theorem

First we will set up a definition for the finite product of the age-to-age factors in the inverse power curve model.

Definition: $F_n(a,b,c) = \prod_{d=1}^{n}\left(1 + a(d+c)^b\right)$ where $a > 0$, b, and $c \geq 0$ are real numbers and n is a positive integer.

Note, this definition includes cases where *d* begins at a higher value than 1, as the *c* parameter can be increased to handle such cases. It is also worth noting that $a(d+c)^b > 0$, a key fact that will be used in subsequent derivations.

Theorem 1

(i) If $b \geq -1$ then $\lim_{n\to\infty} F_n(a, b, c) = +\infty$.

(ii) If $b < -1$ then $\lim_{n\to\infty} F_n(a, b, c) = F(a, b, c) < +\infty$ exists.

Proof:

(i) For any sequence of numbers $x_i > 0$ where $i = 1, \ldots, n$ and $n \geq 2$ the inequality $\prod_{i=1}^{n}(1 + x_i) > 1 + \sum_{i=1}^{n} x_i$ holds according to Lemma A.3. Applying this we have $F_n(a,b,c) = \prod_{d=1}^{n}\left(1 + a(d+c)^b\right) > 1 + \sum_{d=1}^{n} a(d+c)^b = 1 + a\sum_{d=c+1}^{n+c} d^b$. If $b \geq -1$ then $\lim_{n\to\infty}\sum_{d=c+1}^{n+c} d^b = +\infty$ according to Lemma A.1, and consequently $\lim_{n\to\infty} F_n(a, b, c) = +\infty$.

(ii) By Lemma A.2, $\log(1 + x) < x$ for any $x > 0$, so $\log(1 + a(d + c)^b) < a(d + c)^b$. Summing over d gives $\log F_n(a,b,c) = \sum_{d=1}^{n}\log\left(1 + a(d+c)^b\right) < \sum_{d=1}^{n} a(d+c)^b = a\sum_{d=c+1}^{n+c} d^b$. If $b < -1$ then $L = a\left(\lim_{n\to\infty}\sum_{d=c+1}^{n+c} d^b\right)$ exists and is less than $+\infty$ according to Lemma A.1. Now note that $\log F_n(a, b, c)$ is an increasing sequence, because $1 + a(d + c)^b > 1$ implies that $\log(1 + a(d + c)^b) > 0$, and is bounded by L. Consequently, $\lim_{n\to\infty}\log F_n(a, b, c)$ exists and is less than $+\infty$. So $\lim_{n\to\infty} F_n(a, b, c) = F(a, b, c)$ exists and is less than $+\infty$.

2.2. An interval estimate for the infinite product limit

For the convergent case of $b < -1$, it is possible to construct a useful interval estimate for the infinite product. The following definitions are convenient for specifying interval estimates.

Definition: The *tail upper bound factor* is $U_n(a, b, c) = \exp\left(-a\frac{(n+c)^{b+1}}{b+1}\right)$.

Definition: The *tail lower bound factor* is $L_n(a, b, c) = 1 - a\frac{(n+c+1)^{b+1}}{b+1}$.

Theorem 2

Let $F(a, b, c) = \lim_{n\to\infty} F_n(a, b, c)$. If $b < -1$ then:

(i) $\lim_{n\to\infty} U_n(a, b, c) = 1$.

(ii) $\lim_{n\to\infty} L_n(a, b, c) = 1$.

(iii) $F(a, b, c) \in (L_n(a, b, c)F_n(a, b, c), U_n(a, b, c)F_n(a, b, c))$.

Proof:

(i) $b + 1 < 0$ implies that $\lim_{n\to\infty}\left(-a\frac{(n+c)^{b+1}}{b+1}\right) = 0$ and consequently $\lim_{n\to\infty}\exp\left(-a\frac{(n+c)^{b+1}}{b+1}\right) = 1$.

(ii) $b + 1 < 0$ implies $\lim_{n\to\infty}\left(1 - a\frac{(n+c+1)^{b+1}}{b+1}\right) = 1 - \lim_{n\to\infty}\left(a\frac{(n+c+1)^{b+1}}{b+1}\right) = 1$.

(iii) $F(a, b, c) = F_n(a, b, c)\prod_{d=n+1}^{\infty}\left(1 + a(d+c)^b\right)$.

Taking the logarithm of the tail factor and applying bounding techniques described in Lemmas A.1 and A.2, $\log\left(\prod_{d=n+1}^{\infty}\left(1 + a(d+c)^b\right)\right) < \sum_{d=n+1}^{\infty} a(d+c)^b = a\sum_{d=n+c+1}^{\infty} d^b < -a\frac{(n+c)^{b+1}}{b+1}$.

Exponentiating produces $\prod_{d=n+1}^{\infty}\left(1 + a(d+c)^b\right) < \exp\left(-a\frac{(n+c)^{b+1}}{b+1}\right)$. Consequently, $F(a, b, c) < U_n(a, b, c)F_n(a, b, c)$.

Similarly, using techniques from Lemmas A.1 and A.3 produces $\prod_{d=n+1}^{\infty}\left(1 + a(d+c)^b\right) > 1 + \sum_{d=n+1}^{\infty} a(d+c)^b = 1 + a\sum_{d=n+c+1}^{\infty} d^b > 1 - a\frac{(n+c+1)^{b+1}}{b+1}$. Consequently, $F(a, b, c) > L_n(a, b, c)F_n(a, b, c)$. This completes the proof of Theorem 2.

The lower endpoint of the estimation interval is always a better estimate of the infinite product $F(a, b, c)$ than simply using the finite product $F_n(a, b\ c)$, since $L_n(a, b, c) > 1$ and consequently $F_n(a, b, c) < L_n(a, b, c)F_n(a, b, c) < F(a, b, c)$. The tail bound factors are computationally simple even for large values of n and give a measure of the relative width of the estimation interval prior to doing the computationally intense calculation of the finite product. For example, to achieve a certain target U for the upper bound requires $n \approx -c + \left(-\frac{(1+b)\log(U)}{a}\right)^{\frac{1}{1+b}}$. A more relevant measure of relative error, but without any simple formula for n that the author is aware of, is the ratio of the tail upper bound factor to the tail lower bound factor $U_n(a, b, c)/L_n(a, b, c) = \exp\left(-a\frac{(n+c)^{b+1}}{b+1}\right)\left(1 - a\frac{(n+c+1)^{b+1}}{b+1}\right)^{-1}$.

Example 1: An upper bound factor target set at $U = 1.01$ for the parameter values $a = 545540$, $b = -4.0$, and $c = 84.9422$ requires $n \approx 178$. However, by $n = 29$ the ratio of the tail upper bound factor to the tail lower bound factor is about 1.01.

2.3. More numerical examples

Table 2 shows six different sets of parameters, each of which produces an age-to-age factor at $d = 1$ of 1.01 and a cumulative factor from $d = 1$ to 100 of 1.30. The parameter sets are indexed by a set of values {−2.0, −1.5, −1.1, −1.0, −0.9, −0.6} for the power parameter b. For $b = -1$ the divergence happens very slowly, but for $b = -1.1$ the convergence happens remarkably slowly. To achieve $U_n(a, b, c) \approx 1.01$ for the $b = -1.1$ parameter set would require $n \approx 2.7 \times 10^{22}$, although by $n \approx 5.3 \times 10^{10}$ $U_n(a, b, c)/L_n(a, b, c) \approx 1.01$, still an astronomically slow rate of convergence.

Table 2. Examples of finite development factor products and interval estimates for infinite development factor products

	Parameter Values		
Parameters	Example 3	Example 4	Example 5
a	12.1209748535112	1.07747300550919	0.174451676891596
b	–2.0	–1.5	–1.1
c	33.815190439679	21.6431893821624	12.4522704340826
	Cumulative Development Factor Product (Infinite Product Lower Bound, Infinite Product Upper Bound)		
n	Example 3	Example 4	Example 5
1	1.010 (1.352, 1.431)	1.010 (1.458, 1.589)	1.010 (2.359, 3.877)
10	1.083 (1.375, 1.428)	1.081 (1.488, 1.585)	1.078 (2.449, 3.868)
100	1.300 (1.417, 1.423)	1.300 (1.553, 1.581)	1.300 (2.713, 3.858)
1,000	1.406 (1.423, 1.423)	1.477 (1.576, 1.580)	1.610 (3.017, 3.856)
10,000	1.421 (1.423, 1.423)	1.546 (1.579, 1.580)	1.926 (3.263, 3.856)
100,000	1.423 (1.423, 1.423)	1.569 (1.580, 1.580)	2.221 (3.447, 3.856)
1,000,000	1.423 (1.423, 1.423)	1.576 (1.580, 1.580)	2.488 (3.578, 3.856)
10,000,000	1.423 (1.423, 1.423)	1.579 (1.580, 1.580)	2.723 (3.670, 3.856)
100,000,000	1.423 (1.423, 1.423)	1.580 (1.580, 1.580)	2.925 (3.733, 3.856)
1,000,000,000	1.423 (1.423, 1.423)	1.580 (1.580, 1.580)	3.096 (3.776, 3.856)
	Parameter Values		
Parameters	Example 6	Example 7	Example 8
a	0.112891979103701	0.0737384147594275	0.0219230164116958
b	–1.0	–0.9	–0.6
c	10.2891979090266	8.20670480785112	2.69970572509898
	Cumulative Development Factor Product		
n	Example 6	Example 7	Example 8
1	1.010	1.010	1.010
10	1.077	1.075	1.069
100	1.300	1.300	1.300
1,000	1.668	1.744	2.185
10,000	2.161	2.550	8.118
100,000	2.803	4.119	219.782
1,000,000	3.635	7.534	8.72E+05
10,000,000	4.714	16.111	9.55E+14
100,000,000	6.113	41.946	4.86E+37
1,000,000,000	7.928	139.919	5.27E+94

Acknowledgment

The author is greatly thankful to John Robertson, Dan Corro, and Len Herk for review and comments on this paper.

Appendix A—Lemmas

Lemma A.1

Let n be a positive integer and $l > 0$.

(i) If $b \geq -1$ then $\lim_{n\to\infty} \sum_{k=l}^{l+n} k^b = +\infty$.

(ii) If $b < -1$ then $\lim_{n\to\infty} \sum_{k=l}^{l+n} k^b < +\infty$ exists.

Proof:

It suffices to show convergence or divergence for $\lim_{n\to\infty} \sum_{k=l+1}^{l+n} k^b$ since l^b is a finite number.

For $k > 1$ and $b \geq 0$, $k^b \geq 1$, and therefore $\lim_{n\to\infty} \sum_{k=l+1}^{l+n} k^b = +\infty$.

For $k > 1$ and $b < 0$, k^b is a strictly decreasing function of k, and therefore there is a sandwich inequality $\int_k^{k+1} t^b\,dt < k^b < \int_{k-1}^{k} t^b\,dt$ and consequently $\int_{l+1}^{l+n+1} t^b\,dt < \sum_{k=l+1}^{l+n} k^b < \int_{l}^{l+n} t^b\,dt$.

Solving the integrals when $b \neq -1$

$$\frac{(l+n+1)^{b+1} - (l+1)^{b+1}}{b+1} < \sum_{k=l+1}^{l+n} k^b < \frac{(l+n)^{b+1} - l^{b+1}}{b+1}.$$

For $b < -1$, taking limits produces $\frac{-(l+1)^{b+1}}{b+1} < \lim_{n\to\infty} \sum_{k=l+1}^{l+n} k^b < \frac{-l^{b+1}}{b+1}$.

In this case, the upper bound of the inequality is a finite number and implies convergence to a finite number since the sequence of partial sums in the middle is non-decreasing.

For $-1 < b < 0$, taking limits results in $\lim_{n\to\infty} \sum_{k=l+1}^{l+n} k^b = +\infty$, since in this case the lower bound of the earlier inequality diverges $\lim_{n\to\infty} \frac{(l+n+1)^{b+1} - (l+1)^{b+1}}{b+1} = +\infty$.

For the case $b = -1$, integration of the earlier inequality leads to $\log\left(\frac{l+n+1}{l+1}\right) < \sum_{k=l+1}^{l+n} k^b < \log\left(\frac{l+n}{l}\right)$.

Once again taking limits leads to $\lim_{n\to\infty} \sum_{k=l+1}^{l+n} k^b = +\infty$ from the lower bound of the inequality diverging $\lim_{n\to\infty} \log\left(\frac{l+n+1}{l+1}\right) = +\infty$.

Lemma A.2

If $x > 0$, then $\log(1 + x) < x$.

Proof:

If $t > 1$ then $1/t - 1 < 0$, and consequently $\int_1^{1+x} (1/t - 1)\,dt < 0$.

So, $\int_1^{1+x} dt/t - \int_1^{1+x} dt < 0$ and solving the integrals produces $\log(1 + x) - x < 0$.

Lemma A.3

$\prod_{i=1}^{n}(1+x_i) > 1 + \prod_{i=1}^{n} x_i$ for any sequence of numbers $x_i > 0$, $i = 1, \ldots, n$, and integer $n \geq 2$.

Proof:

We proceed by induction.

For $n = 2$, since $x_1x_2 > 0$, it follows that $1 + x_1 + x_2 + x_1x_2 > 1 + x_1 + x_2$.

Assume the conclusion of the lemma is true for n where $n \geq 2$. We will show that the lemma is then true for $n + 1$.

In general $\prod_{i=1}^{n+1}(1+x_i) = (1+x_{n+1})\prod_{i=1}^{n}(1+x_i) = \prod_{i=1}^{n}(1+x_i) + x_{n+1}\prod_{i=1}^{n}(1+x_i)$

But $x_{n+1}\prod_{i=1}^{n}(1+x_i) > x_{n+1}$ so $\prod_{i=1}^{n}(1+x_i) + x_{n+1}$

$\prod_{i=1}^{n}(1+x_i) > 1 + \sum_{i=1}^{n} x_i + x_{n+1} = 1 + \sum_{i=1}^{n+1} x_i$, which establishes the lemma.

References

Boor, J., "Estimating Tail Development Factors: What to Do When the Triangle Runs Out," Casualty Actuarial Society *Forum,* Winter 2006.

CAS Tail Factor Working Party, "The Estimation of Loss Development Tail Factors: A Summary Report," Casualty Actuarial Society *Forum,* Fall 2013.

Lowe, S.P., and D.F. Mohrman, "Discussion of 'Extrapolating, Smoothing and Interpolating Development Factors'," *Proceedings of the Casualty Actuarial Society* 72, 1985, pp. 182–189.

Rudin, W., *Principles of Mathematical Analysis* (3rd ed.), New York: McGraw-Hill, 1976.

Sherman, R.E., "Extrapolating, Smoothing and Interpolating Development Factors," *Proceedings of the Casualty Actuarial Society* 71, pp. 122–155, 1984.