Assessing Extreme Risk using Stochastic Simulation

joint with Juliette Legrand (Université de Bretagne Occidentale) and Nisrine Madhar (Université Paris Cité & Natixis)

Maud Thomas

ISFA, Université Lyon 1

Introduction and motivation

Risk management

- **Risk management** = Crucial in various sectors, specific to each sector
- **Risk factor** = any variable that could result in a loss or damage
  → can be represented by random variables that quantify the magnitude of potential losses
- **Examples**
  - Climatology
    → Meteorological and marine hazards can cause significant damages
    → e.g., drought, floods, landslides
    → Risk factor = any physical quantity (wave heights, wind gusts, precipitation)
  - Finance
    → Market movements can result to substantial losses
    → Risk factor = typically, market parameters, interest rates or exchange rates
- **Tail risk**: Events with very severe magnitudes and that occurred with very low probability

Aim

For a given target risk factor X_j, accurately quantify its risk through the estimation of tail risk metrics (TRM).

Tail risk metrics

- $\mathbf{X} = (X_1, \dots, X_d) \in \mathbb{R}^d$ vector of risk factors, with X_j of density f_j
- We consider 3 TRMs defined as
  - **Expected Shortfall** [Artzner et al., 1999] at level $\alpha \in (0,1)$

$$\mathrm{ES}_\alpha(X_j) = \mathbb{E}\left[X_j \mid X_j > \mathrm{VaR}_\alpha(X_j)\right] = \frac{1}{1-\alpha}\int_{\mathrm{VaR}_\alpha(X_j)}^{\infty} x_j f(x_j)\mathrm{d}x_j$$

    where $\mathrm{VaR}_\alpha(X_j) := \inf\{x_j \in \mathbb{R} : \mathbb{P}\left(X_j \le x_j\right) \ge 1-\alpha\}$.
  - **Multivariate marginal ES** of X_j at level α

$$\mathrm{MMES}_\alpha(X_j; \boldsymbol{X}) = \mathbb{E}\left[X_j \mid \boldsymbol{X}_{-j} \ge \boldsymbol{v}^\alpha_{-j}\right] = \int_{\mathbb{R}} x_j f_{X_j \mid \boldsymbol{X}_{-j} \ge \boldsymbol{v}^\alpha_{-j}}(x_j)\mathrm{d}x_j,$$

    with $\boldsymbol{v}^\alpha = \mathrm{VaR}_\alpha(\boldsymbol{X}) \in \mathbb{R}^d$
  - **Dependent conditional tail expectation** of X_j at level α

$$\mathrm{DCTE}_\alpha(X_j; \boldsymbol{X}) = \mathbb{E}\left[X_j \mid \boldsymbol{X} \ge \boldsymbol{v}^\alpha\right] = \int_{v^\alpha_j}^{\infty} x_j f_{X_j \mid \boldsymbol{X} \ge \boldsymbol{v}^\alpha}(x_j)\mathrm{d}x_j,$$

Tail risk management

Aim

For a given target risk factor X_j, accurately quantify its risk through the estimation of tail risk metrics (TRM).

- The TRMs are computed at extreme levels
- $\boldsymbol{X}$ may exhibit dependence
  $\Rightarrow$ Leverage from this dependence to quantify the risk of X_j given $\boldsymbol{X}_{-j} = (X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_d)$
  $\Rightarrow$ Multivariate Extreme Value Theory (EVT)
- Parametric approaches based on EVT [McNeil et al., 2015]
  - Model selection among the several parametrizations proposed by [Rootzén et al., 2018a] is necessary, challenging and time-consuming
  - Dependence modeling [Nelsen, 2006] of extremes through copula implies uniform dependence across considered risk factors
  - In extreme regions, the number of available tail observations becomes limited making any estimation a challenging task
  - Parametric framework can be restrictive

Non-parametric simulation algorithms

Our approach

Extension of the two non-parametric simulation algorithms for multivariate extremes, developed by Legrand et al. [2023] in the dimension 2 case to larger dimensions

- **Joint simulation of multivariate extremes**
  → estimation of TRMs
- **Conditional simulation of multivariate extremes**
  → estimation of quantities involving some conditional tail distribution

Both algorithms are based on the **multivariate Generalized Pareto distribution**

Non-parametric simulation algorithms

Our approach

Extension of the two non-parametric simulation algorithms for multivariate extremes, developed by Legrand et al. [2023] in the dimension 2 case to larger dimensions

- **Joint simulation of multivariate extremes**
  → estimation of TRMs
- **Conditional simulation of multivariate extremes**
  → estimation of quantities involving some conditional tail distribution

Both algorithms are based on the **multivariate Generalized Pareto distribution**

Elements of extreme value theory

Goals of Extreme Value Theory

Goals of Extreme Value Theory

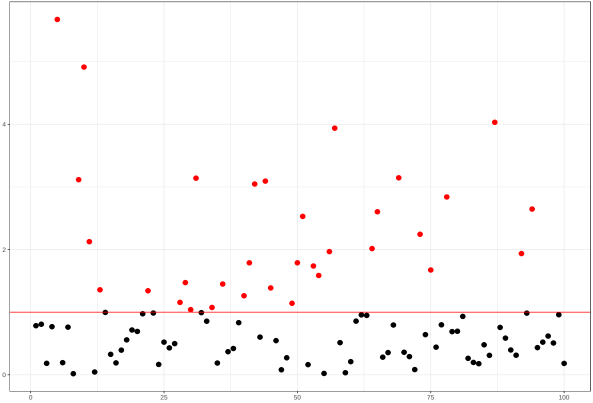
1. Estimate the probability of occurrence of an event more severe/extreme than previously observed
2. Estimate an extreme quantile

⇒ Inference outside the sample support

Univariate Peaks-over-Threshold method

- $Y_1, Y_2, \ldots$ a series of i.i.d. random variables
- Fix a (high) threshold u
- Extreme event = Y_i exceeds u

  → Given that $Y_i > u$, an excess is defined by $Z_i = Y_i - u$



Univariate Peaks-over-Threshold method

- $Y_1, Y_2, \ldots$ a series of i.i.d. random variables
- Fix a (high) threshold u
- Extreme event = Y_i exceeds u

  $\rightarrow$ Given that $Y_i > u$, an excess is defined by $Z_i = Y_i - u$
- Excess distribution

$$\overline{F}_u(z) = P[Y_1 - u > z \mid Y_1 > u] = \frac{\overline{F}(u+z)}{\overline{F}(u)}, \; z > 0\,.$$

Balkema et de Haan (1974), Pickands (1975)

Under certain conditions, the distribution of excesses F_u converges, as $u \to \infty$, to a generalized Pareto distribution (GPD) whose distribution function is

$$H_{\sigma,\gamma}(z) = \begin{cases} 1 - \left(1 + \frac{\gamma}{\sigma} z\right)^{-1/\gamma} & \text{if } \gamma \neq 0 \\ 1 - \exp\left(-\frac{z}{\sigma}\right) & \text{if } \gamma = 0 \end{cases}$$

- Families of possible distributions for excesses = parametric family

  $\hookrightarrow$ Generalized Pareto distributions (GPD)

Generalized Pareto distributions

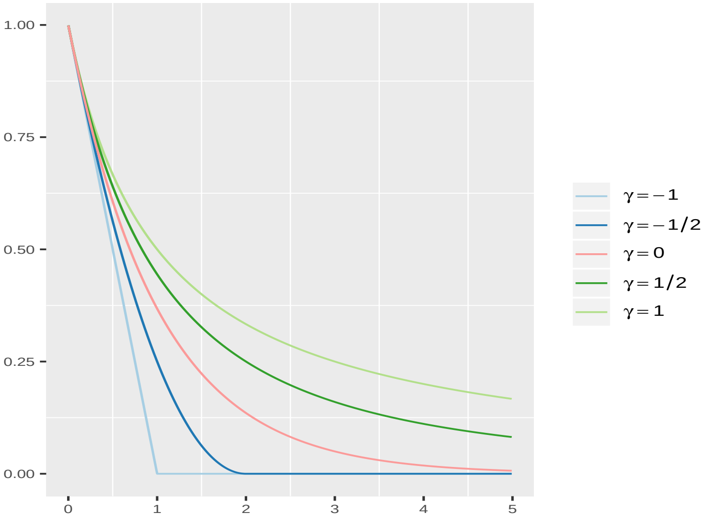


Figure: GPD survival functions

3 domains of attraction

1. Fréchet domain ($\gamma > 0$): **heavy-tailed distributions**

$$1 - H_\gamma(z) \underset{+\infty}{\sim} \gamma^{-1/\gamma} z^{-1/\gamma}$$

   Examples: Cauchy, Log-gamma, Student

2. Gumbel domain ($\gamma = 0$): **thin tail distributions**

$$1 - H_0(z) \underset{+\infty}{\sim} \exp(-z)$$

   Examples: Gaussian, Gamma, Exponential

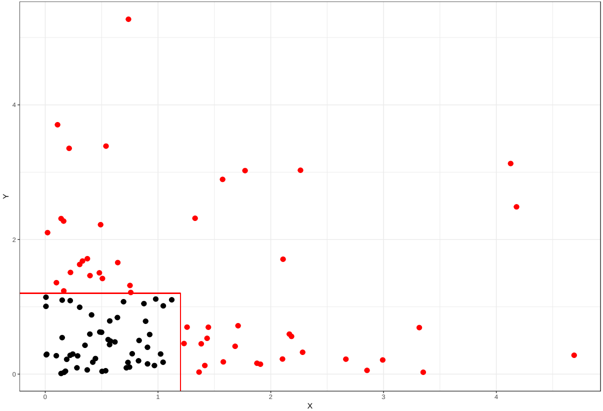
3. Weibull domain ($\gamma < 0$): **finite tail distributions**

$$1 - H_\gamma(z) = 0 \quad \text{for } x \geq -1/\gamma$$

   Examples: Uniform, Beta

Multivariate Generalized Pareto Distributions

- $\mathbf{X} = (X_1, \ldots, X_d)$ observations
- Choose (high) thresholds $\mathbf{u} = (u_1, \ldots, u_d)$
- Extreme event = AT LEAST one of the X_j exceeds its threshold u_j



Multivariate Generalized Pareto Distributions

- $\mathbf{X} = (X_1, \ldots, X_d)$ observations
- Choose (high) thresholds $\mathbf{u} = (u_1, \ldots, u_d)$
- Extreme event = AT LEAST one of the X_j exceeds its threshold u_j
- Theory: asymptotically (when $\boldsymbol{u} \to \infty$), exceedances occur according to a Poisson process and $\mathbf{Z} = \mathbf{X} - \mathbf{u} \mid \boldsymbol{X} \nleq \boldsymbol{u}$ follows a multivariate Generalized Pareto Distribution (MGPD) with a scale parameter $\boldsymbol{\sigma}$ and a shape parameter $\boldsymbol{\gamma}$.

- Standard MGPD → $\boldsymbol{\sigma} = \mathbf{1}$ and $\boldsymbol{\gamma} = \mathbf{0}$
  → Exponential marginals

⚠ NO parametric family of limits distributions

- Rootzén et al. [2018b], Kiriliouk et al. [2019] have proposed explicit density formula for specific models
- Non parametric models are difficult to fit (lack of data)

Multivariate generalized Pareto Vectors

- Let $\boldsymbol{X}$ be a d-dimensional random vector, the vector of excesses is defined as

  (1) $$\boldsymbol{Z} = \boldsymbol{X} - \boldsymbol{u} \mid \boldsymbol{X} \nleq \boldsymbol{u}$$

  where $\boldsymbol{u} \in \mathbb{R}^d$ is a vector of suitably chosen thresholds and $\nleq$ means that at least one of the components of $\boldsymbol{X} - \boldsymbol{u}$ is positive.

- Rootzén et al. [2018b] have shown that a standard MGP vector $\boldsymbol{Z}$ can be decomposed as follows

  $$\boldsymbol{Z} = E + \boldsymbol{T} - \max(\boldsymbol{T}),$$

  where
  - E is a unit exponential variable ;
  - $\boldsymbol{T}$ a d-dimensional random vector
  - $\boldsymbol{T}$ and E are independent.

Non-parametric joint MGP simulation in dimension 2

Legrand et al. [2023]

- From Rootzén et al. [2018b],

$$\begin{cases} Z_1 & = E + T_1 - \max(T_1, T_2) \\ Z_2 & = E + T_2 - \max(T_1, T_2) \end{cases}$$

- Noting $\Delta = Z_1 - Z - 2 = T_1 - T_2$,

$$\begin{cases} Z_1 & = E + \Delta \mathbf{1}_{\Delta < 0} \\ Z_2 & = E - \Delta \mathbf{1}_{\Delta \geq 0} \end{cases}$$

- Simulate values of Δ and E independently
- Simulate E is trivial
- Difficulty : simulate Δ
  $\Rightarrow$ **Bootstrapping on** Δ

Algorithms cornerstone

- Needs to generalize the previous slide on dimension d
- From

$$\boldsymbol{Z} = E + \boldsymbol{T} - \max(\boldsymbol{T}), \tag{2}$$

define

$$\Delta^{j,k} = Z_j - Z_k = T_j - T_k, \text{ for all } j,k = 1,\ldots,d.$$

- Equation (2) can be rewritten as follows

$$Z_j = E + \sum_{k=1,k\neq j}^{d} \Delta^{j,k} \prod_{\ell=1,\ell\neq k}^{d} \mathbf{1}_{\Delta^{\ell,k}<0}, \text{ for all } j = 1,\ldots,d, \tag{3}$$

where $\mathbf{1}_{\cdot}$ denotes the indicator function.

Non-parametric joint MGP simulation

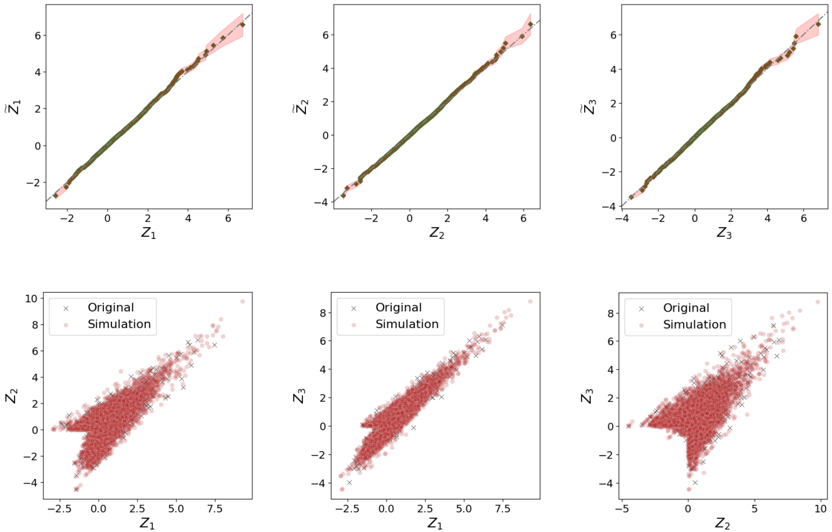
Input: Observations $(\boldsymbol{Z}_i)_{1\le i\le n} = \left(Z_{i,1},\dots,Z_{i,d}\right)_{1\le i\le n}$ from a standard MGPD vector

1. Compute $\Delta_i^{1,k} \leftarrow Z_{i,1} - Z_{i,k}$, for $1 \le i \le n$ and $1 \le k \le d$
   $\rightarrow$ Obtain the vector $\left(\boldsymbol{\Delta}_i^{(1)}\right)_{1\le i\le n}$
2. Generate $E_1,\dots,E_m \overset{\text{i.i.d.}}{\sim} \text{Exp}(1)$
3. Generate a m-bootstrap sample $\left(\widetilde{\boldsymbol{\Delta}}_\ell^{(1)}\right)_{\ell=1,\dots,m}$ from $\left(\boldsymbol{\Delta}_i^{(1)}\right)_{1\le i\le n}$
4. $\widetilde{\Delta}_\ell^{r,s} \leftarrow \widetilde{\Delta}_\ell^{1,s} - \widetilde{\Delta}_\ell^{1,r}$, for $1 \le \ell \le m$ and all $1 \le r,s \le d$
5. $\widetilde{Z}_{\ell,j} \leftarrow E_\ell + \sum_{s=1,s\ne j}^{d} \widetilde{\Delta}_\ell^{j,s} \prod_{r=1,r\ne s}^{d} \mathbf{1}_{\widetilde{\Delta}_\ell^{r,s}<0}$ for all $1 \le \ell \le m$ and $1 \le j \le d$

Output: A standard MGP simulated sample $\left(\widetilde{\boldsymbol{Z}}_m\right)_{1\le \ell\le m} = \left(\widetilde{Z}_{\ell,1},\dots,\widetilde{Z}_{\ell,d}\right)_{1\le \ell\le m}$

Non-parametric joint MGP simulation

Bivariate representations of the original sample a MGP vector $\boldsymbol{Z} \in \mathbb{R}^3$ (black) and the simulated sample $\widetilde{\boldsymbol{Z}}$ (red) through QQ plots and scatter plots.



Simulation framework

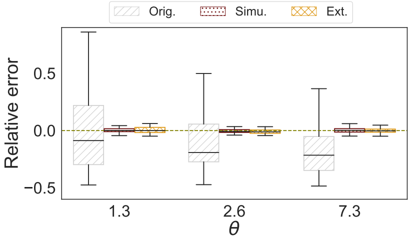
- Let $\boldsymbol{X} = (X_1, X_2, X_3)$ be a random vector with marginals distributed as a Student t-distribution with degrees of freedom $\nu_1 = 2$, $\nu_2 = 3$, $\nu_3 = 2.5$.
- An underlying assumption of our simulation framework is that the components of $\boldsymbol{X}$ are asymptotically dependent
- To ensure that this hypothesis is satisfied, we consider the Gumbel copula [Nelsen, 2006] to obtain dependent extremes in the upper tail

$$\mathscr{C}(\boldsymbol{y}) := \exp\left(-\left(\sum_{i=1}^{3}\left[-\log(y_i)\right]^{\theta}\right)^{1/\theta}\right),$$

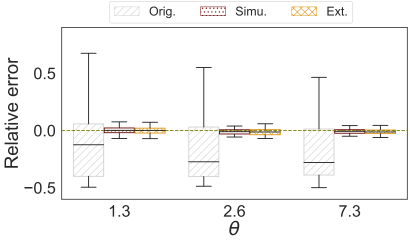
  where $\theta \geq 1$ is the copula parameter. The larger θ, the stronger the asymptotic dependence structure between the components of $\boldsymbol{X}$.
- The numerical experiments are performed on simulated data sets $\mathscr{D} \in \mathbb{R}^{1500\times 3}$,

TRMs estimation through joint simulation algorithm

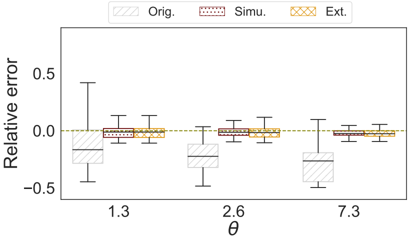
Estimations of TRMs on 50 original samples *(grey)*, **50 simulated samples** *(red)* **and 50 extended samples** *(yellow)*



a) $\mathrm{ES}_{0.9975}$

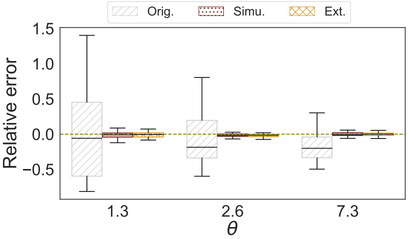


b) $\mathrm{ES}_{0.999}$

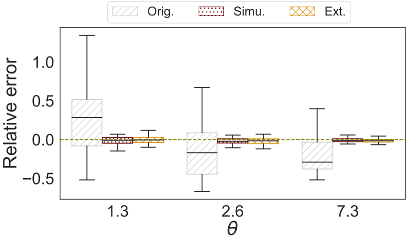


TRMs estimation through joint simulation algorithm

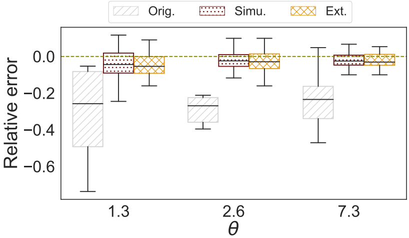
Estimations of TRMs on 50 original samples *(grey)*, **50 simulated samples** *(red)* **and 50 extended samples** *(yellow)*



d) $\text{MMES}_{0.9975}$

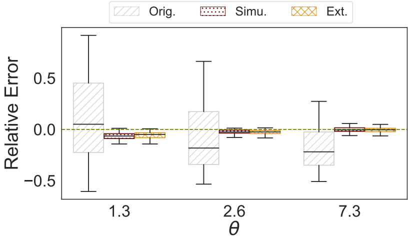


e) $\text{MMES}_{0.999}$

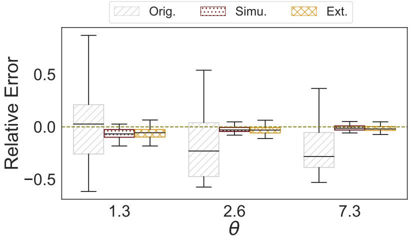


TRMs estimation through joint simulation algorithm

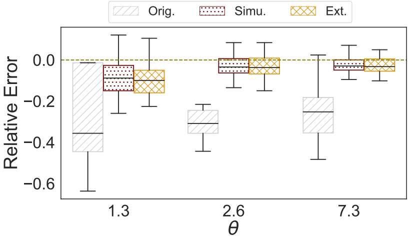
Estimations of TRMs on 50 original samples *(grey)*, **50 simulated samples** *(red)* **and 50 extended samples** *(yellow)*



g) $\mathrm{DCTE}_{0.9975}$



h) $\mathrm{DCTE}_{0.999}$



Conclusion

- A primary concerns when estimating risks at high levels is the data sparseness
- This issue was addressed by the development of two non-parametric simulation approaches of multivariate extremes

Main contribution of the suggested non-parametric approaches

- Expands the number of observation above extreme level
- Ensures more reliable estimations
- Enables extrapolation beyond the range of observed data

- **Conditional simulation of multivariate extremes**
  $\rightarrow$ estimation of quantities involving some conditional tail distribution e.g. $\mathbb{E}[X_j \mid \boldsymbol{X}_{-j} = \boldsymbol{x}_{-j}]$

Thank you for your attention!

Reference I

P. Artzner, F. Delbaen, J.-M. Eber, and D. Heath. Coherent measures of risk. *Mathematical finance*, 9(3):203–228, 1999.

A. Kiriliouk, H. Rootzén, J. Segers, and J. L. Wadsworth. Peaks over thresholds modeling with multivariate generalized pareto distributions. *Technometrics*, 61 (1):123–135, 2019.

J. Legrand, P. Ailliot, P. Naveau, and N. Raillard. Joint stochastic simulation of extreme coastal and offshore significant wave heights. 2023.

A. J. McNeil, R. Frey, and P. Embrechts. *Quantitative risk management: concepts, techniques and tools-revised edition*. Princeton university press, 2015.

R. B. Nelsen. *An introduction to copulas*. Springer, 2006.

H. Rootzén, J. Segers, and J. L. Wadsworth. Multivariate peaks over thresholds models. *Extremes*, 21(1):115–145, 2018a.

H. Rootzén, J. Segers, and J. L. Wadsworth. Multivariate generalized pareto distributions: Parametrizations, representations, and properties. *Journal of Multivariate Analysis*, 165:117–131, 2018b.