# A Cost of Capital Approach to Credit and Liquidity Spreads

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**Abstract**[2]

The Market Cost of Capital approach has emerged as the standard for estimating risk margins for insurers' fair value balance sheets. This paper takes some of the ideas developed for valuing life insurance liabilities and applies them to the problem of valuing credit risky bonds. The basic idea is that credit spreads should cover the cost a) best estimate defaults plus the cost of holding capital for b) contagion risk (e.g. a credit crunch) and c) parameter risk, this risk that the best estimate is wrong and must be revised. We argue that the margins required for parameter risk can capture liquidity issues. In addition, the models developed here allow the cost of capital rate itself to be a random variable which allows credit spreads to open and close stochastically.

Finally, the paper argues that it is reasonable to include something like AA best estimate default rates and liquidity spreads when valuing insurance liabilities. The main rationale for doing so is the idea that there are elements of the total credit risk issue that can be hedged between the assets and liabilities.

## Introduction

It is now 20 years since the SOA held its first research conference dedicated to the topic of how one should determine the fair value of life insurance liabilities. One particularly vexing issue has always been the question of what yield curve one should use to discount some appropriately risk-adjusted set of liability cash flows.

Some have argued that if a life insurer that wants to be considered an AA risk then it should use an AA yield curve to discount its liabilities. While this makes some intuitive sense it can lead to the conclusion that the lower a life insurer's credit rating is, the lower its liabilities. This is controversial.

If we try to fix this problem by ignoring all credit spread issues in the liability valuation, we create another problem that many companies trying to develop market consistent reporting models experienced during the financial crisis of 2008. At the height of the crisis, the flight to quality decreased yields on risk free instruments (raising liability values) while credit spreads increased enough to more than offset the drop in risk free yields. The result, for many companies, was that the market value of their assets dropped while the fair value of their liabilities increased. Many people, including this author, consider that result to be somewhat inappropriate, especially if assets and liabilities were reasonably well "matched" going into the crisis.

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[2] The views and opinions expressed in this paper are those of the author and not GGY AXIS.

The Solvency II QIS 5 specification tried to address the issues outlined above by allowing insurers to use inter-bank swap rates plus a spread designed to remove credit risk and capture the idea that many insurance obligations are fairly illiquid[3]. To the extent the allowed liquidity spread varies with market conditions there is an element of credit spread hedging going on between the assets and liabilities. This goes some way to resolving issues raised above.

The models developed in this paper ultimately lead to a liquidity adjustment for liability valuation as in the Solvency II QIS 5 approach. However, the path we take to get there and the resulting pattern of liquidity adjustments are quite different.

Following this introduction, the paper is divided into three main sections

1. We develop the key ideas in a simple two state model where a bond is either in good standing or in default. This is where the analogy to life insurance is clearest. We are able to formulate a reasonably tractable affine model where the cost of capital rate is a random variable. For simplicity, we assume the risk free interest rate is deterministic.

   The main conclusion is that the forward default rates for a credit risky bond can be decomposed into the sum of

   i. A best estimate default rate (i.e. one that assumes the law of large numbers for a portfolio of bond applies).
   ii. A spread for contagion risk – the risk that current experience differs materially from best estimate
   iii. A dynamic spread for parameter risk – the risk that the best estimate is wrong and must be revised. We argue this spread also captures liquidity issues
   iv. A final spread that arises if the cost of capital rate itself is stochastic

   Each of items ii – iv is engineered to provide for holding a specific amount of economic capital.

   After this step we have enough theory to develop the author's views on how credit risk issues should affect insurance liability valuation. The main points are

   a) On the asset side the insurer should hold capital for each of risks ii – iv in the list above.
   b) For long liabilities we can take a capital offset for risks iii-iv. Because these risks are being effectively hedged. The contagion risk issue cannot be hedged.
   c) As a result of (b) it makes sense to build spreads iii – iv into the liability valuation. We also argue that best estimate defaults should be included

   One consequence of taking this point of view is that the net credit risk capital requirement for an insurer whose assets and liabilities are "matched" in the sense defined here is reduced to the contagion risk requirement. On the other hand a financial

[3] "QIS 5 Technical Specification: Risk-free interest rates", CRO/CFO Forum 2010. http://ec.europa.eu/internal_market/insurance/docs/solvency/qis5/cfo-forum-cro-forum-paper-risk-free-rates_en.pdf

institution, such as a bank, that does not have long liabilities would presumably have to hold more credit risk capital due the larger mismatch between assets and liabilities.

2. The next step is to generalize the simple model to a multi-state world where the best estimate model is represented by a typical credit transition matrix. While there are many different ways in which the two-state model can be generalized we choose an approach to minimize the technical details. The end result is a model in the same general family as a model published in 1997 by Jarrow, Lando and Turnbull[4].

   We don't develop this model in a lot of detail. Instead, the point is to show that the additional complexity can be handled in a reasonable way.

There are two issues that are deliberately out of scope for this paper. These are

1. How to calibrate the model to observed market data
2. A detailed comparison with the current Solvency II approach to liability spreads

These are both significant issues that deserve discussion. Unfortunately, an appropriate treatment of these issues could easily triple the length of the current paper. The author hopes that other interested risk professionals will rise to the occasion and engage in this important discussion.

## The Two-State Model

The starting point for our model is the actuary's cost of capital approach. This is usually a three step process where we start with a best estimate model of the default process and then adjust that model for two kinds of risk which are a) the risk that current experience differs materially from the best estimate and b) the risk that the best estimate is wrong and must be revised.

A key difference between the mortality model and the credit risk model developed here is that we also take market sentiment into account by allowing the cost of capital rate to be stochastic rather than a constant. This is a fourth step.

The author's life insurance version of this idea[5] was developed in detail in a paper presented at the 2014 ERM Symposium held in Chicago in October. A very short summary of that paper's main conclusion is that we can capture the essential elements by taking our best estimate forward rates of default and then add a static margin for current experience risk and a dynamic margin for parameter risk.

While this model was originally developed to value non-hedgeable insurance risk there are insights that can be gained by applying the model to valuing credit risk. We will use those insights to justify our approach to applying credit spreads to the valuation of insurance liabilities.

*The Best Estimate Model*

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[4] Jarrow, R.A., Lando, D., Turnbull, R.M., "A Markov Model for the Term Structure of Credit Risk Spreads", *The Review of Financial Studies,* Vol. 10, No. 2, pp. 481-523.

[5] Manistre, B.J., "Down but Not Out: A Cost of Capital Approach to Fair Value Risk Margins". This paper can be found on the Society of Actuaries' (SOA) website. A short summary of the paper appeared in the fall 2014 issue of the SOA's section newsletter *Risk Management*.

Let $V_0 = V_0(t,T)$ be the value at time $t$ of a credit risky bond that is scheduled to mature at time $T$ for $1 if it has not already defaulted. No other cash flows are assumed.

In a simple two-state model the first key credit risk parameter is the best estimate force of default $\mu_0(t)$ and the second key assumption is the residual value $RV$ of the credit risky bond once it has actually gone into default. A valuation equation that captures these two assumptions is

$$\frac{dV_0}{dt} + \mu_0(RV_0 - V_0) = rV_0, \qquad V_0(T,T) = 1.$$

What this equation says is that the total rate of change of the bond's value due to both the passage of time and the occurrence of defaults is equal to the risk free rate $r$, which we assume is constant for now.

The resulting value $V_0$ is then calculated by discounting the maturity value with interest $r$ and an adjusted force of default $\mu_0(1-R)$ i.e.

$$V_0(t,T) = \exp[-\int_t^T \{r + \mu_0(1-R)\} ds].$$

This looks a lot like traditional actuarial discounting and could apply to a portfolio of bonds that was structured so that he law of large numbers could be applied to average out the experience.

*Adverse Current Experience - Static Margins*

The only thing we know for sure about our best estimate model is that it is wrong. No matter how much effort we put into developing our best estimate assumptions we cannot predict default costs precisely. For our two-state example, we will assume that, even if our model is correct in the long run, it is still possible to experience $n$ years of best estimate credit risk losses in a single year.

In order to protect its solvency, a risk enterprise that owns a portfolio of risky bonds should hold enough economic capital that it could withstand the adverse event if it occurred. The kind of event we have in mind is one where the law of large numbers would not help. In the mortality application this might the onset of a pandemic. In the bond application this could be an economic crisis where many bonds are suddenly down-graded.

If $V(t,T)$ is the new, risk adjusted, value of the bond then the risk enterprise needs to hold economic capital in the amount $n\mu_0(1-R)V$. The new valuation equation, which reflects the cost of holding this amount of risk capital, is now

$$\frac{dV}{dt} + \mu_0(RV - V) = rV + \pi n \mu_0(1-R)V, \qquad V(T,T) = 1.$$

Here $\pi$ is a deterministic cost of capital rate that we will discuss in more detail later. To the extent that the economic capital was invested at the risk free rate $r$, the total expected return to an investor putting up the risk capital is $r + \pi$.

A little algebra shows that the valuation equation above is equivalent to

$$\frac{dV}{dt} + \mu_0(1 + n\pi)(RV - V) = rV, \qquad V(T,T) = 1.$$

The important observation is that taking short term risk issues into account is equivalent to adding a simple loading $\pi c = \pi n \mu_0$ to the best estimate default rate. The resulting risk adjusted value can then be calculated as

$$V(t,T) = \exp[-\int_t^T \{r + (\mu_0 + \pi c)(1 - R)\} ds].$$

*Assumption Risk and Liquidity - Dynamic Margins*

The previous analysis assumed our model was basically right over the long term but was vulnerable to adverse short term fluctuations. We now relax the long term assumption and ask what happens if we decide our best estimate default cost $\mu = \mu_0(1 - R)$ is wrong and must be revised to a new value $\hat{\mu} = (\mu_0 + \Delta\mu)(1 - R)$. The economic loss that would occur if this happens is the difference between two bond values $V - \hat{V}$, each based on its own default assumption. The idea now is to work out what it means to build in the cost of holding this amount of capital into the bond value.

In a principles based insurance model, assumptions are being examined and revised all the time as new information becomes available. To the extent that the insurance models are well understood, and based on credible experience, the potential assumption shock $\Delta\mu$ should be small. However, if the business is not that well understood then the assumption shock should be larger. At a high level, the size of the assumption shock therefore reflects the core elements of the liquidity issue.

We now examine the mathematical consequences of building in risk margins for holding this kind of economic capital. Once we have done this, the case for taking the assumption shock approach to modelling liquidity risk gets even stronger.

If we continue on with the same path that we have already started, we would write down a valuation equation of the form

$$\frac{dV}{dt} + \mu_0(RV - V) = rV + \pi c(1 - R)V + \pi(V - \hat{V}), \qquad V(T,T) = 1.$$

Unfortunately, this approach raises the thorny issue of how to calculate the shocked value $\hat{V}(t,T)$. The obvious next step is to write down a valuation equation for $\hat{V}$ of the form

$$\frac{d\hat{V}}{dt} + (\mu_0 + \Delta\mu)\left(R\hat{V} - \hat{V}\right) = r\hat{V} + \pi c(1 - R)\hat{V} + \pi(\hat{V} - \hat{\hat{V}}), \qquad \hat{V}(T,T) = 1.$$

This equation makes the reasonable assumption that the shocked default rate is $\mu_0 + \Delta\mu$ but has introduced a second shocked value $\hat{\hat{V}}$ which presumably depends on some secondary shocked default rate like $\mu_0 + \Delta\mu + \Delta\mu$. Writing down a valuation equation for $\hat{\hat{V}}$ simply leads to the same problem. This is called the circularity problem in the insurance literature.

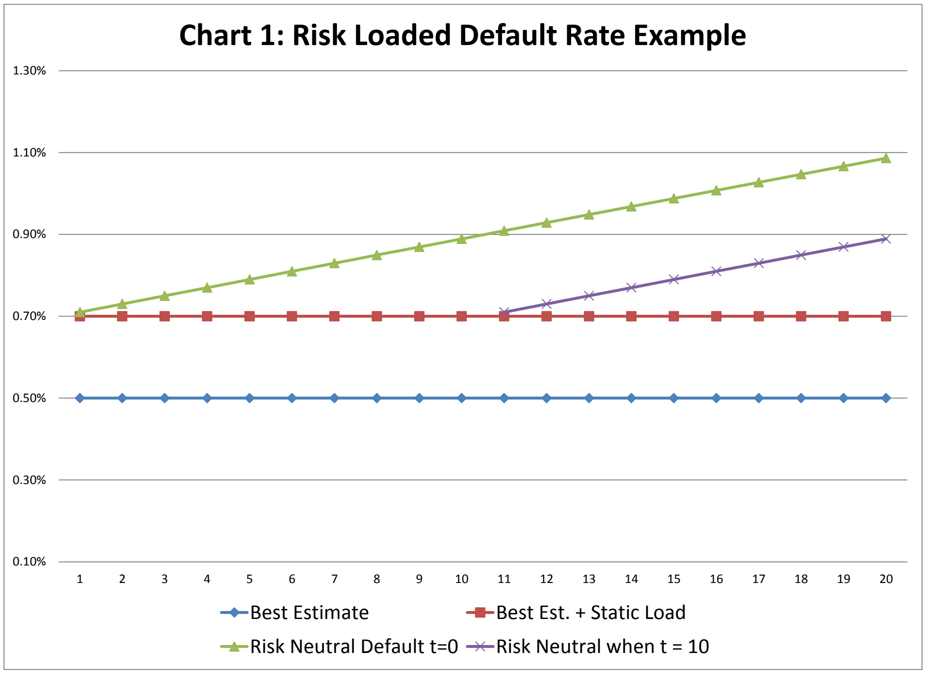
A number of practical ways to resolve the circularity problem were discussed in a paper presented at the 2014 ERM Symposium[6]. The approach taken here was called the explicit margin method in that paper. We assume there is a margin variable $\beta$ which allows the risk loaded default rate to be written for times $s > t$ as

$$\mu(s) = \mu_0(s) + \pi c(s) + \beta(t,s)\Delta\mu(s).$$

The margin variable $\beta$ is assumed to be 0 in the real world so $\beta(t,t) = 0$, but for $s > t$ it is assumed to evolve according to the dynamical rule

$$d\beta = [\pi - \beta\Delta\mu(1-R)]ds.$$

When we come to do a new valuation at some later point in time $t' > t$ the margin variable resets to zero at that time. The graphic below illustrates the idea just described[7].




In the example above, the best estimate forward default rate is $\mu_0$ =.50%. The static load assumes a short term shock equal to 4 years' worth of best estimate defaults and a cost of capital rate equal to $\pi = 10\%$. The resulting load is then $.10 \times 4 \times .005 = .20\%$.

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[6] See footnote 5.

[7] Under the stated assumptions, we can calculate the margin variable in closed form as $\beta(t,s) = \pi\frac{1-e^{-\Delta\mu(1-R)(s-t)}}{\Delta\mu(1-R)}$ which is approximately $\pi(s-t)$ when $\Delta\mu(1-R)$ is small. Later in this paper it will be convenient to approximate the dynamics of $\beta$ by $d\beta = \pi ds$.

The dynamic load calculation assumes the same 10% cost of capital rate but then uses a parameter shock of $\Delta\mu = .20\%$ and a recovery rate of $R = 50\%$. Under these assumptions, the margin variable $\beta$ grades from 0.00 to roughly 1.98 over the 20 year projection. When we come to do a new valuation 10 years later the static load has not changed but the dynamic load has been pushed out.

Deriving the dynamic load model from first principles is beyond the scope of this paper but it is fairly easy to show that the resulting structure has the desired properties. To do this, we think of the value of the risky bond, in the valuation measure, as a function $V = V(s, \beta, T)$ that for $s \geq t$ it satisfies the following evolution equation,

$$\frac{\partial V}{\partial s} + [\pi - \beta\Delta\mu(1-R)]\frac{\partial V}{\partial \beta} = [r + (\mu_0 + \pi c + \beta(t,s)\Delta\mu)(1-R)]V, \qquad V(T,\beta,T) = 1.$$

What this equation says is that the total expected rate of change of the value, in the valuation measure, is equal to the risk free interest rate plus the appropriate risk loaded default cost. On the valuation date, when $s = t$ and $\beta(t,t) = 0$ the real world expected rate of change is

$$\frac{\partial V}{\partial s}|_{s=t} + \mu_0(t)(R-1)V(t) = [r(t) + \pi c(t)(1-R)]V(t) - \pi\frac{\partial V}{\partial \beta}|_{s=t}.$$

This result makes sense if the greek $\delta = \frac{\partial V}{\partial \beta}|_{s=t}$ can be interpreted as a reasonable negative amount of economic capital to hold for parameter risk i.e. $\delta = \hat{V} - V$. That this is in fact the case can be shown by differentiating the valuation equation above with respect to the margin variable $\beta$ to get

$$\frac{\partial \delta}{\partial s} + [\pi - \beta\Delta\mu(1-R)]\frac{\partial \delta}{\partial \beta} = [r + (\mu_0 + \Delta\mu + \pi c + \beta(t,s)\Delta\mu)(1-R)]\delta + \Delta\mu(1-R)V.$$

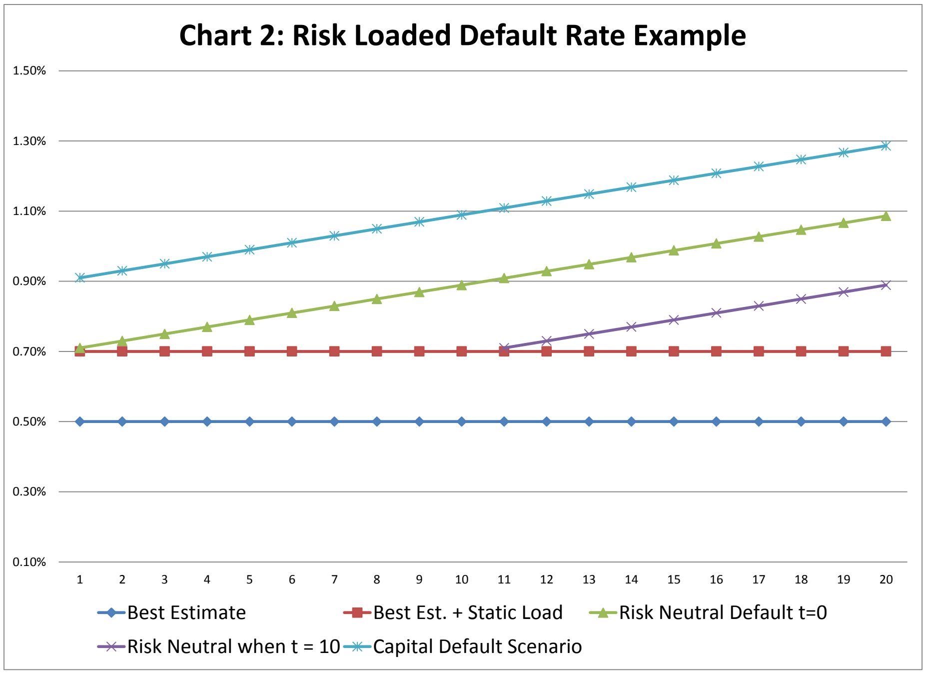
The solution to this equation at time $t$ is

$$\delta(t) = -\int_t^T e^{-\int_t^s [r + (\mu_0 + \Delta\mu + \pi c + \beta(t,v)\Delta\mu)(1-R)]dv}\,\Delta\mu(1-R)V(s)ds.$$

This result tells us that the greek $\delta$, at the valuation date, is minus the present value of losses that would occur if we valued the risky bond assuming the best estimate default rate was $\mu_0$ but the actual experience turned out to be $\mu_0 + \Delta\mu$. This means $\delta = \hat{V} - V$ where $\hat{V}$ is calculated using shocked best estimate default rates but with the same static and dynamic default rate loadings that were deemed appropriate for the base case. Variations on this theme are possible.

We conclude that the dynamic margin mechanism is releasing margin into income in a way that is consistent with the cost of capital concept provided the “greek” method outlined above is used to estimate the economic capital for parameter risk.

Chart 2 below is the same as Chart 1 except that we have added the “capital” forward default rate scenario used to compute $\hat{V}$ at the valuation date.




We can now develop the argument that we can capture bond liquidity issues by varying the size of the parameter shock. Liquid bonds would have small parameter shocks while illiquid bonds should have larger parameter shocks, even if they have the same short term best estimate default rates. This suggests a bond valuation model in which a risky bond's value is driven by both its rating (a proxy for current best estimate default rate) and a liquidity rating factor embedded in the shock $\Delta\mu$.[8]

A short summary of the argument so far: The valuation model developed has three components:

1. A best estimate default assumption
2. A capital requirement and associated static risk loading to deal with the risk that the best estimate could be wrong in the short run i.e. a credit crunch.
3. A capital requirement and associated dynamic risk loading to deal with the risk that the best estimate itself could be wrong and require revision.

We argued that we can capture liquidity issues with this last element.

One thing this model cannot explain yet is the way credit spreads open and close in a seemingly random fashion. A classic example of such a phenomenon would be a "flight to quality". Note that this is different from a contagion event because it does not necessarily imply any ratings down-grades.

[8] For most bonds it would be logical to assume $\Delta\mu > 0$ but for some applications this might not be the case. For example, people often take the swap curve to define the "risk free" rate. If we do this, we might have to use a negative parameter shock to model US Treasury bonds. Note that this leads to the conclusion that the economic capital required for holding Treasury bonds could be negative. This makes sense in the context of the current model. It makes even more sense once we consider "flight to quality" issues in the next section.

*Market Sentiment and Stochastic Cost of Capital Rates*

When the model described above is applied to value life insurance underwriting risk one normally assumes a constant cost of capital rate such as $\pi = 6.00\%$. For the credit risk example it makes more sense to allow the cost of capital rate itself to be a random quantity in order to allow the credit risk spreads to open and close with changing market sentiment.

We therefore extend the model by allowing the cost of capital rate $\pi$ to follow a Cox, Ingersoll &Ross process of the form

$$d\pi = \kappa(\pi_\infty - \pi)dt + \xi\sqrt{\pi}dz.$$

This will lead to a tractable affine model with a non-negative cost of capital rate. For the moment, we continue to assume a deterministic risk free interest rate $r$ and we will also be deliberately ambiguous as to whether these are P measure (real world) or Q measure (risk adjusted) parameters.

If the cost of capital rate is stochastic, then we have to think of the risky bond's value $V$ as a function of time and two state variables $\beta, \pi$ i.e. $V = V(s, \beta, \pi, T)$. The fundamental valuation equation then generalizes to

$$\begin{aligned}&\frac{\partial V}{\partial s} + [\pi - \beta\Delta\mu(1-R)]\frac{\partial V}{\partial \beta} + \kappa(\pi_\infty - \pi)\frac{\partial V}{\partial \pi} + \frac{\xi^2\pi}{2}\frac{\partial^2 V}{\partial \pi^2} \\ &= [r + (\mu_0 + \pi c + \beta(t,s)\Delta\mu)(1-R)]V, \qquad V(T,\beta,\pi,T) = 1.\end{aligned}$$

This equation can be solved by using the well-known financial engineering trick of assuming an affine solution of the form

$$V(t,\beta,\pi,T) = \exp[A(t,T) + \beta B(t,T) + \pi P(t,T)].$$

We can then derive the following system of ordinary differential equations for the quantities $A, B, P$, as a function of $t$,

$$\begin{aligned}&\dot{A} + \mu_0(R-1) + \kappa\pi_\infty P = r, \;\; AT(t,T) = 0, \\ &\dot{B} + \Delta\mu(R-1) = \Delta\mu(1-R)B, \;\; B(T,T) = 0, \\ &\dot{P} - \kappa P + \frac{\xi^2}{2}P^2 = c(1-R) - B, \;\; P(T,T) = 0.\end{aligned}$$

The middle equation for $B(t,T)$ does not depend on any of the other variables so we discuss that first. This equation is simple enough that it can be solved in closed form. The result is

$$B(t,T) = -(1 - \exp[-\Delta\mu(1-R)(T-t)]) \approx -\Delta\mu(1-R)(T-t).$$

As we saw in the previous section, the economic capital required for parameter risk is

$$-\frac{\partial V}{\partial \beta} = -B(t,T)V(t,\beta,\pi,T) \approx \Delta\mu(1-R)(T-t)V(t,\beta,\pi,T).$$

This is a reasonable formula for the assumed liquidity capital that varies in a simple way by maturity. The longer the bond, the riskier it appears to be. This makes intuitive sense and differs from the contagion risk capital factor $c(1-R)$ which does not depend on the maturity of the risky bond.

For a more complex bond with coupon payments, the formula for liquidity risk capital clearly generalizes to

$$EC \approx \Delta\mu\,(1-R)DV,$$

where $D$ is the usual (modified) duration of the bond.

Now that we know $B(t,T)$ we can substitute it into the third equation for $P(t,T)$ to get

$$\dot{P} - \kappa P + \frac{\xi^2}{2}P^2 = c(1-R) + (1 - \exp[-\Delta\mu(1-R)(T-t)]).$$

*The case* $\xi = 0$

When $\xi = 0$, this equation can also be solved in closed form to get a function we will call $P_0(t,T)$. The result is

$$P_0(t,T) = \frac{e^{-\Delta\mu(1-R)(T-t)} - e^{-\kappa(T-t)}}{\kappa - \Delta\mu(1-R)} - (c(1-R)+1)\frac{1-e^{-\kappa(T-t)}}{\kappa}.$$

For short maturity bonds we have

$$\begin{aligned} P_0(t,T) &\approx \frac{\left(\Delta\mu(1-R)\right)^2 - \kappa^2}{\kappa - \Delta\mu(1-R)}\frac{(T-t)^2}{2} - c(1-R)(T-t) \\ &= -\left\{c(1-R)(T-t) + \left(\kappa + \Delta\mu(1-R)\right)\frac{(T-t)^2}{2}\right\}. \end{aligned}$$

This is clearly negative if $\Delta\mu > 0$.

For long maturity bonds we clearly have

$$P_0(t,T) \approx \begin{cases} -\dfrac{c(1-R)+1}{\kappa}, if\ \Delta\mu > 0, \\ \dfrac{e^{-\Delta\mu(1-R)(T-t)}}{\kappa - \Delta\mu(1-R)}, if\ \Delta\mu < 0. \end{cases}$$

The final equation for $A(t,T)$ can be written as an integral

$$A(T,T) - A(t,T) = \int_t^T \frac{dA(s,T)}{ds}\,ds,$$
$$= \int_t^T [r + \mu_0(1-R) - \kappa\pi_\infty P_0(s,T)]\,ds.$$

This is enough information to allow us to compute the risk adjusted forward discount rates at the valuation date when $\beta = 0$. The result is

$$
\begin{aligned}
F_0(t,T) &= -\frac{\partial}{\partial T}\ln(V(t,\beta,\pi,T), \\
&= -\left[\frac{\partial A}{\partial T} + \beta\frac{\partial B}{\partial T} + \pi\frac{\partial P_0}{\partial T}\right], \\
&= \left[r + \mu_0(1-R) - \kappa\pi_\infty P_0 + \pi\frac{\partial P_0}{\partial t}\right].
\end{aligned}
$$

Using the first ordinary differential equation for $A(t,T)$, and the solutions already derived, we can write

$$
\begin{aligned}
F_0(t,T) = r + {} & \mu_0(1-R) + \pi c(1-R) \\
& + \pi\left(1 - e^{-\Delta\mu(1-R)(T-t)}\right) + \kappa(\pi - \pi_\infty)P_0(t,T).
\end{aligned}
$$

Each term in the equation above makes sense for the problem at hand. The total forward discount rate is the sum of

1. A risk free interest rate (there would be other terms if $r$ were stochastic).
2. A best estimate recovery adjusted default rate
3. A static spread for contagion risk
4. A dynamic spread for liquidity risk that starts from 0 at the valuation date and then grades to $\pi$ at a rate determined by the size of the parameter shock $\Delta\mu(1-R)$. This is consistent with the pattern of liquidity capital requirements. It is not consistent with the current Solvency II model referred to earlier.
5. A final spread that vanishes if the current cost of capital rate is equal to its long term mean reversion target $\pi_\infty$ on the valuation date.

If you substitute the known closed form expression for $P_0(t,T)$, we get the same basic model but with some reasonable adjustments for the time varying cost of capital rate.

A quantity of some interest is the sensitivity $-\frac{\partial \ln(V)}{\partial \pi} = -P$ which is another duration-like quantity that measures the impact on value if the cost of capital rate itself were to change. We will therefor call $-P$ the *capital duration*. In the simple case where $\pi = \pi_\infty$ we can see that the capital duration is just the total economic capital required for contagion and liquidity risk.

Asset/Liability Management in a world driven by this kind of model would want to keep the difference between the capital and liquidity durations of assets and liabilities under close scrutiny.

*The case $\xi \neq 0$*

We now discuss the case where the volatility $\xi$ of the cost of capital rate is non-zero. In this situation the differential equation for $P(t,T)$ does not have a simple closed for solution although it can be simplified by introducing a new unknown function $U(t,T)$ such that $P(t,T) = \frac{2}{\xi^2}\frac{\dot{U}}{U}$.[9] This will give us the right behaviour as long as $U$ satisfies the linear second order differential equation

$$\ddot{U} - \kappa\dot{U} = \frac{\xi^2}{2}\{c(1-R) + (1 - \exp[-\Delta\mu(1-R)(T-t)]\}U.$$

This equation can be attacked numerically, or, if $\Delta\mu$ is constant, by assuming a power series solution of the form

$$U(t,T) = \sum_{j=0}^{\infty} a_j[T-t]^j,\ a_0 = 1,\ a_1 = 0.$$

A recurrence relation for the constant coefficients $a_j$ is

$$a_{j+2} = -\kappa\frac{a_{j+1}}{j+2} + \frac{\xi^2}{2}\frac{1}{(j+2)(J+1)}\left[c(1-R)a_j - \sum_{k=1}^{j}\frac{\left(-\Delta\mu(1-R)\right)^k}{k!}a_{j-k}\right].$$

This series converges quickly and is fairly easy to implement in a spreadsheet environment.

It is not hard to show that, if $\Delta\mu > 0$, then $0 \geq P(t,T) \geq P_0(t,T)$ and the new forward discount rates are then given by

$$\begin{aligned} F(t,T) &= r + \mu_0(1-R) + \pi c(1-R) \\ &\quad + \pi\left(1 - e^{-\Delta\mu(1-R)(T-t)}\right) + (\pi - \pi_\infty)\kappa P - \frac{\xi^2}{2}P^2, \\ &= F_0(t,T) + \kappa(\pi - \pi_\infty)(P - P_0) - \pi\frac{\xi^2}{2}P^2,\ (P > P_0). \end{aligned}$$

Interestingly, there is no simple conclusion which states that allowing $\xi > 0$ makes the forward default rates go consistently up or down. The last term $\pi\frac{\xi^2}{2}P^2$ clearly reduces the forward default rates but that may, or may not, be offset by the term $\kappa(\pi - \pi_\infty)(P - P_0)$. Only when $\pi < \pi_\infty$, is it clear that using a non-zero volatility $\xi$ reduces the forward default rates.

One definite conclusion we can draw is that assuming a stochastic cost of capital makes risky bond values less sensitive to a change in the cost of capital rate.

Chart 3 below shows the impact of using a non-zero cost of capital rate volatility under the assumptions that $\pi = \pi_\infty = 10\%$, $\kappa = 15\%$ and $\xi = 50\%$.

[9] This is a standard applied math trick for solving differential equations of the Ricatti type.

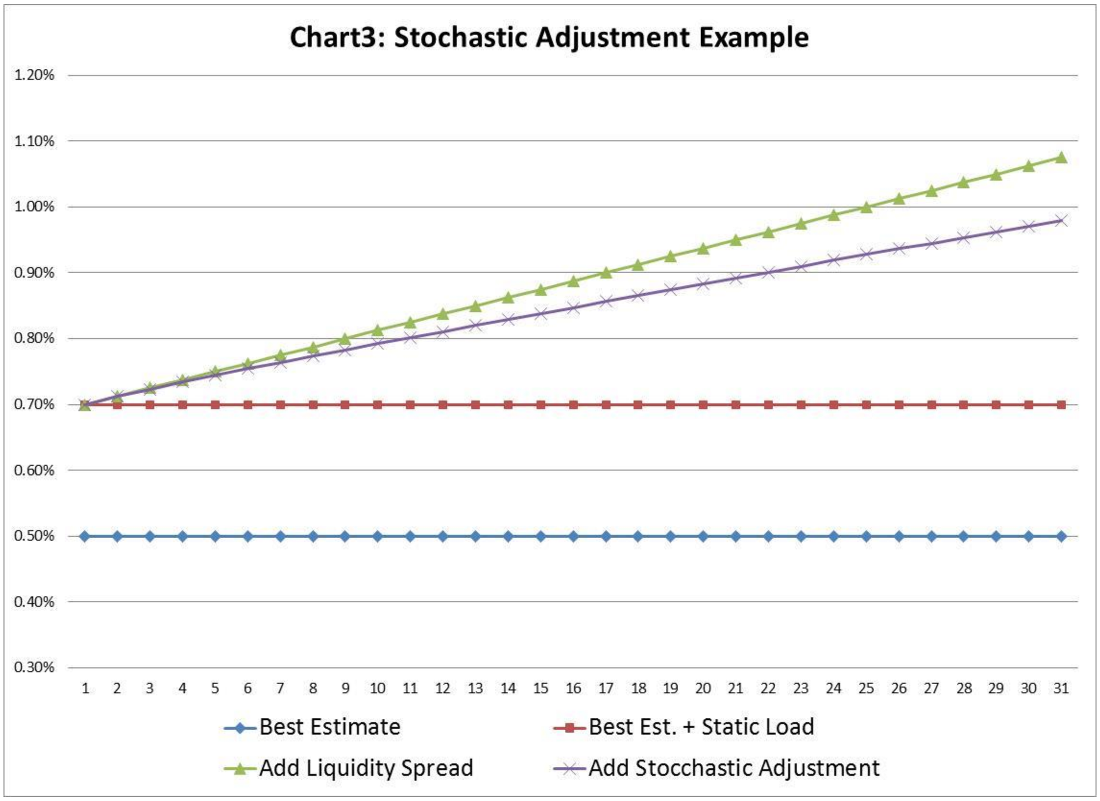
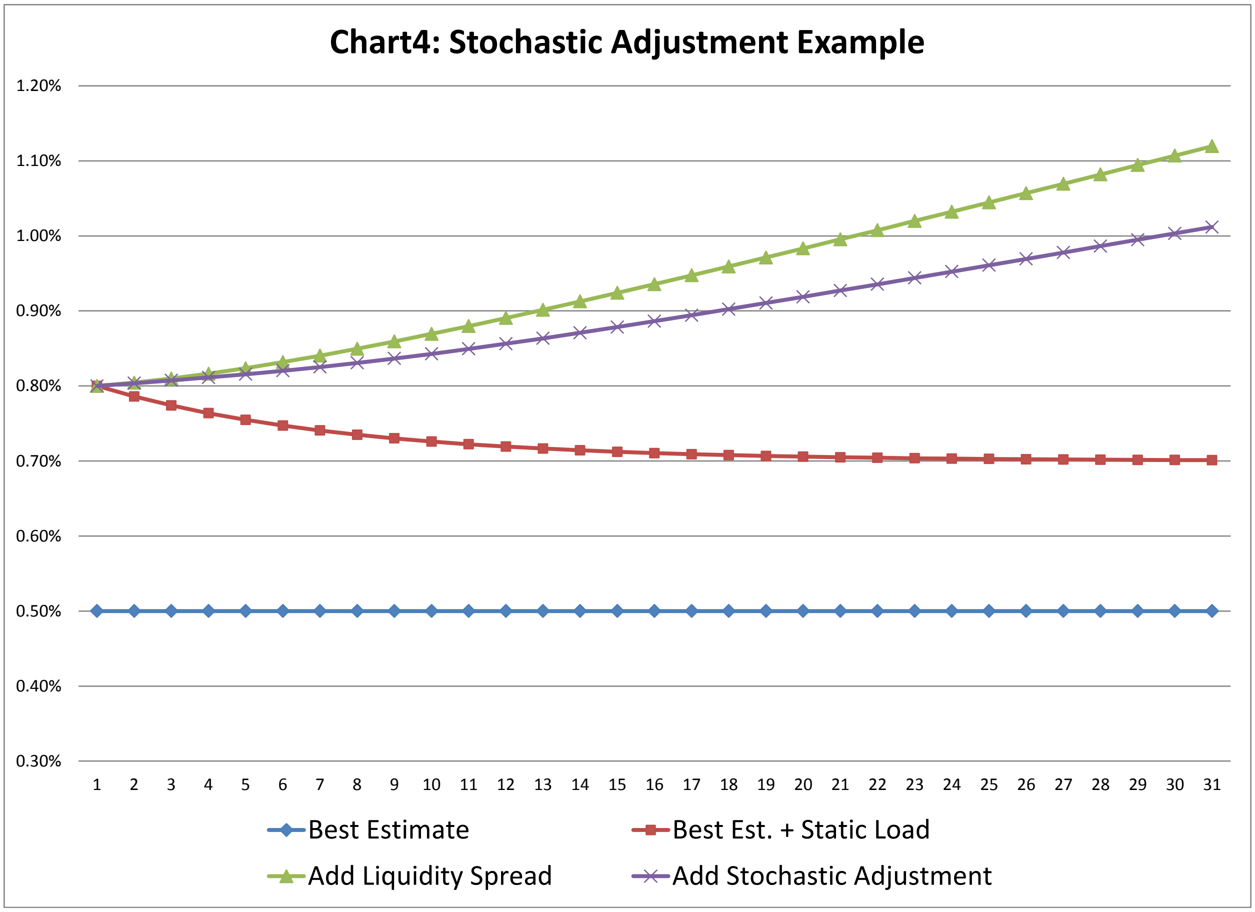



Chart 4 is presented to show what happens if the cost of capital rate at the valuation date is not equal to the long term mean. This particular example assumes $\pi = 15\%$ at the valuation date with all other parameters unchanged from Chart 3.




As expected, the long term default spreads have not changed.

While cost of capital volatility may not have that much impact on model values, it does have significant risk management implications. Imagine, for simplicity, that the two-state model we have been developing is actually good enough to describe the real world of risky bonds. Assume also that we have found a reasonable way to calibrate the model so that we know the key parameters and state variables.

An investor holding a credit risky bond is then subject to a number of risks

1. Best Estimate credit default experience – portfolio risk diversified away by the law of large numbers
2. Short term credit crunch (correlated ratings downgrades in a more sophisticated model)
3. A change in the bond's perceived liquidity
4. Fluctuations in the risk free yield curve
5. Fluctuations in market sentiment

To the extent that the risky bond is being used to back a long term insurance liability we can ask which, if any, of these risks can be naturally hedged between the asset and liability. If an insurer is holding capital and risk margins for all of risks 2-5, and the bond's value reflects that, then we are entitled to do two things for risk that can be hedged:

- Take a capital offset for any risk that can be reduced by taking on a matching liability.

- Take credit for the cost of capital savings when putting a fair value on the liability.

The author's point of view is that it is appropriate to take credit for items 1, 3, 4, 5 in the list above when valuing life insurance liabilities. We discuss each item in turn.

1. Best Estimate default experience. As noted near the beginning of this paper this idea is controversial and has been debated for many years. The author's point of view is that insurance company customers are taking some credit risk when buying a life insurance product and are entitled to some form of premium for taking that risk. The best estimate default probability makes sense in this case.

2. To the extent a credit crunch occurs, and $n$ years' worth of defaults and credit downgrades happened over night, this is not the policyholder's problem. The contagion spread should not be used when valuing an insurance liability.

3. If the market suddenly changes its point of view about bond liquidity (at a portfolio level) then it makes sense for this risk to be passed through to the liability side by introducing a similar adjustment to liability values. A liquidity spread should therefore be included when valuing insurance liabilities. At a high level this is consistent with Solvency II in Europe. As mentioned earlier the details of the liquidity model developed here are different from the details of the current Solvency II model[10].

4. Most insurers already assume that fluctuations in the risk free yield curve can be hedged between assets and liabilities. The usual way to handle this issue is to hold capital for the net mismatch between assets and liabilities. When pricing liabilities some companies assume a mismatch budget to take account of the fact that matching can never be perfect. In principle the idea of a mismatch budget could be expanded to cover the broader sense of "match" discussed in this paper.

5. The last issue requires more discussion because we have not stated, yet, what the risk associated with a change in market sentiment really is. Our point of view though is that this risk can be hedged between assets and liabilities.

At the beginning of this section we stated that our assumption for the dynamics of the cost of capital rate was

$$d\pi = \kappa(\pi_\infty - \pi)dt + \xi\sqrt{\pi}dz.$$

We will now argue that the development above makes sense if this is the risk neutral process. To see this, assume we start with a real world process of the form

$$d\pi = \kappa'(\pi'_\infty - \pi)dt + \xi'\sqrt{\pi}dz.$$

A sudden change in market sentiment $\pi \to \pi + \Delta\pi$ causes the risky bond's value to change by

[10]The current Solvency II mode for this issue takes observed CDS spreads as an input. This is reasonable in that it takes observable market data into account. However, the resulting patter of liquidity adjustments is flat for a certain period and then drops to zero after a fixed time horizon. This is not consistent with the risk insights that come out of the model described in this paper.

$$V(t,\beta,\pi+\Delta\pi,T)-V(t,\beta,\pi,T)\approx\Delta\pi\frac{\partial V}{\partial\pi}+\frac{(\Delta\pi)^2}{2}\frac{\partial^2 V}{\partial\pi^2}.$$

Now assume that the bond's owner is holding economic capital to cover this potential loss. The cost of capital concept then says the fundamental valuation equation should be

$$\frac{\partial V}{\partial s}+[\pi-\beta\Delta\mu(1-R)]\frac{\partial V}{\partial\beta}+\kappa'(\pi'_\infty-\pi)\frac{\partial V}{\partial\pi}+\frac{\xi'^2\pi}{2}\frac{\partial^2 V}{\partial\pi^2}+\mu_0(R-1)V$$
$$=rV+[(\pi c+\beta(t,s)\Delta\mu)(1-R)]V-\pi\left[\Delta\pi\frac{\partial V}{\partial\pi}+\frac{(\Delta\pi)^2}{2}\frac{\partial^2 V}{\partial\pi^2}\right].$$

On rearranging, this becomes the risk neutral equation studied earlier

$$\frac{\partial V}{\partial s}+[\pi-\beta\Delta\mu(1-R)]\frac{\partial V}{\partial\beta}+\kappa(\pi_\infty-\pi)\frac{\partial V}{\partial\pi}+\frac{\xi^2\pi}{2}\frac{\partial^2 V}{\partial\pi^2}$$
$$=[r+(\mu_0+\pi c+\beta(t,s)\Delta\mu)(1-R)]V,\qquad V(T,\beta,\pi,T)=1$$

provided the following relationships hold between risk neutral and real world parameters

$$\kappa=\kappa'-\Delta\pi,$$
$$\pi_\infty=\pi'_\infty\kappa'/(\kappa'-\Delta\pi),$$
$$\xi^2=\xi'^2+\Delta\pi^2.$$

These are all reasonable results. Risk adjustment reduces the rate of mean reversion, increases the long term cost of capital assumption and also increases the assumed volatility.

Using the risk neutral parameters to value insurance liabilities is then equivalent to assuming we take economic capital credit for the market sentiment risk on the liability side of the balance sheet. That this is reasonable is the current paper's main argument[11].

**The Multi-State Model and Other Enhancements**

Once a modelling process has started there is never an end to the enhancements that could be made. The main point of this section is to show that some important issues ignored so far do not really change any of the important risk management conclusions derived in the context of the two-state model.

1. Stochastic risk free rates. The model developed here could easily be incorporated into any affine model of the risk free rate. Examples of affine models are the Hull & White model and its higher dimensional cousins such as G2++. One would have to consider how the risk free rate and cost of capital rate processes are correlated.

2. Additional parameter risk. The model developed here assumed the parameters governing the cost of capital process were known with certainty. Since this is certainly not the case one could argue that an important parameter such as $\pi_\infty$ should get the same kind of

[11] Note that if $\Delta\mu<0$ then a "flight to quality" event where $\Delta\pi>0$ could cause the bond's value to rise. This makes sense as long as we allow very high quality bonds to have negative liquidity capital requirements.

treatment that we gave to $\mu_0$. This is certainly possible and should be considered if the issue is material to the problem at hand.

3. Income Tax Effects. To the extent one thinks of an income tax system as a risk sharing arrangement, it may be appropriate to tax effect the both required economic capital and the associated risk margins described in this document[12]. The models in this document ignore income tax issues.

4. Multi state bond ratings. There are a number of commercial bond rating services that publish their analysis of the credit worthiness of individual bond issues. In most situations these ratings are intended to indicate a given bond's probability of default over some relatively short time frame such a year or less.

   Many authors have studied the process of bond transitions where a rating agency changes a bond's rating as new information about the bond issuer's credit worthiness becomes available. The simplest version of a model which takes this multi-state issue into account is an annual ratings transition matrix $T$ which we assume most readers of this paper are already familiar with.

   The two state model building process described earlier can then be generalized as follows:

   - Best Estimate Defaults are modelled by a transition intensity matrix $M$ such that the annual transition probabilities are given by a matrix $T = \exp[M]$. The value function $V$ gets generalized to a vector of values $\boldsymbol{V} = (V^1, \dots, V^n)$ . Here $V^i$ is the value of the risky bond give that it is in rating class $i$. Realistic examples of such matrices are given in Tables 1 and 2 below[13].

| | | | **Table 1: Best Estimate Transition Matrix *T*** | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | |
| | **AAA** | **AA** | **A** | **BBB** | **BB** | **B** | **C** | **D** |
| **AAA** | 91.95% | 7.24% | 0.77% | 0.00% | 0.03% | 0.00% | 0.00% | 0.00% |
| **AA** | 1.10% | 90.99% | 7.56% | 0.26% | 0.07% | 0.01% | 0.00% | 0.00% |
| **A** | 0.05% | 2.40% | 91.91% | 5.01% | 0.48% | 0.12% | 0.01% | 0.02% |
| **BBB** | 0.05% | 0.24% | 5.27% | 88.52% | 4.82% | 0.78% | 0.16% | 0.17% |
| **BB** | 0.01% | 0.04% | 0.50% | 5.69% | 84.99% | 6.98% | 0.54% | 1.25% |
| **B** | 0.01% | 0.03% | 0.13% | 0.43% | 6.63% | 83.15% | 3.15% | 6.47% |
| **C** | 0.00% | 0.00% | 0.00% | 0.57% | 1.71% | 4.35% | 68.21% | 25.16% |
| **D** | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 100.00% |

[12] The author's views on this subject were laid out in the article "An ERM Approach to Income Tax Risk", which appeared in the spring 2009 edition of the SOA newsletter *Risk Management*.

[13] This particular matrix was adapted by the author from a study by Moody's covering the decade of the 1990's. All we claim here is that it is broadly representative of what a realistic transition matrix looks like.

| | | | Table 2: Transition Intensity Matrix $M$ | | | | | |
|---|---|---|---|---|---|---|---|---|
| | AAA | AA | A | BBB | BB | B | C | D |
| AAA | -8.4% | 7.9% | 0.5% | 0.0% | 0.0% | 0.0% | 0.0% | 0.0% |
| AA | 1.2% | -9.6% | 8.3% | 0.1% | 0.1% | 0.0% | 0.0% | 0.0% |
| A | 0.0% | 2.6% | -8.7% | 5.5% | 0.4% | 0.1% | 0.0% | 0.0% |
| BBB | 0.1% | 0.2% | 5.8% | -12.5% | 5.5% | 0.7% | 0.2% | 0.1% |
| BB | 0.0% | 0.0% | 0.4% | 6.6% | -16.8% | 8.3% | 0.5% | 1.0% |
| B | 0.0% | 0.0% | 0.1% | 0.2% | 7.9% | -18.9% | 4.2% | 6.5% |
| C | 0.0% | 0.0% | 0.0% | 0.6% | 2.0% | 5.7% | -38.4% | 30.1% |
| D | 0.0% | 0.0% | 0.0% | 0.0% | 0.0% | 0.0% | 0.0% | 0.0% |

The state $i = 'D'$ is usually assumed to mean actual default. A typical simplifying assumption is that $V^D(t,T) = Re^{-r(T-t)}$ where $R$ is the recovery rate.

The best estimate equation of value is then

$$\frac{d\mathbf{V}}{dt} + M\boldsymbol{V} = r\boldsymbol{V}.$$

Based on this equation one can develop time dependent best estimate forward default rates that take into account a scenario where a risky bond gradually declines from a high credit rating to default.

Putting $r = 0$ for convenience, the solution to the equation above can be written as

$$V^i(t,t) = (1,..,1,R)'$$
$$V^i(t,T+1) = \sum_j T^i_j V^j(t,T).$$

This could be one way to develop a best estimate default assumption $\mu_0 = \mu_0(t)$ that reflects the multi-state environment. Table 3 below shows the forward default rates implied by the assumed best estimate transition matrix. These rates were calculated using

$$d^j_T(t) = \ln[\frac{V^j(t,T-1)}{V^j(t,T)}].$$

**Table 3: Best Estimate Forward Default Rates**

| Time | AAA | AA | A | BBB | BB | B | C |
|---|---|---|---|---|---|---|---|
| 1 | 0.00% | 0.00% | 0.01% | 0.08% | 0.63% | 3.29% | 13.44% |
| 2 | 0.00% | 0.00% | 0.02% | 0.15% | 0.84% | 3.29% | 10.53% |
| 3 | 0.00% | 0.00% | 0.04% | 0.21% | 1.00% | 3.18% | 8.08% |
| 4 | 0.00% | 0.01% | 0.05% | 0.27% | 1.12% | 3.02% | 6.13% |
| 5 | 0.00% | 0.01% | 0.07% | 0.32% | 1.20% | 2.82% | 4.61% |
| 10 | 0.02% | 0.05% | 0.17% | 0.51% | 1.26% | 1.87% | 1.24% |
| 15 | 0.04% | 0.11% | 0.27% | 0.59% | 1.09% | 1.23% | 0.52% |
| 20 | 0.08% | 0.18% | 0.34% | 0.60% | 0.90% | 0.86% | 0.31% |
| 25 | 0.13% | 0.23% | 0.38% | 0.58% | 0.74% | 0.63% | 0.22% |
| 30 | 0.18% | 0.28% | 0.41% | 0.54% | 0.61% | 0.48% | 0.16% |

Note that if we want to assume insurance liabilities have a AA rating we are talking about a fairly small best estimate default spread. The behavior of the C bond's forward default rates can be explained by noting that such a bond will only survive to 30 years by migrating back to a higher rating class at some future point in time.

- Contagion risk can still be modelled by assuming a capital requirement equal to, approximately, $n$ years' worth of best estimate credit transitions overnight. The capital requirement would be, if the eigen values of $T$ are small enough,

$$\boldsymbol{EC} = (I - T^n)\boldsymbol{V} \approx -nM\boldsymbol{V}.$$

The new valuation equation becomes

$$\frac{d\mathbf{V}}{dt} + (1 + n\pi)M\boldsymbol{V} = r\boldsymbol{V}$$[14].

This equation can be solved for a set of risk loaded forward default rates by using the same approach as was used for the best estimate case except that we now use a risk loaded transition matrix $\hat{T} = \exp[M(1 + n\pi)]$. The table below shows the impact on default rates of assuming $n = 4$ and $\pi = .10$. This table shows the difference between the forward default rates consistent with the loaded model and the forward default rate in Table 3.

[14] At this point we are close to the model of Jarrow, Lando and Turnbull referenced in footnote 4. Their model could be understood as a version of the current model where the cost of capital is a vector $\boldsymbol{\pi}(t)$ which varies by rating class and with time. They then use the time dependence of $\boldsymbol{\pi}$ to calibrate the model to observed yield curves by rating class. Their derivation does not use cost of capital concepts. The author's main critique of this approach is that the concept of say, a AA yield curve, does not really exist since many similar bonds, with the same current rating, can have different prices due to liquidity considerations.

**Table 4: Contagion Loaded Forward Default Spreads**

| | AAA | AA | A | BBB | BB | B | C |
|---|---|---|---|---|---|---|---|
| 1 | 0.00% | 0.00% | 0.01% | 0.05% | 0.31% | 1.32% | 4.53% |
| 2 | 0.00% | 0.00% | 0.02% | 0.11% | 0.48% | 1.23% | 2.10% |
| 3 | 0.00% | 0.01% | 0.04% | 0.17% | 0.56% | 1.04% | 0.52% |
| 4 | 0.00% | 0.01% | 0.06% | 0.21% | 0.59% | 0.82% | -0.34% |
| 5 | 0.01% | 0.02% | 0.08% | 0.24% | 0.58% | 0.61% | -0.70% |
| 10 | 0.03% | 0.08% | 0.17% | 0.30% | 0.33% | 0.03% | -0.39% |
| 15 | 0.08% | 0.15% | 0.22% | 0.25% | 0.13% | -0.10% | -0.11% |
| 20 | 0.14% | 0.19% | 0.22% | 0.19% | 0.03% | -0.10% | -0.05% |
| 25 | 0.18% | 0.22% | 0.21% | 0.14% | 0.00% | -0.09% | -0.03% |
| 30 | 0.21% | 0.22% | 0.19% | 0.11% | -0.01% | -0.07% | -0.02% |

According to the author's point of view, this is the component of the forward default rate that should not be used to discount insurance liabilities. It is very similar to the best estimate forward default rate if $n = 4$.

The fact that some of the risk adjusted forward default rates can be lower is not an error. This table simply shows that adding a contagion loading to the best estimate default matrix simply exaggerates the survival issue we saw in Table 3.

We can also use these calculations to derive economic required capital factors for contagion risk. Table 5 below shows the factors that should apply by rating class and cash flow maturity

$$c_{T-t}^{i} = \frac{n \sum_j M_j^i V^j(t,T)}{V^i(t,T)}.$$

**Table 5: Contagion Capital%**

| | AAA | AA | A | BBB | BB | B | C |
|---|---|---|---|---|---|---|---|
| 1 | 0.0% | 0.0% | 0.1% | 0.6% | 3.3% | 13.2% | 43.1% |
| 2 | 0.0% | 0.0% | 0.2% | 0.9% | 4.2% | 12.6% | 29.7% |
| 3 | 0.0% | 0.0% | 0.3% | 1.2% | 4.7% | 11.5% | 20.0% |
| 4 | 0.0% | 0.1% | 0.4% | 1.5% | 5.0% | 10.4% | 13.5% |
| 5 | 0.0% | 0.1% | 0.5% | 1.7% | 5.1% | 9.3% | 9.2% |
| 10 | 0.2% | 0.4% | 1.0% | 2.3% | 4.4% | 5.1% | 2.2% |
| 15 | 0.4% | 0.8% | 1.4% | 2.4% | 3.4% | 3.1% | 1.1% |
| 20 | 0.6% | 1.1% | 1.6% | 2.2% | 2.6% | 2.1% | 0.7% |
| 25 | 0.9% | 1.3% | 1.7% | 2.0% | 2.0% | 1.5% | 0.5% |
| 30 | 1.1% | 1.5% | 1.7% | 1.8% | 1.7% | 1.2% | 0.4% |
| OSFI | 0.25% | 0.5% | 1.0% | 2.0% | 4.0% | 8.0% | 16.0% |

The bottom row of this table shows the current Canadian[15] regulatory capital requirements by bond rating class. They would appear to be reasonable for a mix of bond maturities if contagion risk were the only issue on the agenda.

- The idea of parameter or liquidity risk can be incorporated by assuming a potential shock $\Delta M = \varphi M$. Assuming the shock is proportional to the best estimate is merely the simplest place to start. More complex models are possible.

  Given this particular approach to liquidity risk, the valuation equation becomes, for $\boldsymbol{V} = \boldsymbol{V}(t, \beta, T)$,

$$\frac{\partial \mathbf{V}}{\partial t} + \pi \frac{\partial \boldsymbol{V}}{\partial \beta} + (1 + n\pi + \beta\varphi) M \boldsymbol{V} = r \boldsymbol{V}^{16}.$$

  This equation can be solved by assuming we know the left eigen vectors of the matrix $M$. This is a matrix $L$ such that

$$LM = -DM,$$

  i.e. each row of $L$ is an eigen row of $M$ and $D$ is a diagonal matrix of the form

$$D = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \dots & 0 \\ 0 & 0 & \mu_n \end{pmatrix}.$$

  For this particular example, the eight diagonal eigen-values are, in increasing order

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0.0% | 1.0% | 5.9% | 9.0% | 13.3% | 18.0% | 26.6% | 39.6% |

and the eigen-row matrix is given by

| | | | **Eigen Row Matrix *L*** | | | | | |
|---|---|---|---|---|---|---|---|---|
| | **AAA** | **AA** | **A** | **BBB** | **BB** | **B** | **C** | **D** |
| **AAA** | 0.125 | 0.125 | 0.125 | 0.125 | 0.125 | 0.125 | 0.125 | 0.125 |
| **AA** | 0.259 | 0.229 | 0.197 | 0.154 | 0.096 | 0.050 | 0.015 | - |
| **A** | 1.184 | 0.379 | (0.001) | (0.170) | (0.210) | (0.141) | (0.041) | - |
| **BBB** | 0.898 | (0.051) | (0.135) | 0.013 | 0.129 | 0.115 | 0.031 | - |
| **BB** | 3.118 | (1.930) | 0.405 | 0.593 | (0.390) | (0.636) | (0.160) | - |
| **B** | 1.273 | (1.634) | 1.480 | (1.737) | 0.023 | 1.290 | 0.305 | - |
| **C** | (0.045) | 0.131 | (0.254) | 0.873 | (2.209) | 1.907 | 0.597 | - |
| **D** | 0.000 | (0.001) | 0.003 | (0.017) | 0.076 | (0.272) | 1.211 | - |

[15] These factors are very similar to the ones used by US regulators. The acronym OSFI stands for Office of the Super-Intendant of Financial Institutions which is the Canadian federal government regulator for both banks and insurers.

[16] This particular model assumes $d\beta = \pi dt$ without any adjustment. This is a simplification that is usually immaterial in most practical examples. One alternative is to assume the eigen-vectors of $M$ are known with certainty and then apply the two state model to each eigen value of $M$ sepateately. The next level of complexity is to allow for uncertainty in the eigen vectors of $M$. Such considerations are beyond the scope of this paper.

If we now define a new vector by $\boldsymbol{W} = L\boldsymbol{V}$ then the fundamental valuation equation becomes

$$\frac{\partial \boldsymbol{W}}{\partial t} + \pi \frac{\partial \boldsymbol{W}}{\partial \beta} = \big(r + D(1 + n\pi + \beta\varphi)\big)\boldsymbol{W}.$$

This equation is subject to the boundary condition $\boldsymbol{W}(T,T) = L\boldsymbol{V}(T,T)$. Based on the particular matrix chosen[17] above we find $\boldsymbol{W}(T,T) = (.9375,1,1,\ldots,1)'$.

Because the matrix $D$ is diagonal, we only need to consider the diagonal components which are very similar to the two-state model described earlier. Each diagonal component is of the form

$$\frac{dW^{\boldsymbol{i}}}{dt} + \pi \frac{\partial W^{\boldsymbol{i}}}{\partial \beta} = \big(r + \mu_i(1 + n\pi + \beta\varphi)\big)W^{\boldsymbol{i}}.$$

If the cost of capital rate $\pi$ is a constant, then $\beta(t,s) = \pi(s-t)$ and the closed form solution for $W^{\boldsymbol{i}}(t,T)$ is

$$W^{\boldsymbol{i}}(t,T) = W^{\boldsymbol{i}}(T,T)\exp[-\int_t^T \big(r + \mu_i(1 + \pi(n + (s-t)\varphi)\big)ds],$$

$$= W^{\boldsymbol{i}}(T,T)\exp[-\left[r(T-t) + \mu_i(T-t)\left(1 + \pi\left(n + \varphi\frac{(T-t)}{2}\right)\right)\right].$$

We can recover the original vector value from $\boldsymbol{V} = L^{-1}\boldsymbol{W}$ and then compute forward default spreads as before. Table 6 below shows the results that follow from the assumptions stated earlier along with $\varphi = .25$ i.e. $\Delta M = .25\ M$.

**Table 6: Parameter/Liquidity Risk Spreads** $\phi = 25\%$

| | AAA | AA | A | BBB | BB | B | C |
|---|---|---|---|---|---|---|---|
| 1 | 0.00% | 0.00% | 0.00% | 0.00% | 0.01% | 0.04% | 0.13% |
| 2 | 0.00% | 0.00% | 0.00% | 0.01% | 0.04% | 0.12% | 0.23% |
| 3 | 0.00% | 0.00% | 0.01% | 0.02% | 0.08% | 0.17% | 0.19% |
| 4 | 0.00% | 0.00% | 0.01% | 0.04% | 0.12% | 0.19% | 0.10% |
| 5 | 0.00% | 0.01% | 0.02% | 0.06% | 0.15% | 0.20% | 0.03% |
| 10 | 0.02% | 0.05% | 0.09% | 0.15% | 0.20% | 0.13% | 0.00% |
| 15 | 0.07% | 0.12% | 0.17% | 0.20% | 0.16% | 0.08% | 0.02% |
| 20 | 0.16% | 0.21% | 0.23% | 0.21% | 0.13% | 0.05% | 0.02% |
| 25 | 0.26% | 0.28% | 0.27% | 0.22% | 0.12% | 0.05% | 0.01% |
| 30 | 0.34% | 0.33% | 0.29% | 0.22% | 0.12% | 0.05% | 0.02% |

Interestingly, a 25% parameter shock can have more impact on high quality bond spreads than it does on low quality bond spreads as the bonds get longer.

[17] The rows of the matrix $L$ can be permuted and scaled arbitrarily without affecting the end results.

- Allowing the cost of capital rate to be stochastic still does not change very much.

  The model becomes

$$\frac{d\mathbf{V}}{dt} + \kappa(\pi_\infty - \pi)\frac{\partial \boldsymbol{V}}{\partial \pi} + \frac{\xi^2 \pi}{2}\frac{\partial^2 \boldsymbol{V}}{\partial \pi^2} + \pi \frac{\partial \boldsymbol{V}}{\partial \beta} + (1 + n\pi + \beta\varphi)M\boldsymbol{V} = r\boldsymbol{V}.$$

  As before we let $\boldsymbol{W} = L\boldsymbol{V}$ and $\boldsymbol{W}(T,T) = (.9375,1,1,\ldots,1)'$. The diagonal component equations are then

$$\frac{dW^{\boldsymbol{i}}}{dt} + \pi\frac{\partial W^{\boldsymbol{i}}}{\partial \beta} + \kappa(\pi_\infty - \pi)\frac{\partial W^{\boldsymbol{i}}}{\partial \pi} + \frac{\xi^2 \pi}{2}\frac{\partial^2 W^{\boldsymbol{i}}}{\partial \pi^2} = \big(r + \mu_i(1 + n\pi + \beta\varphi)\big)W^{\boldsymbol{i}}.$$

This is slightly different from the equation studied in the two-state model section but the main technical conclusions do not change. We will not go further into the details of solving this model here.

**Conclusion**

The main risk management conclusion of this paper is that there are several credit related risks that can be naturally hedged between the asset and liability side of a life insurer's balance sheet. In particular, we have argued that the risks associated with credit loss parameter risk are essentially the same as liquidity risk and that the issues associated with market sentiment can also be passed through to the liability side of the balance sheet. If the reader accepts this point of view, then it makes sense to include those components of a credit risky bond yield curve into the valuation of life insurance liabilities. Taking this approach to liability valuation would go a long way towards resolving the issues raised in the introduction to this paper.

To the extent a life insurer matches its long assets with its long liabilities the only remaining risk that it needs to hold capital for is the contagion risk element (credit crunch). At a high level, this is already where most regulatory capital models are[18].

A second conclusion is that institutions that do not have long liabilities (such as banks) should be holding more regulatory capital for risky bonds because there is no capital offset with their liabilities.

[18] To be fair, most of these regulatory models were developed for a book value accounting world where the risks associated with short term market fluctuations are swept under the rug by accounting tricks such as book value accounting. If you take the risk management point of view, as we do here, that this is inappropriate, then the capital models need to be adjusted. This may include allowing very high quality bonds to have negative capital requirements.