The Marginal Cost of Risk in a Multi-Period Risk Model

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Chapter 1

Introduction

Capital allocation is used widely within the insurance industry for purposes of pricing and performance measurement. The practice, however, inspires controversy on several levels. Some question its necessity (Phillips, Cummins, and Allen, 1998; Sherris, 2006). Others argue that it leads to economically suboptimal decisions (Venter, 2002; Gründl and Schmeiser, 2007). Many thought leaders in the actuarial community have picked up on this disparity providing various vantage points why the allocation problem may be misguided or what the debate may be missing (Mango, 2005; Kreps, 2005; Venter, 2010; D'Arcy, 2011).

In this report, we start in Chapter 2 by reviewing the various approaches to capital allocation and identifying the circumstances under which pricing based on capital allocation is economically optimal. To preview, in relatively simple settings, where the objective is to maximize firm value in a single period, capital allocation can be consistent with marginal cost pricing. This results trivially if the problem is cast as expected profit maximization subject to a risk measure constraint (in which case gradient allocation methods applied to the constraining risk measure are appropriate for pricing purposes). It also results in more complex specifications of the single period model, although the correct risk measure is similarly complex (Bauer and Zanjani, 2013a).

In Chapter 3 of the the report, we perform numerical analysis using simulated data provided by a catastrophe reinsurer to get a sense of the differences between the methods and to assess their stability. Here, we confirm findings elsewhere in the literature that tail measures, despite their current popularity with regulators and practitioners, are rather unstable, especially when using thresholds consistent with current capitalization of reinsurance companies.

In Chapter 4, we consider the theoretical impact of extending the canonical model of a profit maximizing insurer beyond a single period and in particular consider the effect of having opportunities to raise external financing in future periods. Then, outside of special cases, capital allocation as currently conceived (i.e., the allocation of the accounting surplus of the firm to business line) *cannot* be done in a way to produce prices consistent with marginal cost. Once the company has resources other than its current capital to absorb its risks, allocating current capital no longer produces prices that cover the marginal cost of risk. Thus, as we introduce real world complications into our model of the firm, traditional capital allocation fails as a pricing technique.

Conventional capital allocation may fail in theory, but does it fail in practice? In Chapter 5, we compare the capital allocations and RAROC generated by those traditional techniques to the economically correct values based on the model in Chapter 4. We find that traditional techniques can be resurrected *if* 1) an appropriate definition of capital is used (and the appropriate definition in our model is a much broader conception of all financial resources—including ones that have not yet been tapped) and 2) a broader conception of the cost of underwriting business—including, for example, penalties for the impact of risk on the continuation value of the firm, as well as an allocation for future costs of external financing—is used when evaluating the profitability of a contract, and 3) an appropriate risk measure—connected to the fundamentals of the underlying business—is used.

The main message of this paper lies in the analyses of Chapters 4 and 5. As we consider more realistic models of firm value, accurate pricing requires us to think carefully about defining capital and accounting for the impact of risk on the firm. Standard allocation techniques are not necessarily inconsistent with this guidance, but they can only be correctly implemented if the user has an understanding of the economic context in which the firm is operating. This essentially echoes Venter's criticism (Venter, 2010) of allocation as an attempt "to do risk pricing while avoiding the rigors of the pricing project." While allocation can be deployed thoughtfully, we see no alternative to careful consideration of the firm's objectives, constraints, and institutional context in setting prices.

Chapter 6 concludes and identifies some avenues for future research.

Chapter 2

A Review on Capital Allocation Theory and Practice

This chapter reviews the theory and practice for the allocation of capital to business units or lines. We commence by briefly describing *why*, or rather *when*, this problem is of interest (Section 2.1), and by defining *what* exactly we mean by a "capital allocation" (Section 2.2). Next, we review approaches of *how* to allocate capital, first from a conceptual perspective in Section 2.3 and then from a practical perspective in Section 2.4. We discuss an array of different allocation methods that were proposed in the literature—which will the be analyzed in the context of example applications in Chapter 3.

2.1 Why Allocate Capital?

We must first establish *why* we allocate capital. The simple answer from the practitioner side is that allocation is a necessity for pricing and performance measurement. When setting benchmarks for lines of business within a multi-line firm, one must ensure that the benchmarks put in place are consistent with the firm's financial targets, specifically the target return on equity.

This seems logical at first glance, yet some of the academic literature has been skeptical. Phillips, Cummins and Allen (1998) noted that a "financial" approach to pricing insurance in a multi-line firm rendered capital allocation unnecessary, a point reiterated by Sherris (2006). The "financial" approach relies on applying the usual arbitrage-free pricing techniques in a complete market setting without frictional costs. In such a setting, one simply pulls out a market consistent valuation measure to calculate the fair value of insurance liabilities. Capital affects this calculation in the sense that the amount of capital influences the extent to which insurance claims are actually paid in certain states of the world, but, so long as the actuary is correctly evaluating the extent of claimant recoveries in various states of the world (including those where the insurer is defaulting), there is no need to apportion the capital across the various lines of insurance.

Once frictional costs of capital are introduced, the situation changes. Frictions open up a gap between the expected profits produced by "financial" insurance prices and the targetted level of profits for the firm. In such a case, the "gap" becomes a cost that must be distributed back to business lines, like overhead or any other common cost whose distribution to business lines is not immediately obvious.

As a practical example, consider catastrophe reinsurers. Natural catastrophe risk is often argued to be "zero beta" in the sense of being essentially uncorrelated with broader financial markets. If we accept this assessment, basic financial theory such as the CAPM would then imply that a market rate of return on capital exposed to such risk would be the risk-free rate. Yet, target ROEs at these firms are surely well in excess of the risk-free rate. The catastrophe reinsurer thus has the problem of allocating responsibility for hitting the target ROE back to its various business lines without any guidance from the standard no arbitrage pricing models.

Viewed in this light, "capital allocation" is really shorthand for "capital cost allocation." Capital itself, absent the segmentation of business lines into separate subsidiaries, is available for all lines to consume. A portion allocated to a specific line is not in any way segregated for that line's exclusive use. Hence, the real consequence of allocation lies in the assignment of responsibility for capital cost: A line allocated more capital will have higher target prices.

An important point, to which we shall return later, is the economic meaning of the allocation. Merton and Perold (1993) debunked the notion that allocations could be used to guide business decisions involving inframarginal or supramarginal changes to a risk portfolio (e.g., entering or exiting a business line). The more common argument is that allocation is a marginal concept—offering accurate guidance on small, infinitesimal changes to a portfolio. As we will see, many methods do indeed have a marginal interpretation, but the link to marginal *cost* is not always a strong one.

2.2 Capital Allocation Defined

We first start with notation and by defining capital allocation.[1] Consider a one period model with N business lines with loss realizations $L^{(i)}$, $1 \leq i \leq N$, modeled as square-integrable random-variables in an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. At the beginning of the period, the insurer decides on a quantity of exposure in each business line and receives a corresponding premium $p^{(i)}$, $1 \leq i \leq N$, in return. The exposure is an indemnity parameter $q^{(i)} \in \Phi^{(i)}$, where $\Phi^{(i)}$ are compact choice sets, so that the actual exposure to loss $i \in \{1, 2, \ldots, N\}$ is

$$I^{(i)} = I^{(i)}(L^{(i)}, q^{(i)}).$$

We assume that an increase in exposure shifts the distribution of the claim random variable so that the resulting distribution has first order stochastic dominance over the former:

$$\mathbb{P}\left(I^{(i)}(L^{(i)}, \hat{q}^{(i)}) \geq z\right) \geq \mathbb{P}\left(I^{(i)}(L^{(i)}, q^{(i)}) \geq z\right) \ \forall z \geq 0,\ \hat{q}_i \geq q_i.$$

[1] This subsection and the next borrows notation and approach from Bauer and Zanjani (2013b).

For simplicity, we typically consider $q^{(i)}$ representing an insurance company's quota share of a customer i's loss:

$$I^{(i)} = L^{(i)} \times q^{(i)}.$$

Other specifications could be considered, but the specification above implies that the claim distribution is homogeneous with respect to the choice variable $q^{(i)}$. This simplifies capital allocation, although it should be noted that insurance claim distributions are not always homogeneous (Mildenhall (2004)), and the "adding up" property associated with a number of methods depends on homogeneity.

We denote company assets as a and *capital* as k, where to fix ideas we adopt a common specification of the difference between the fair value of assets and the present value of claims. We denote by I the aggregate claims for the company, with the sum of the random claims over the sources adding up to the total claim:

$$\sum_{i=1}^{N} I^{(i)} = I.$$

However, actual payments made only amount to $\min\{I, a\}$ because of the possibility of default. We can also decompose actual payments, where the typical assumption in the literature is of equal priority in bankruptcy, so that the payment to loss i is:

$$\min\left\{I^{(i)}, \frac{a}{I} I^{(i)}\right\} \Rightarrow \sum_{i=1}^{N} \min\left\{I^{(i)}, \frac{a}{I} I^{(i)}\right\} = \min\{I, a\}.$$

Allocation is simply a division of the company's capital or assets across the N sources of risk, with $k^{(i)}$ representing the capital per unit of exposure assigned to the i-th source (and $a^{(i)}$ representing a similar quantity for assets). Of course, a full allocation requires that the individual amounts assigned to each of the lines "add up" to the total amount for the company:

$$\sum_{i=1}^{N} q^{(i)}\, k^{(i)} = k \text{ and } \sum_{i=1}^{N} q^{(i)}\, a^{(i)} = a.$$

It is worth noting that the question of *what* to allocate is not necessarily straightforward. Are we to allocate the book value of equity? The market value of equity? Assets? In general, the answer to this question is going to be guided by the nature of costs faced by the firm. Even then, the costs may be difficult to define, as the decomposition of capital costs offered by Mango (2005) (see below) suggests.

2.3 Conceptual Approaches to Allocating Capital

Assuming we have answered the question of *what* to allocate, the remaing question is how to do it. Unfortunately, the answer is not straightforward: There is a bewildering variety of peddlers in the capital allocation market. Mathematicians bearing axioms urge us to adhere to their methods—failure to do so will result in some immutable law of nature being violated. Economists assure us that only their methods are "optimal." Game theorists insist that only their solution concepts can be trusted. Practitioners wave off all of the foregoing as the raving of ivory tower lunatics, all the while assuring us that only their methods are adapted to the "real world" problems faced by insurance companies. Everyone has a "pet method," perhaps one that has some intuitive appeal, or one that is perfectly adapted to some particular set of circumstances.

Given such variety, it is not surprising that allocation methods defy easy categorization. Many do end up in essentially the same place—the so-called Euler or Gradient Principle—a convergence noted by Urban, Dittrich, Klüppelber, and Stölting (2003) and Albrecht (2004). But others do not. In the following, we attempt to give an overview on the primary approaches. We keep the focus on concepts, and delay examples and implementation to the next section.

2.3.1 The Euler Method and Some Different Ways to Get There

Consider setting capitalization based on a differentiable risk measure $\rho(I) = k$ and further imagine allocating capital to line i based on:

$$k^{(i)} = \frac{\partial \rho(I)}{\partial q^{(i)}} \tag{2.1}$$

This allocation is commonly referred to as g*radient* or *Euler* allocation, the latter being a reference to *Euler's homogeneous function theorem*. This theorem states that for every positive homogeneous function of degree one $(q^{(1)}, \ldots, q^{(N)}) \mapsto \rho(q^{(1)}, \ldots, q^{(N)})$—which is equivalent to requiring that the risk measure $\rho(I) = \rho(\sum_i q^{(i)} L^{(i)})$ be homogeneous—we automatically obtain the "adding up" property: $\rho(I) = \sum_{i=1}^{N} q^{(i)} \frac{\partial \rho(I)}{\partial q^{(i)}}$. The basic Euler approach can be found in Schmock and Straumann (1999) and Tasche (2004), among others.

The Euler or gradient allocation can also be implemented without requiring that $\rho(I) = k$ by normalizing:

$$\frac{k^{(i)}}{k} = \frac{\frac{\partial \rho(I)}{\partial q^{(i)}}}{\rho(I)}. \tag{2.2}$$

One of the major advantages of the Euler allocation is that it is possible to directly calculate (approximative) allocations given that one has an Economic Capital framework available that allows to derive $\rho(I)$ and k.[2] More specifically, we can approximate the derivative

[2] This is not at all to say that this task is simple. In fact, the computational complexity associated with evaluating economic capital presents a serious problem for financial institutions and frequently leads them to adopt second-best calculation techniques (Bauer, Reuss, and Singer, 2012). However, the availability of a suitable model for the different risk within a company's portfolio and their interplay clearly is a necessity for the derivation for any coherent allocation of capital.

occurring in the allocation rule by simple finite differences (although more advanced approaches may be used), that is:

$$\frac{\partial \rho(I)}{\partial q^{(i)}} \approx \frac{\rho(I + \Delta L^{(i)}) - \rho(I)}{\Delta}, \tag{2.3}$$

where $\Delta > 0$ is "small."

A number of different paths lead to the Euler allocation.

Denault (2001) proposes a set of axioms that define a *coherent* capital allocation principle when $\rho(I) = k$. His axioms required:

1. Adding up - The sum of allocations must be k.

2. No undercut - Any sub-portfolio would require more capital on a stand-alone basis.

3. Symmetry - If risk A and risk B yield the same contribution to capital when added to any disjoint subportfolio, their allocations must coincide.

4. Riskless allocation - a deterministic risk receives zero allocation in excess of its mean (see also Panjer (2002)).

Denault showed that the risk measure must necessarily be linear in order for a coherent allocation to exist. This result essentially echoes the findings of Merton and Perold (1993), but shows that allocation based on a linear risk measure constitutes an exception to their indictment of using allocations to evaluate inframarginal or supramarginal changes to a portfolio. Linear risk measures are obviously of limited application, but Denault (2001) found more useful results when analyzing *marginal* changes in the portfolio. In particular, he used five axioms to define a "fuzzy" coherent allocation principle that exists for any given coherent, differentiable risk measure—and this allocation is given by the Euler principle applied to the supplied risk measure.

Kalkbrener (2005) used a different set of axioms:

1. Linear aggregation - which combines axioms 1 and 4 of Denault

2. Diversification - which corresponds to axiom 2 of Denault

3. Continuity - Small changes to the portfolio should only have a small effect on the capital allocated to a subportfolio.

He finds that the unique allocation under these axioms is given by the Gâteaux derivative in the direction of the subportfolio, which again collapses to the Euler allocation:

$$k^{(i)} = \lim_{\varepsilon \to 0} \frac{\rho(I + \varepsilon\, L^{(i)}) - \rho(I)}{\varepsilon} = \frac{\partial \rho(I)}{\partial q^{(i)}}.$$

Some have approached the capital allocation problem from the perspective of game theory. Lemaire (1984) and Mango (1998) both noted the potential use of the Shapley Value, which rests on a different set of axioms, in solving allocation problems in insurance. The Shapley Value (Shapley, 1953) is a solution concept for cooperative games that assigns each player a unique share of the cost. Denault (2001) formally applied this idea to the capital allocation problem, in particular by relying on the theory of fuzzy cooperative games introduced by Aubin (1981). The key idea here is that the cost functional c of a cooperative game is defined via the risk measure ρ:

$$c(q^{(1)}, q^{(2)}, \dots, q^{(N)}) = \rho(q^{(1)}, q^{(2)}, \dots, q^{(N)})$$

The problem is then to allocate shares of this "cost" to the players, with the set of valid solutions being defined as (see also Tsanakas and Barnett (2003)):

$$\begin{aligned} C \;=\; & \Big\{(k^{(1)}, k^{(2)}, \dots, k^{(N)}) \Big| c(q^{(1)}, q^{(2)}, \dots, q^{(N)}) = \sum k^{(i)}\, q^{(i)} \\ & \qquad \& \;\; c(u) \geq \sum k^{(i)}\, u_i,\; u \in [0, q^{(1)}] \times \ldots \times [0, q^{(N)}]\Big\} \end{aligned}$$

Thus, for allocations in this set, any (fractional) subportfolio will feature an increase in aggregated per-unit costs, which connects to the usual solution concept in cooperative games requiring any solution to be robust to defections by subgroups of the players. The Aumann-Shapley solution is:

$$k^{(i)} = \frac{\partial}{\partial u_i} \int_0^1 c(\gamma\, u)\, d\gamma \Big|_{u_j = q^{(j)}\, \forall j}$$

If the risk measure is subadditive, positively homogeneous, and differentiable, the solution boils down to the Euler method when loss distributions are homogeneous.[3]

The Euler method is also recovered in a number of "economic" approaches to capital allocation, where the risk measure is either embedded as a constraint in a profit maximization problem (e.g., Meyers (2003) or Stoughton and Zechner (2007)) or embedded in the preferences of policyholders (Zanjani, 2002). In either case, the marginal cost of risk ends up being defined in part by the gradient of the risk measure. To illustrate, consider the optimization problem adapted from Bauer and Zanjani (2013a):

$$\max_{k, q^{(1)}, q^{(2)}, \dots, q^{(N)}} \underbrace{\left\{ \sum_{i=1}^{N} p^{(i)}(q^{(i)}) - V\left(\min\{I, a\}\right) - C \right\}}_{=\Pi} \tag{2.4}$$

[3]Aumann-Shapley values can also be used to cope with the problem of inhomogeneous loss distributions. In this case, Powers (2007) demonstrates that although the Euler principle will not apply, the Auman-Shapley value can be used for the risk-allocation problem. Similarly, it may offer a solution if the underlying risk measure does not satisfy the homogeneity condition. For instance, Tsanakas (2009) shows how to allocate capital with convex risk measures, although the absence of homogeneity is shown to potentially produce an incentive for infinite fragmentation of portfolios. The intuition for this rather undesirable feature are risk aggregation penalties within inhomogeneous convex risk measures.

subject to

$$\rho(q^{(1)}, q^{(2)}, \dots, q^{(N)}) \le k,$$

From the optimality conditions associated with this problem, one can obtain:

$$\frac{\partial \Pi}{\partial q^{(i)}} = \left(-\frac{\partial \Pi}{\partial k}\right) \times \frac{\partial \rho}{\partial q^{(i)}} \tag{2.5}$$

at the optimal exposures and capital level. Hence, for the optimal portfolio, the risk adjusted marginal return $\frac{\frac{\partial \Pi}{\partial q^{(i)}}}{\frac{\partial \rho}{\partial q^{(i)}}}$ for each exposure i is the same and equals the cost of a marginal unit of capital $\frac{-\partial \Pi}{\partial k}$. More to the point, the right hand side of 2.5 allocates a portion of the marginal cost of capital to the i-th risk, an allocation that is obviously equivalent to the Euler allocation. In this sense, the Euler allocation is indeed "economic," but it is important to stress that any economic content flows from the imposition of a risk measure constraint.

2.3.2 Distance-Minimizing Allocations

Not all approaches lead to the Euler principle. Laeven and Goovaerts (2004), whose work was later extended by Dhaene, Gooverts, and Kaas (2003) and Dhaene et al. (2012), derived allocations based on minimizing a measure of the deviations of losses from allocated capital. Specifically, Laeven and Goovaerts proposed solving:

$$\begin{cases} \min_{k^{(1)},^{(2)},\dots,^{(N)}} \rho\left(\sum_{i=1}^{N} \left(I^{(i)}(L^{(i)}, q^{(i)}) - q^{(i)}\, k^{(i)}\right)^{+}\right) \\ \text{s.th. } \sum_{i=1}^{N} q^{(i)}\, k^{(i)} = k \end{cases}$$

to identify an allocation, whereas Dhaene et al. (2012) considered:

$$\begin{cases} \min_{k^{(1)},^{(2)},\dots,^{(N)}} \sum_{i=1}^{N} q^{(i)} \mathbb{E}\left[\theta^{(i)} D\left(\frac{I^{(i)}(L^{(i)}, q^{(i)})}{q^{(i)}} - k^{(i)}\right)\right] \\ \text{s.th. } \sum_{i=1}^{N} q^{(i)}\, k^{(i)} = k \end{cases}$$

where D is a (distance) measure and $\theta^{(i)}$ are weighting random variables with $\mathbb{E}[\theta^{(i)}] = 1$.

In the approach by Dhaene et al. (2012), certain choices for D and $\theta^{(i)}$ can reproduce various allocation methods. For instance, for $D(x) = x^2$ and $k = \sum \mathbb{E}[\theta^{(i)} I^{(i)}]$, they arrive at so-called *weighted risk capital allocations* $k^{(i)} = \mathbb{E}[\theta^{(i)} L^{(i)}]$ studied in detail by Furman and Zitikis (2008b). Other choices lead to other allocation principles, including several that can be derived from the application of the Euler principle.

2.3.3 Allocations by Co-Measures and the RMK Algorithm

The Ruhm-Mango-Kreps (RMK) algorithm (Ruhm and Mango, 2003; Kreps, 2005) is a popular approach of capital allocation in practice, partly due to its ease of implementation.

According to Kreps (2005), it commences by defining $k = \sum_i q^{(i)}\, k^{(i)} = \mathbb{E}[I] + R$ as the *total capital* to support the company's aggregate loss I, where $\mathbb{E}[I]$ is the mean (reserve) and R is the risk load. Then the capital allocations $q^{(i)}\, k^{(i)}$ for risks i emanating from the asset or the liability side are defined as:

$$\begin{aligned} q^{(i)}\, k^{(i)} &= \mathbb{E}[I^{(i)}] + R_i \\ &= \mathbb{E}\left[I^{(i)}\right] + \mathbb{E}\left[\left(I^{(i)} - \mathbb{E}[I^{(i)}]\right)\, \phi(I)\right], \end{aligned} \tag{2.6}$$

where ϕ is the *riskiness leverage*, and "all" that one needs to do is to find the appropriate form of ϕ. This allocation method adds up by definition, it scales with a currency change if $\phi(\lambda\, x) = \phi(x)$ for a positive constant λ.

Different interpretations are possible, but key advantage ease of implementation since it solely relies on taking "weighted averages" (Ruhm, 2013):

Algorithm 2.3.1. *RMK Algorithm*

- *Simulate possible outcomes by component and total.*
- *Calculate expected values* $\mathbb{E}[I^{(i)}]$ *by taking simple averages.*
- *Select a risk measure on total company outcomes and express the risk measure as leverage factors.*
- *Calculate risk-adjusted expected values* $\mathbb{E}[I^{(i)}\, \phi(I)]$ *by taking "weighted averages".*
- *Allocate capital in proportion to risk, by:*[4]

$$\frac{q^{(i)}\, k^{(i)}}{k} = \frac{\mathbb{E}[I^{(i)}\, \phi(I)] + \mathbb{E}[I^{(i)}](1 - E[\phi(I)])}{\mathbb{E}[I\, \phi(I)] + \mathbb{E}[I](1 - E[\phi(I)])}.$$

Of course, the RMK algorithm only presents the general framework. The crux lies in the determination of the riskiness leverage ϕ. Various examples are presented in Kreps (2005), some of which result in familiar allocation principles that can be alternatively derived by the gradient principles.

More generally, Venter (2004) and Venter, Major, and Kreps (2006) introduce so-called *co-measures*. Specifically, consider the risk measure[5]

$$\rho(I) = \mathbb{E}\left[\sum_j h_j(I)\phi_j(I)\Big|\text{Condition on } I\right],$$

where the h are linear functions. They then define the *co-measure* as

$$r(I^{i}) = \mathbb{E}\left[\sum_{j=1}^{J} h_j(I^{(i)})\phi_j(I)\Big|\text{Condition on } I\right],$$

[4] We adjust Ruhm's formula here to be in line with the allocation above.

[5] The definition in Venter (2010) allows for different conditions for the different j.

which obviously satisfy $\sum_{i=1}^{n} r(I^{(i)}) = \rho(I)$ and thus serve as an allocation. Obviously, this allocation can also be conveniently implemented with a similar algorithm as above.

As Venter (2010) points out, even for one risk measure there may be different co-measures, i.e. the representation is not unique. Some of them yield representations that are equivalent to the gradient allocation, but this is not necessarily the case.

2.3.4 Capital Allocation by Percentile Layer

Bodoff (2009) argues that allocations according to Value at Risk or according to tail risk measures do not consider loss realizations at smaller percentiles, even though the firm's capital obviously supports these loss levels as well. Thus, in order to allocate, he advocates considering *all* loss layers *up to* the considered confidence level. His approach considers allocating capital to loss events, but since we are interested in allocating capital to lines we follow the description from Venter (2010).

Assume the capital k is determined by some given risk measure (VaR in Bodoff (2009)). Then the allocation for the layer of capital $[z, z + dz]$ is:

$$\mathbb{E}\left[\frac{I_i}{I}\big| I \geq z\right] \times dz.$$

Going over all layers of capital, we obtain the allocation:

$$q^{(i)}\, k_i = \int_0^k \mathbb{E}\left[\frac{I_i}{I}\big| I \geq z\right]\, dz,$$

where obviously:

$$\sum_{i=1}^{N} q^{(i)}\, k_i = \int_0^k \mathbb{E}\left[\frac{I}{I}\big| I \geq z\right]\, dz = \int_0^k dz = k.$$

As Venter (2010) points out, even if k is set equal to a risk measure and allowed to change with the volume of the writings, the resulting allocation does not collapse to the gradient allocation in any known cases.

When implementing the approach based on a sample of size N, obviously it is necessary to approximate the integral formulation above. When we base it on VaR_α and use the simple empirical quantile for its estimation, we can set:

$$q^{(i)}\, k_i = \sum_{j=1}^{\alpha N} \mathbb{E}\left[\frac{I_i}{I}\big| I \geq I_{(j)}\right]\, \left[I_{(j)} - I_{(j-1)}\right],$$

where we set $I_{(0)} = 0$. Since the conditional expectations within the sum have to be also approximated by taking averages, the implementation in a spreadsheet may be cumbersome (or even infeasible) for large samples.

2.4 Capital Allocation in Practice

In this section, we review several specific examples of capital allocation. Most of the examples are tied to a specific risk measure. In particular, the first four examples directly result from an application of the Euler principle to Standard Deviation, Value-at-Risk, Expected Shortfall, and an exponential risk measure.

An overview of all the considered allocation methods is presented in Figure 2.1

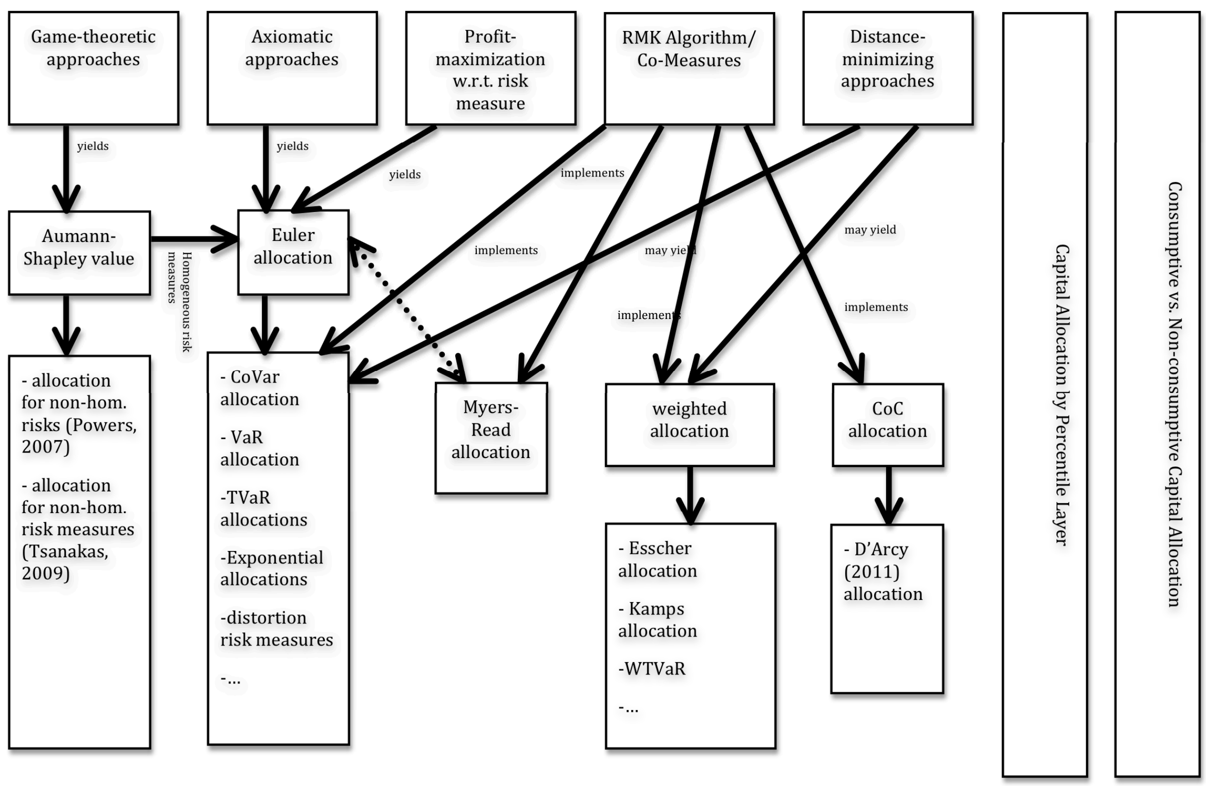


Figure 2.1: Overview of capital allocation methods

2.4.1 Covariance-type Allocations

The most basic risk measure of course are *Standard-deviation* and its square, *Variance*. In particular, standard deviation is a homogeneous risk measure so that it possible to immediately apply the Euler principle. However, it is common to more generally consider a risk measure of the following form, which directly derives from the so-called *standard deviation premium principle* (Deprez and Gerber, 1985):

$$\rho(I) = \mathbb{E}[I] + \beta \text{StDev}[I] = \mathbb{E}[I] + \beta\sqrt{\text{Var}[I]}.$$

Applying the Euler principle (that is, taking the derivative), we obtain:

$$k^{(i)} = \mathbb{E}[L^{(i)}] + \beta \frac{\mathbb{E}[(L^{(i)} - \mathbb{E}[L^{(i)}]) \times (I - \mathbb{E}[I])]}{\sqrt{\text{Var}[I]}} = \mathbb{E}[L^{(i)}] + \beta \frac{\text{Cov}(L^{(i)}, I)}{\sqrt{\text{Var}[I]}},$$

which obviously "adds up":

$$\sum_{i=1}^{N} q^{(i)}\, k^{(i)} = \sum_{i=1}^{N} \mathbb{E}[I^{(i)}] + \beta \frac{\text{Cov}(I^{(i)}, I)}{\sqrt{\text{Var}[I]}} = \rho(I).$$

Of course, it is possible to immediately rely on this equation to determine allocations. Alternatively, we can always rely on the approximation from Equation (2.3) or use the RMK algorithm; specifically, here we can set:

$$\phi(I) = \beta \frac{I - \mathbb{E}[I]}{\text{StDev}(I)}. \tag{2.7}$$

to yield the allocation above.

2.4.2 Value-at-Risk Allocation

The most common risk measure in financial application is Value-at-Risk at the confidence level $\alpha \in (0, 1)$, defined as the α-quantile of the aggregate loss distribution (McNeil et al., 2005):

$$\text{VaR}_\alpha(I) = \inf\{x \in \mathbb{R} : \mathbb{P}(I \geq x) \leq 1 - \alpha\} = \inf\{x \in \mathbb{R} : F_I(x) \geq \alpha\},$$

where F_I is the cumulative distribution function of the aggregate loss I. Since VaR is homogeneous, we can apply the Euler principle. We obtain (Gourieroux, Laurent and Scailet, 2000):

$$k^{(i)} = \mathbb{E}\left[L^{(i)} \middle| I = \text{VaR}_\alpha(I)\right].$$

Again, we could rely on the approximation from Equation (2.3) or use the RMK algorithm with

$$\phi(I) = \frac{\delta(I - \text{VaR}_\alpha(I))}{f_I(\text{VaR}_\alpha(I))}, \tag{2.8}$$

where $\delta(\cdot)$ is the Dirac Delta function to derive that allocation. However, as pointed out by Kalkbrener (2005) approximating the directional derivative in (2.3) will be highly unstable, and of course weighting one point by infinity as suggested by the dirac delta function is not feasible. Instead, in our numerical approximations, we rely on a Bell-curve around the required quantile.

2.4.3 TVaR-Based Allocations

One of the most widespread allocation principles in practice are based on *Expected Shortfall* or *Tail-Value-at-Risk* at the confidence level α:[6]

$$\mathrm{ES}_\alpha(I) = \mathrm{TVaR}_\alpha(I) = \mathbb{E}\left[I \middle| I \geq \mathrm{VaR}_\alpha(I)\right] = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_u(I)\,du.$$

Applying Euler, we obtain:

$$k^{(i)} = \mathbb{E}\left[L^{(i)} \middle| I \geq \mathrm{VaR}_\alpha(I)\right],$$

which can again be estimated via approximation (2.3) or via the RMK algorithm with:

$$\phi(I) = \frac{1_{\{I \geq \mathrm{VaR}_\alpha(I)\}}}{1-\alpha}. \tag{2.9}$$

One criticism of these allocations is that they only consider extreme tail losses, so one approach is to consider a weighted sum of these allocations at difference confidence levels. This is similar in spirit to the allocation by percentile layer as suggested by Bodoff (2009) (see Sec. 2.3.4).

There are various examples of allocations that derive from the ES. For instance, we may consider a combination of ES and conditional Standard Deviation as a risk measure in the spirit of Section 2.4.1 above:

$$\rho(I) = \mathbb{E}[I|I \geq \mathrm{VaR}_\alpha(I)] + \beta \mathrm{StDev}[I|\mathrm{VaR}_\alpha(I)],$$

which is *risk-adjusted TVaR* (RTVaR) as introduced by Furman and Landsman (2006) under a different name (Venter, 2010). An application of Euler yields:

$$q^{(i)}\,k^{(i)} = \mathbb{E}\left[I_i|I \geq \mathrm{VaR}_\alpha(I)\right] + \beta\,\frac{\mathrm{Cov}\left(I_i, I|I \geq \mathrm{VaR}_\alpha(I)\right)}{\mathrm{StDev}\left(I|I \geq \mathrm{VaR}_\alpha(I)\right)}, \tag{2.10}$$

and it can be implanted as a co-measure by choosing $J = 2$, $h_1(x) = h_2(x) = x$, $\phi_1 \equiv 1$, $\phi_2 = \beta\frac{I-\mathbb{E}[I]}{\mathrm{StDev}(I|I\geq \mathrm{VaR}_\alpha(I))}$ with condition $I \geq \mathrm{VaR}_\alpha(I)$.

Another alternative is to calculate the TVaR under a transformed distribution yielding what is referred to as *weighted TVaR* (see Section 2.4.5).

2.4.4 Exponential Allocations

A less conventional risk measure that nonetheless satisfies the homogeneity property and this allows for an application of the Euler principle is the so-called *exponential risk measure*:

$$\rho(I) = \mathbb{E}\left[I \exp\left\{\frac{c\,I}{\mathbb{E}[I]}\right\}\right].$$

[6] Aside from minor subtleties in the case of non-continuous distributions, the *Expected Shortfall* is identical to the *Tail-Value-at-Risk* (TVaR) or *Conditional Tail Expectation* (CTE). We will treat them as synonyms for the purpose of this report.

As indicated by Venter, Major, and Kreps (2006), one may be tempted to define *weighted allocations* as considered in Section 2.4.5 below via:

$$k^{(i)} = \mathbb{E}\left[L^{(i)} \exp\left\{\frac{c\,I}{\mathbb{E}[I]}\right\}\right],$$

which again can be immediately implemented via the RMK algorithm. However, an application of Euler yields the slightly more intricate form:

$$k^{(i)} = \frac{\partial\rho(I)}{\partial q^{(i)}} = \mathbb{E}\left[L^{(i)} \exp\left\{\frac{c\,I}{\mathbb{E}[I]}\right\}\right] + c\frac{\mathbb{E}[L^{(j)}]}{\mathbb{E}[I]} \times \mathbb{E}\left[I\,\exp\left\{\frac{c\,I}{\mathbb{E}[I]}\right\} \times \left(\frac{L^{(i)}}{\mathbb{E}[L^{(i)}]} - \frac{I}{\mathbb{E}[I]}\right)\right]. \tag{2.11}$$

Nonetheless, it can be implemented as a co-measure by choosing $J = 2$, $h_1(I) = I$, $\phi_1(I) = e^{\frac{cI}{\mathbb{E}[I]}} + c\frac{Ie^{\frac{cI}{\mathbb{E}[I]}}}{\mathbb{E}[I]}$, $h_2(I) = -\mathbb{E}[I]$, and $\phi_2(I) = \frac{c\,I^2\,e^{\frac{cI}{\mathbb{E}[I]}}}{\mathbb{E}[I^2]}$.

2.4.5 Allocation based on Distortion Risk Measures

Distortion risk measures (Denneberg, 1990; Wang, 1996) are defined as

$$\rho(I) = \int_0^\infty g(\bar{F}_I(x))\,dx,$$

where $\bar{F}_I(x) = \mathbb{P}(I > x)$ is the survival function and $g : [0,1] \mapsto [0,1]$ is an increasing and concave *distortion function* with $g(0) = 0$ and $g(1) = 1$. Distortion risk measures are coherent and, therefore, positive homogeneous so that we can derive allocations via the Euler principle.

Distortion risk measures and allocations on their basis have been studied by Tsanakas and Barnett (2003); Tsanakas (2004, 2008). In particular, by integration-by-parts we obtain,

$$\rho(I) = \mathbb{E}\left[I\,g'(\bar{F}_I\,(I))\right],$$

which takes the form of a *spectral risk measure* (Acerbi, 2002). Tsanakas and Barnett (2003) show that an application of the Euler principle then yields:

$$k^{(i)} = \ = \mathbb{E}\left[L^{(i)}\,g'(\bar{F}_I\,(I))\right],$$

which are indeed in the class of so-called *weighted allocations* (Furman and Zitikis, 2008b) considered below.

Popular choices for g are the *proportional hazards transform* $g(p) = p^a$ or the *Wang transform* $g(p) = 1 - \Phi\left(\Phi^{-1}(1-p) - \lambda\right)$ (Wang, 2000), where Φ denotes the cumulative distribution function of the standard Normal disribution and λ is a parameter that is often associated with a "risk premium." In particular, the Wang transform falls in the class of *complete* and *adapted* risk measures as defined by Balbas, Garrido, and Mayoral (2009). In case of the Wang transform, we obtain:

$$\begin{aligned} g'(p) &= \varphi\left(\Phi^{-1}(1-p)-\lambda\right) \times \frac{1}{\varphi\left(\Phi^{-1}(1-p)\right)} \\ &= \frac{\varphi\left(\Phi^{-1}(1-p)-\lambda\right)}{\varphi\left(\Phi^{-1}(1-p)-\lambda\right)} \\ &= \exp\left\{\Phi^{-1}(1-p)\,\lambda - \frac{1}{2}\,\lambda^2\right\}. \end{aligned}$$

For implementation in a Monte Carlo setting, we may approximate $\bar{F}_I\,(I)$ by its empirical counterpart.

2.4.6 Myers-Read Approach

Myers and Read (2001) argue that, given complete markets, default risk can be measured by the default value, i.e. the premium the insurer would have to pay for guaranteeing its losses in the case of a default. They then propose that "sensible" regulation will require companies to maintain the same default value per dollar of liabilities and effectively choose this latter ratio as their risk measure.

More precisely, following Mildenhall (2004), the default value can be written as:[7]

$$\begin{aligned} & D(q^{(1)}, q^{(2)}, \ldots, q^{(N)}) \\ = \;& \mathbb{E}\left[\mathit{1}_{\{I \geq a\}}(I-a)\right] \\ = \;& \mathbb{E}\left[\mathit{1}_{\{I \geq \mathbb{E}[I]+k^{(1)}\,q^{(1)}+\ldots+k^{(N)}\,q^{(N)}\}}\left(I - [\mathbb{E}[I] + k^{(1)}\,q^{(1)} + \ldots + k^{(N)}\,q^{(1)}]\right)\right], \end{aligned}$$

and the company's default-to-liability ratio is:

$$c = \frac{D}{\mathbb{E}[I]} = \frac{\mathbb{E}[\mathit{1}_{\{I \geq a\}}(I-a)]}{\mathbb{E}[I]}.$$

Myers and Read (2001) verify the "adding up" property for D—which again shows a relationship to the Euler principle. They continue to demonstrate that in order for the default value to remain the same as an exposure is expanded, it is necessary that:

$$c = \frac{\frac{\partial D}{\partial q^{(i)}}}{\mathbb{E}[L^{(i)}]},$$

which in turn yields:

$$\begin{aligned} c\,\mathbb{E}[L^{(i)}] &= \mathbb{E}\left[\left(L^{(i)} - \left(\mathbb{E}[L^{(i)}] + k^{(i)}\right)\right)\, \mathit{1}_{\{I \geq a\}}\right] \\ \Rightarrow k^{(i)} &= \mathbb{E}\left[\left(L^{(i)} - \mathbb{E}[L^{(i)}]\right)\big| I \geq a\right] - c\,\frac{\mathbb{E}[L^{(i)}]}{\mathbb{P}(I \geq a)}. \end{aligned}$$

[7] In contrast to Myers and Read (2001), we ignore the asset side and possible adjustments in calculating the "option value".

This is similar to the allocation found by Venter, Major, and Kreps (2006), although they allocate assets rather than capital so $\mathbb{E}[L^{(i)}]$ does not occur in the first term. As indicated in their paper, it is possible to represent this allocation as a co-measure using $J = 2$, $h_1(I) = I - \mathbb{E}[I]$, $\phi_1(I) = 1_{\{I \geq a\}}$, $h_2(I) = I$, and $\phi_2(I) = -\frac{c}{\Pr[I \geq a]}$.

2.4.7 Transformed Distributions and Weighted Allocations

Weighted allocations are studied in (Furman and Zitikis, 2008b), and they arise from so-called *weighted* distributions, which are a particular type of transformed distributions (Furman and Zitikis, 2008a). They are defined via:

$$k^{(i)} = \frac{\mathbb{E}\left[L^{(i)} w(I)\right]}{E\left[w(I)\right]}$$

where $w : [0, \infty) \to [0, \infty)$ is a non-decreasing weight function. They naturally "add up" to the risk measure:

$$\rho(I) = \frac{\mathbb{E}\left[I w(I)\right]}{E\left[w(I)\right]}$$

Thus, this class is again imlpementable by (and indeed closely related to) co-measures.

Examples include the allocations based on the Esscher transform

$$k^{(i)} = \frac{\mathbb{E}\left[L^{(i)} e^{tI}\right]}{E\left[e^{tI}\right]}$$

or allocattions based on the premium principle by Kamps (1998):

$$k^{(i)} = \frac{\mathbb{E}\left[L^{(i)}(1 - e^{-tI})\right]}{E\left[(1 - e^{-tI})\right]}.$$

As pointed out by Venter (2010), such transformed distributions can be applied to other statistics than the mean. For instance, the ES (TVaR) under a transformed distribution is called *weighted TVaR* (WTVaR). Since it is no longer linear in the loss, one can choose a lower threshold thus addressing that the ES only depends on "extreme" scenarios.

2.4.8 D'Arcy (2011) Allocation

D'Arcy (2011) considers allocations by the RMK algorithm but identifies the flexibility in choosing the riskiness leverage as its "greatest flaw." To uniquely identify the "right" function, he proposes to use capital market concepts, particulatly cost-of-capital to "reflect the actual cost of recapitalizing the firm."[8] Specifically, he allows the riskiness leverage to

[8] This also serves as motivation for model developed in Section 4.1 although there we account for capital costs explicitly in a multi-period setting.

depend both on the size of the loss realization as well as on the type of shock leading to the loss (idiosyncratic, industry-wide, or systemic). The riskiness leverage factor is the ratio of the cost of capital divided by the normal cost of capital, where the "realized" cost of capital, in addition to systemic factors, depends additively on the ratio of aggregate losses to the firm's actual capital:

$$\phi(I) = \frac{1_{\{I \geq C\}}(\mathrm{CoC}_{\mathrm{market}} + \frac{I-a}{a})}{\mathrm{CoC}_{\mathrm{normal}}}. \tag{2.12}$$

It is important to note that D'Arcy (2011) only proposes the RMK algorithm for the "consumptive" aspect of capital allocation, whereas he also includes a "non-consumptive" allocation in the spirit of Mango (2005) (see Section 2.4.9).

2.4.9 Consumptive vs. Non-consumptive Capital

Mango (2005) argues that capital costs consist of two parts: On the one hand, an insurer's capital stock can be depleted if a loss realization exceeds the reserves for a certain segment or line, or when reserves are increased. He refers to this as a *consumptive* use of capital since in this case, funds are transferred from the (shared) capital account to the (segment-specific) reserve account. The second, *non-consumptive* component arises from a "capacity occupation cost" that compensates the firm for preclusion of other opportunities. It is thought to originate from rating agency requirements in the sense that taking on a certain liability depletes the underwriting capacity.

The importance of this distinction for our purposes is that it complicates practice in cases where the two sources of costs require different approaches to allocation. For example, D'Arcy (2011) follows Mango's suggestion by first allocating consumptive capital via the RMK algorithm, where the riskiness leverage or *capital call cost factor* ϕ is associated with the cost of capital (see also Bear (2006) and D'Arcy (2011)). He then allocates capital according to regulatory rules, and the final allocation ends up as an average of the two allocations. Thus, the two different motivations for holding capital are reflected in a hybridization of allocation methods.

2.4.10 Some Connections between the Allocations

There are several connections between the various allocation methods beyond what has been pointed out so far in this section. We list them here:

- For elliptical distributions, the Euler allocation yields to the same relative amounts of capital allocated to each line, irrespective of which (homogeneus) risk measure we use (McNeil et al., 2005, Corollary 6.27).

- Asimit, Furman, Tang, and Vernic (2012) show that risk capital allocation based on TVaR is asymptotically proportional to the corresponding Value-at-Risk (VaR) risk measure as the confidence level goes to 1.

2.5 Discussion

As laid out in Section 2.1, the practice of capital allocation serves a clear purpose: optimal risk pricing and measuring the performance of different lines. Hence, many of the conceptual approaches presented in Section 2.3—though possibly mathematical appealing—miss the point. From the perspective of the insurer, capital should be allocated as to maximize profits, and, as noted at the end of Section 2.3.1, the Euler method can claim to meet this standard under some circumstances. This has already been acknowledged by (Venter, Major, and Kreps, 2006), who point out that a marginal allocation appears to be the only method that is in line with profit maximization.

Even the Euler method, however, suffers from dependence on the choice of risk measure—a dependence shared by most capital allocation methods. The Euler method delivers the correct marginal cost if and only if we have identified the appropriate risk measure. As shown by Gründl and Schmeiser (2007), a risk measure constraint, even a seemingly sensible one, will not help a firm improve profitability. This was also pointed out by Venter (2010) he hints that "The allocation of risk measures used to quantify capital risks does not necessarily capture the value of the risk for risk transfer purposes. [...] If risk attribution is used to compare profits of insurer business units, the risk attributed to the business unit should in some way reflect the value of the risk the unit is taking."

In some cases, regulators may dictate the use of a particular risk measure, a situation that would seem to fit the setup of Equation 2.4 neatly. Unfortunately, in reality, regulation is only part of the story, as insurers typically hold far more capital than is required by the regulator (Hanif et al., 2010). Any serious consideration of the constraints facing insurers must take account of market influences, such as those dictated by rating agencies, policyholders, and counterparties of various kinds, and complexity builds as more realism is introduced. As illustrated by Bauer and Zanjani (2013a), even a relatively simple one period model of an insurance company yields a fairly complex picture of risk, and, therefore, a relatively complex risk measure is the appropriate one.

As we will illustrate in Chapters 4 and 5, the problem becomes even more acute when multiple periods are introduced. In such cases, the carrying cost of capital on the balance sheet is no longer a sufficient statistic for the cost of risk. Thus, pricing using traditional capital allocation methods yields an inaccurate representation of the cost of risk. In this more complex situation, we must broaden our notion of "capital" and take care in estimating its cost when pricing risk.

Chapter 3

Stability of Allocation Methods

In this chapter, we analyze the allocation problems and methods discussed in the previous chapters in the context of an example application. We gained access to (scaled) simulated loss data for a global catastrophe reinsurance company. We believe this data offers a degree of realism missing from previous contributions where proposed allocation methods are only studied in the context of stylized examples or based on Normal distributions (which is particularly limiting as discussed in Section 2.4.10).

We begin by describing in detail the data and approach to aggregation in Section 3.1. In particular, for the majority of the analyses we limit the presentation to an aggregation to four lines only in order to facilitate interpretation of the results. In Section 3.2, we compare various conventional allocation methods introduced in Chapter 2, where we also study their robustness by running sensitivity analyses. The subsequent Chapter 5 then studies "optimal" allocations in the context of the model from Chapter 4 and discuss the outcomes.

3.1 Description of the Data

We have 50,000 joint loss realizations for 24 distinct lines differing by peril and geographical region. Figure 3.1 provides a histogram of the aggregate loss distribution, and Table 3.1 lists the lines and provides some descriptive statistics about each line. The largest lines for our reinsurer (by premiums and expected losses) are "US Hurricane", "N American EQ West" (North American Earthquake West), and "ExTropical Cyclone" (Extratropical Cyclone). The expected aggregate loss is \$187,819,998 with a standard deviation of \$162,901,154, and the aggregate premium income is \$346,137,808.

We consider three different levels of aggregation: The first aggregation level that considers all line separately with line numbers listed in Column "Agg1" of Table 3.1. The second level has nine lines, where we aggregate business lines in the same peril category by different geographical region. The line numbers are listed in Column "Agg2." The third level considers an aggregation to four lines, with line numbers listed in Column "Agg3." Here, where we lump together all lines by perils. In particular, we can think of Line 1 as

Line	Statistics			Aggs		
	Premiums	Expected Loss	Standard Deviation	Agg1	Agg2	Agg3
N American EQ East	6,824,790.67	4,175,221.76	26,321,685.65	1	1	1
N American EQ West	31,222,440.54	13,927,357.33	47,198,747.52	2	2	1
S American EQ	471,810.50	215,642.22	915,540.16	3	2	1
Australia EQ	1,861,157.54	1,712,765.11	13,637,692.79	4	3	1
Europe EQ	2,198,888.30	1,729,224.02	5,947,164.14	5	3	1
Israel EQ	642,476.65	270,557.81	3,234,795.57	6	3	1
NZ EQ	2,901,010.54	1,111,430.78	9,860,005.28	7	3	1
Turkey EQ	214,089.04	203,495.77	1,505,019.84	8	3	1
N Amer. Severe Storm	16,988,195.98	13,879,861.84	15,742,997.51	9	4	2
US Hurricane	186,124,742.31	94,652,100.36	131,791,737.41	10	4	2
US Winterstorm	2,144,034.55	1,967,700.56	2,611,669.54	11	4	2
Australia Storm	124,632.81	88,108.80	622,194.10	12	5	2
Europe Flood	536,507.77	598,660.08	2,092,739.85	13	5	2
ExTropical Cyclone	37,033,667.38	23,602,490.43	65,121,405.35	14	5	2
UK Flood	377,922.95	252,833.64	2,221,965.76	15	5	2
US Brushfire	12,526,132.95	8,772,497.86	24,016,196.20	16	6	3
Australian Terror	2,945,767.58	1,729,874.98	11,829,262.37	17	7	4
CBNR Only	1,995,606.55	891,617.77	2,453,327.70	18	7	4
Cert. Terrorism xCBNR	3,961,059.67	2,099,602.62	2,975,452.18	19	7	4
Domestic Macro TR	648,938.81	374,808.73	1,316,650.55	20	7	4
Europe Terror	4,512,221.99	2,431,694.65	8,859,402.41	21	7	4
Non Certified Terror	2,669,239.84	624,652.88	1,138,937.44	22	7	4
Casualty	5,745,278.75	2,622,161.64	1,651,774.25	23	8	4
N American Crop	21,467,194.16	9,885,636.27	18,869,901.33	24	9	3

Table 3.1: Descriptive Statistics

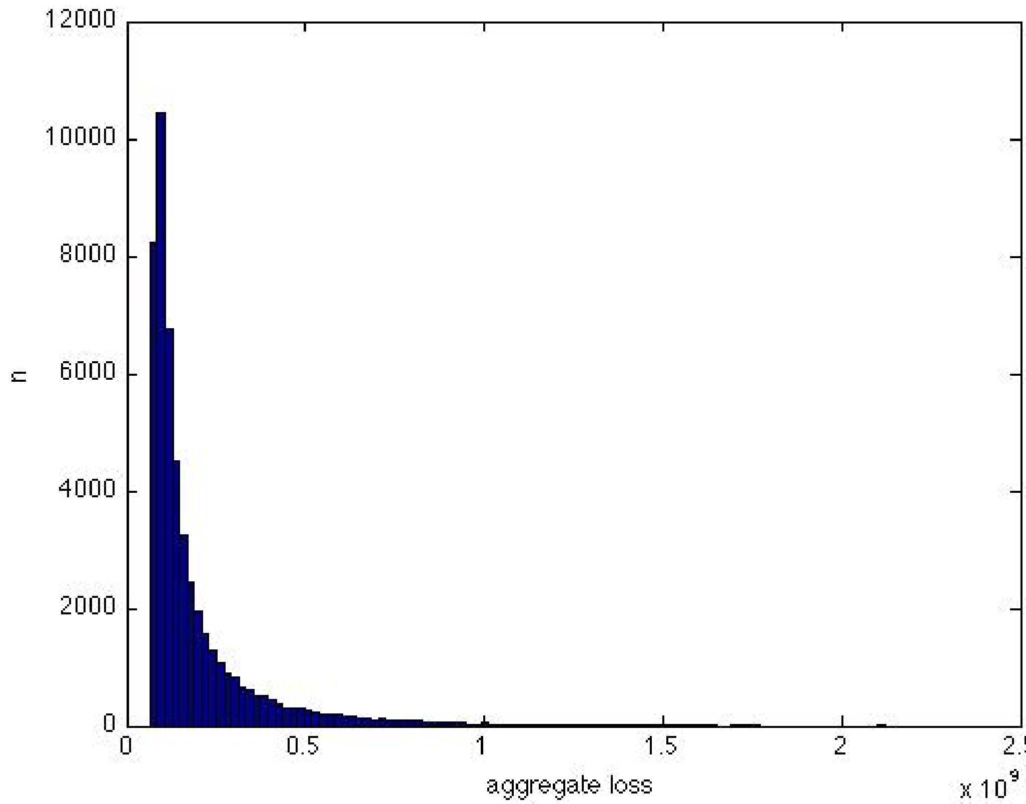


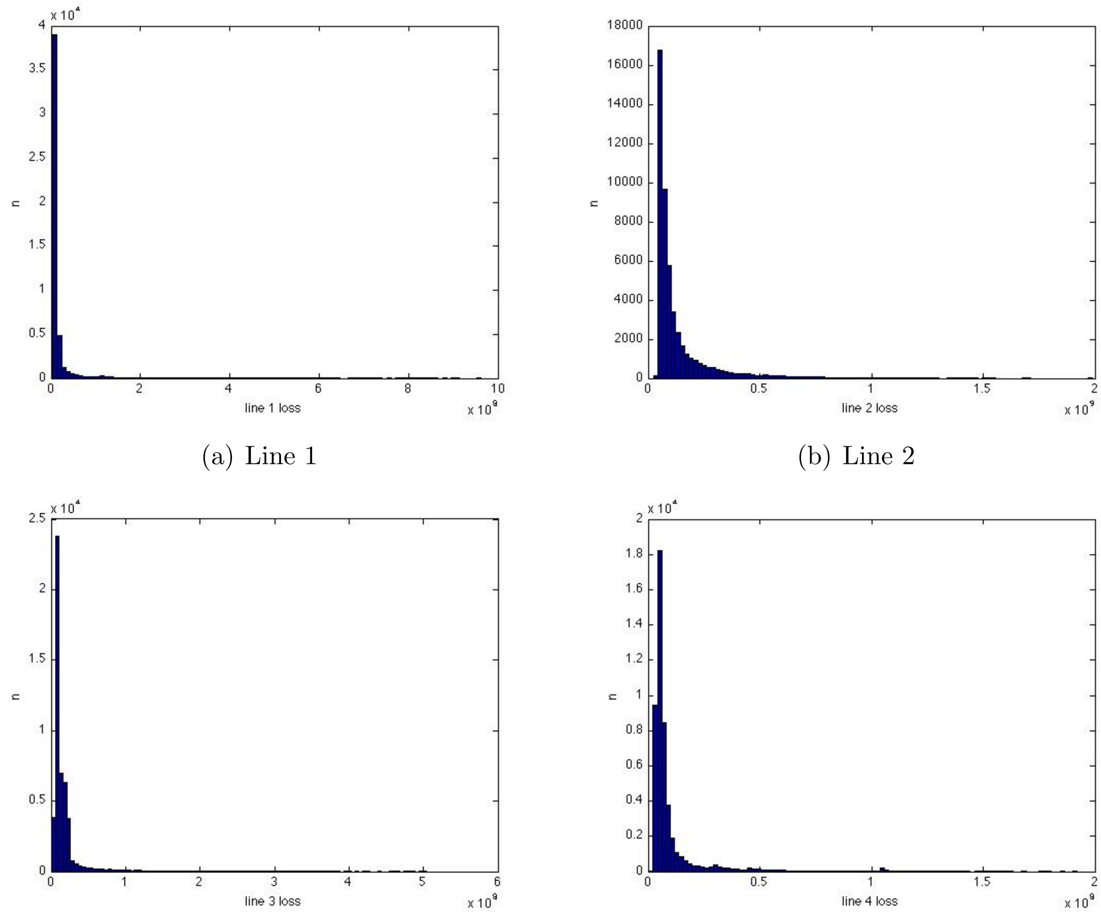
Figure 3.1: Histograms for Aggregate Loss

"earthquake," Line 2 as "storm and flood," Line 3 as "fire & crop," and Line 4 as "terror & casualty." In the text, in order to keep the results comprehensible, for many analyses we limit the exposition to the four-line aggregation level. Figure 3.2 shows histograms for each of these four lines.

We notice that the "earthquake" distribution is concentrated at low loss levels with only relatively few realizations exceeding \$50,000,000 (the 99% VaR only slightly exceeds \$300,000,000). However, the distribution depicts relatively fat tails with a maximum loss realization of only slightly under one billion. The (aggregated) premium for this line is \$46,336,664 with an expected loss of \$23,345,695.

"Storm & flood" is by far the largest line, both in terms of premiums (\$243,329,704) and expected losses (\$135,041,756). The distribution is concentrated around loss realizations between 25 and 500 million, though the maximum loss in our 50,000 realizations is almost four times that size. The 99% VaR is approximately 700 million USD.

In comparison, the "fire & crop" and "terror & casualty" lines are small with an (aggregated) premiums (expected loss) of about 34 (19) million and 22.5 (11) million, respectively. The maximal realizations are around 500 million for "fire & crop" (99% VaR = 163,922,557) and around 190 million for "terror & casualty" (99% VaR = 103,308,358).



(a) Line 1 (b) Line 2

(c) Line 3 (d) Line 4

Figure 3.2: Histograms for Aggregation Level 3 (Agg 3) Lines

3.2 Comparison of Conventional Allocation Methods

We consider most allocation techniques introduced in Section 2.3 and 2.4:

- Allocation by expected values.
- A covariance allocation (Section 2.4.1). Here we choose the parameter $\beta = 2$ due to the similarities of the supporting risk measure to a quantile for a Normal distribution, where 2 (or rather 1.96 for a two-sided confidence interval of 95%) is a common choice.
- TVaR (Expected Shortfall) allocations for confidence levels $\alpha = 75\%$, 90%, 95%, and 99% (Section 2.4.3).

- VaR based allocations for confidence levels $\alpha = 95\%$ and 99% (Section 2.4.2). Here in addition to estimating the allocations based on splitting up the corresponding empirical quantile in its loss components (labeled "simple"), we consider an estimation that takes into account the surrounding realizations by imposing a bell curve centered at the quantile with a standard deviation of three (labeled "bell").

- Exponential allocations for parameters $c = 0.1$, 0.25, and 1 (Section 2.4.4).

- Allocations based on a distortion risk measure–particularly allocations based on the Wang transform (Section 2.4.5). For the transform parameters, we use $\lambda = 0.25$, $\lambda = 0.5$, $\lambda = 0.75$, where we follow Wang (2012) indicating that typical transform parameters in the reinsurance domain range between 0.5 and 0.77, whereas 0.25 is a typical assumption for long-termed Sharpe ratios in the financial market.

- Myers-Read allocations for different capital levels (Section 2.4.6). In particular, we choose the capital equal to the 99.94% quantile, which roughly depends on capital levels to support an AM Best AA+ rating; three times the premium which is roughly consistent with NAIC aggregate levels; and the 99% VaR just for comparison purposes.

- Weighted/transform-based allocations based on the Esscher and Kamps transfrom (Section 2.4.7). Here we choose transform parameters such that an evaluation is possible (non-explosive) but sufficiently different from the expected value allocation (which results for $t = 0$). In particular, we use $t = 1.E - 07/1.E - 09$ for the Esscher transform and $t = 1.E - 08/1.E - 11$ for the Kamps transfrom.

- The D'Arcy (2011) implementation of the RMK algorithm (Section 2.4.8), where we rely two on the same (first) two capital levels as for the Myers-Read allocation.

- Allocations based on the percentile layer (Section 2.3.4), where we allocate the 90%, 95%, and 99% VaR.

- RTVaR allocations with $\alpha = 75\%, 90\%$, and 95% and $\beta = 2$ as for the covariance allocation.

- Allocation on the (simple) average of the four considered TVaRs.

3.2.1 Comparisons for the Unmodified Portfolio

Table 3.2 presents allocation results for the (unmodified) portfolio of the company. Here for each risk measure, we list the capital levels for each line, their sum, as well as the risk measure evaluated for the aggregate loss distributions. Obviously, the last two numbers should coincide—which can serve as a simple check for the calculations.

Obviously these aggregate risk measures vary tremendously, and thus so do the by-line allocations. For instance, it is trivial that the 99% quantile (VaR) is far greater than the

Allocation	Line 1	Line 2	Line 3	Line 4	Sum	RiskMeas
ExpVal	23,345,695	135,041,756	18,658,134	10,774,413	187,819,998	187,819,998
	12.43%	71.90%	9.93%	5.74%	100%	
CovWBeta	62,292,648	398,599,625	38,938,608	13,791,425	513,622,306	513,622,306
$\beta = 2$	12.13%	77.61%	7.58%	2.69%	100%	
CovWBeta/RMK	62,292,258	398,596,990	38,938,405	13,791,395	513,619,048	513,619,048
	12.13%	77.61%	7.58%	2.69%	100%	
$\text{TVaR}_{75\%}$	51,212,126	301,490,365	34,960,054	13,571,979	401,234,524	401,234,524
	12.76%	75.14%	8.71%	3.38%	100%	
$\text{TVaR}_{90\%}$	72,975,560	462,489,152	42,432,578	12,350,749	590,248,038	590,248,038
	12.36%	78.36%	7.19%	2.09%	100%	
$\text{TVaR}_{95\%}$	85,607,782	596,451,028	44,523,108	11,979,729	738,561,647	738,561,647
	11.59%	80.76%	6.03%	1.62%	100%	
$\text{TVaR}_{99\%}$	106,293,324	869,605,928	58,168,596	11,294,985	1,045,362,832	1,045,362,832
	10.17%	83.19%	5.56%	1.08%	100%	
$\text{VaR}_{95\%}$ (simple)	9,776,274	330,811,906	173,984,700	14,118,944	528,691,824	528,691,824
	1.85%	62.57%	32.91%	2.67%	100%	
$\text{VaR}_{95\%}$ (bell)	62,367,597	404,259,189	51,869,109	10,157,414	528,653,309	528,653,309
	11.80%	76.47%	9.81%	1.92%	100%	
$\text{VaR}_{99\%}$ (simple)	7,170,815	816,870,497	39,256,266	4,060,778	867,358,356	867,358,356
	0.83%	94.18%	4.53%	0.47%	100%	
$\text{VaR}_{99\%}$ (bell)	40,819,623	780,226,792	29,835,131	16,807,649	867,689,194	867,689,194
	4.70%	89.92%	3.44%	1.94%	100%	
Exponential Alloc.	27,850,426	166,607,986	20,890,454	10,989,763	226,338,629	226,338,629
$c = 0.1$	12.30%	73.61%	9.23%	4.86%	100%	
Exponential Alloc.	37,216,272	245,941,709	23,451,637	8,755,958	315,365,576	315,365,576
$c = 0.25$	11.80%	77.99%	7.44%	2.78%	100%	
Exponential Alloc.	-3,671,904,055	23,720,252,516	-4,880,193,171	-3,252,109,822	11,916,045,468	11,916,045,468
$c = 1$	-30.81%	199.06%	-40.95%	-27.29%	100%	
Wang	27,893,622	164,362,273	21,313,685	11,452,085	225,021,665	225,021,665
$\lambda = 0.25$	12.40%	73.04%	9.47%	5.09%	100%	
Wang	33,402,026	201,407,441	24,294,793	12,004,713	271,108,972	271,108,972
$\lambda = 0.5$	12.32%	74.29%	8.96%	4.43%	100%	
Wang	39,867,730	247,043,116	27,573,810	12,410,105	326,894,761	326,894,761
$\lambda = 0.75$	12.20%	75.57%	8.44%	3.80%	100%	
MyersRead, $a =$	120,879,204	1,006,221,885	50,599,598	-7,876,720	1,169,823,967	1,169,823,967
1,357,643,965	10.33%	86.01%	4.33%	-0.67%	100%	
MyersRead, $a =$	64,285,407	756,847,100	37,746,242	-8,285,324	850,593,426	850,593,426
1,038,413,423	7.56%	88.98%	4.44%	-0.97%	100%	
MyersRead, $a =$	60,821,986	606,579,733	21,827,406	-9,690,768	679,538,358	679,538,358
867,358,356	8.95%	89.26%	3.21%	-1.43%	100%	
Esscher, $t =$	8,226,240	1,987,777,539	54,262,714	72,291,051	2,122,557,544	2,122,557,544
1.E-07	0.39%	93.65%	2.56%	3.41%	100%	
Esscher, t=	27,359,301	163,201,071	20,690,228	11,031,052	222,281,653	222,281,653
1.E-09	12.31%	73.42%	9.31%	4.96%	100%	
Kamps, t=	26,654,710	155,020,509	20,725,105	11,406,228	213,806,553	213,806,553
1.E-08	12.47%	72.51%	9.69%	5.33%	100%	
Kamps, t=	40,195,291	249,028,113	27,434,525	12,082,021	328,739,950	328,739,950
1.E-11	12.23%	75.75%	8.35%	3.68%	100%	
D'Arcy, $a =$	55,387,701	408,352,811	25,296,290	5,487,442	494,524,244	494,524,244
1,357,643,965	11.20%	82.57%	5.12%	1.11%	100%	
D'Arcy, $a =$	104,209,232	875,467,350	62,492,051	11,388,390	1,053,557,023	1,053,557,023
1,038,413,423	9.89%	83.10%	5.93%	1.08%	100%	
Bodoff,	47,772,088	273,086,001	36,572,490	18,591,385	376,021,964	376,021,964
$\text{VaR}_{90\%}$	12.70%	72.63%	9.73%	4.94%	100%	
Bodoff,	66,892,452	393,236,293	46,970,415	21,592,664	528,691,824	528,691,824
$\text{VaR}_{95\%}$	12.65%	74.38%	8.88%	4.08%	100%	
Bodoff,	104,581,664	670,427,703	66,183,982	26,165,006	867,358,356	867,358,356
$\text{VaR}_{99\%}$	12.06%	77.30%	7.63%	3.02%	100%	
RTVaR	93,544,093	654,561,402	50,055,237	11,458,019	809,618,751	809,618,751
$\alpha = 75\%,\ \beta = 2$	11.55%	80.85%	6.18%	1.42%	100%	
RTVaR	107,255,495	824,861,293	53,516,378	11,599,036	997,232,202	997,232,202
$\alpha = 90\%,\ \beta = 2$	10.76%	82.72%	5.37%	1.16%	100%	
RTVaR	114,157,530	935,144,574	61,454,489	11,804,261	1,122,560,854	1,122,560,854
$\alpha = 95\%,\ \beta = 2$	10.17%	83.30%	5.47%	1.05%	100%	
AvgTVaR	79,022,198	557,509,118	45,021,084	12,299,361	693,851,760	693,851,760
	11.39%	80.35%	6.49%	1.77%	100%	

Table 3.2: Allocations for Aggregation level 3; Basic Results

95% quantile (VaR). Thus, in the second line for each method, Table 3.2 lists the allocations as a percentage of the aggregate risk measure. These are the percentages on which we will base our comparisons. This is not only because it facilitates comparisons, but also because this is in line with practice where the actual capital of a company may not be given in terms of a risk measure at all, or even if it is this may not be the measure used for allocation.

The first observation when comparing the allocations the realizations that many of them look quite similar, which resonates with observations in other studies. For instance, in the context of assumptions used for the CAS DFA modeling challenge ("Bohra-Weist DFAIC distributions"), Vaughn (2007) points out that a variety of methods, including allocations based on "covariance, Myers/Read, RMK with Variance, Mango Capital Consumption, and XTVaR99 are all remarkably similar." We find similar results in the context of an example from life insurance (Bauer and Zanjani, 2013b). There are a few outliers, however, most notably the Exponential Allocation with $c = 1$. The reason is that here there is an extreme weight on the extreme tail—that in turn is driven by very extreme realizations of line 2. Indeed there are various realizations in the aggregate tail where the line realizations for lines 1, 4, and 5 are under the expected loss, which explains the resulting negative allocations to these lines (cf. Equation (2.11)).

For comparing the remaining allocations, we note that each allocation in our four-line context is characterized by *three*—not four—real numbers, since the fourth follows by subtracting the sum of the others from 100%. Hence, we can compare allocations as points in Euclidean space, and moreover we can evaluate the "distance" between two allocations by identifying it with the distance between the two points in terms of its Euclidean norm.

Figure 3.3 plots all of our allocations except for the aforementioned exponential allocation with $c = 1$. From Panel 3.3(a), we see that there are a few other outliers in the sense that the distance to other allocation methods is quite significant: Three value at risk allocations, namely the "simple" calculation (VaR1S, VaR2S) for both confidence levels and the bell-curve based calculation for the higher confidence level (VaR2B); and the Esscher allocation for the (high) parameter of 1E-7 (Essch1). The intuition for the latter is, again, the exponential weight pushing all relevance to the extreme tail where line 2 dominates the others. Hence, both the exponential allocation and the Esscher allocations are extremely sensitive to the choice of the parameter (although this sensitivity does not appear to apply to the Kamps allocation). For VaR, on the other hand, it is well-known that estimation based on Monte Carlo simulation is erratic (Kalkbrener, 2005)—so it may be numerical errors driving these outliers (at least for VaR1S).

Interestingly, aside from the "outlier" allocations mentioned above and two Myers-Read allocations, the points all appear to lie on a parabola-shaped curve in three-dimenional space that is suggestive of a systematic pattern. In order to zoom in on the remaining allocations, Figure 3.3(b) re-plots the same points, but this time we exclude outlying allocations as well as the two outer Myers-Read allocations. Again, the allocations seem related and we find that the expected value allocation (EV) plays an "extreme role." This may not come as a surprise since suitable allocation methods should penalize risk "more than linearly." (Venter, 2010).

A number of allocation methods are very close to the expected value allocation: The

Kamps allocations, the Wang allocations, the Bodoff allocations, and the Covariance allocation are all within 0.06 of the EV allocation. In contrast, all but one TVaR/RTVaR/AvgTVaR allocations, the D'Arcy allocations, and the Myers-Read allocations all range between 0.07 and 0.19—all roughly along the parabola-shaped curve, where the order appears to be driven by the parameters. The former methods are all driven by the entire distribution, whereas the focus of the latter allocation methods is on the tails (though the Myers-Read allocation does depend on the entire distribution).

Hence, the key observation is a dissonance between tail-based allocations and allocations that are based on the entire distribution. But which one is more appropriate? Should we, or should we not, focus on tails? (Venter, 2010) argues that from an economic stance, risk-taking is not risk free—any modification to risk taking should carry some charge, so that a focus on the tails is misguided. He supports using marginal (i.e., Euler-based) methods that are based on the entire distribution such as the Wang transform, since they are most "the most commensurate with pricing theory." However, D'Arcy (2011) and Myers and Read (2001) also present approaches with an economic motivations. Before we follow up this question in the context of "optimal allocations" from the model presented in Chapter 4, we first examine the stability of these allocation methods, which is also an important concern in practice.

3.2.2 Stability of the Methods

In this section, we study the stability of the allocation methods. In particular, we recalculate the allocations from the previous subsection for two distorted portfolios:

- *Sensitivity 1*: We eliminate 1,000 arbitrary sample realizations leaving us with 49,000 realizations.
- *Sensitivity 2*: We replace the five worst case (aggregate) scenarios with the sixth worst aggregate scenario (so that our sample now contains six identical scenarios).

Tables A.1 and A.2 in Appendix A shows the results.

The intuition behind the first stability test is clear: An allocation should be robust to unsystematic changes in the sample. When adding, changing, or subtracting from the sample in an unsystematic way, we would hope to see the allocation staying more or less the same. And since we cannot add to or change the sample because we do not know the data-generating process, we subtract.

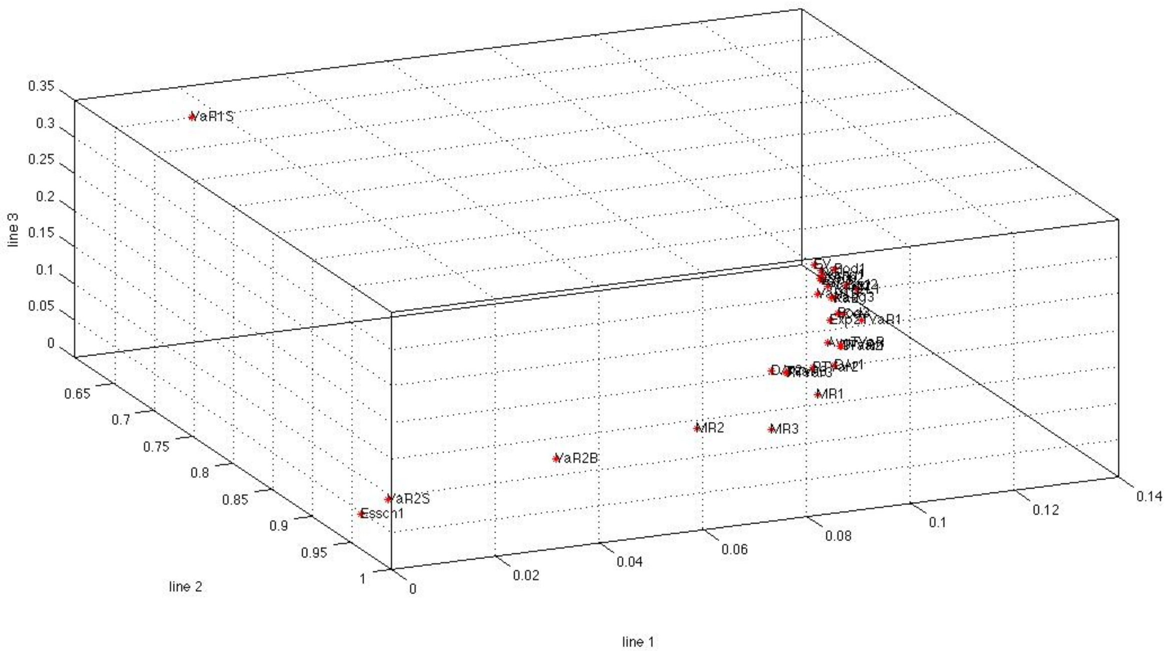
The second test is motivated by ideas from Kou, Peng, and Heyde (2012), who discuss robustness properties of risk measures and—based on the observation that *coherent* risk measures are not always robust—define so-called *natural* risk statistics. It is important to note that our angle is different in that we consider allocations and not risk measures, even though the underlying issues are the same. Specifically, extreme tail scenarios are very hard to assess—for instance, with 5,000 observations one can not distinguish between the Laplace distribution and the T-distributions (Heyde and Kou, 2004). Therefore, modifications in the extreme tail should not have a tremendous impact on the allocation.

As indicated in the previous subsection, we can identify allocations for our four business lines with points in three dimensional space, and we can identify the "difference" between two allocations with the (Euclidean) distance between the corresponding points. For a yardstick when assessing allocations, note that the difference between the 90% TVaR and the 99% TVaR is 0.056, which is thus a sizable difference. The difference between the 95% and the 99% TVaR is 0.026, which is still considerable.

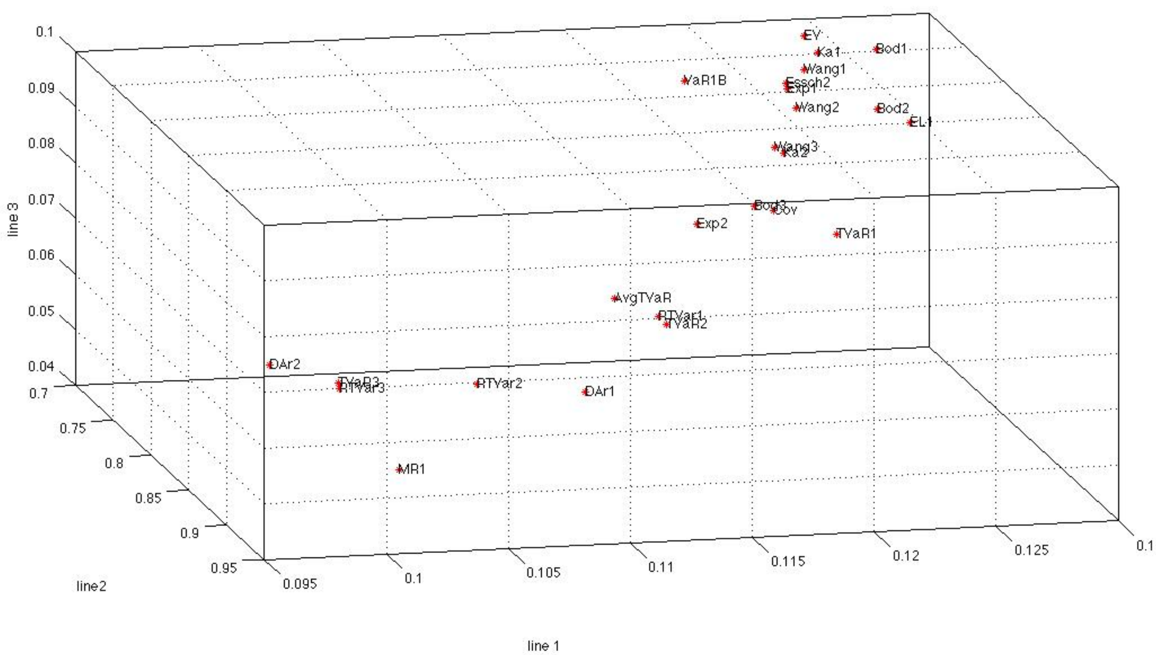
Figure 3.4 plots the distance between the allocations for the original portfolio and the modified portfolio for both sensitivity portfolios and all considered allocations methods. Again, we find that VaR-based allocations and the Exp3 allocation stand out as extreme outliers, though on different tests. More specifically, VaR allocations respond particularly poorly to unsystematic changes in the portfolio, whereas the exponential allocation is particularly sensitive to changes in the tail. This contrasts with Kou, Peng, and Heyde (2012), who argue that VaR has good robustness properties for risk measurement. We eliminate (all) VaR-based and exponential allocations and plot the differences for the remaining methods for both tests separately. Figure 3.5 displays the results.

Figure 3.5(a) shows the results for the (unsystematic) modification via eliminating 1,000 samples. We find that when ignoring VaR-based allocations, all methods are relatively stable. The maximal difference now is about 0.0025 which is not too sizable for the Myers-Read 2 allocation: the corresponding allocation vectors from Tables 3.2 and A.1 are (7.56%;88.98%;4.44%;-0.97%) and (7.78%;88.85%;4.38%;-1.01%), respectively.

In contrast, when eliminating tail scenarios, the impact can be considerable. Figure 3.5(b) shows that in some cases it can amount to more than 0.04. The most sensitive methods are the Myers-Read allocations, the D'Arcy allocations, and the Esscher allocations—all of which are "tail-focused." However, we do not find the same for TVaR based allocations, which again is contrary to the findings from Kou, Peng, and Heyde (2012) for risk measurement. Also noteworthy is the stability of the Wang, the Kamps, and the Bodoff allocations, so it appears that stability is less critical for non-tail-focused methods.



(a) All Methods (w/o Exp3)



(b) Restricted Methods

Figure 3.3: Comparisons of Allocation Methods

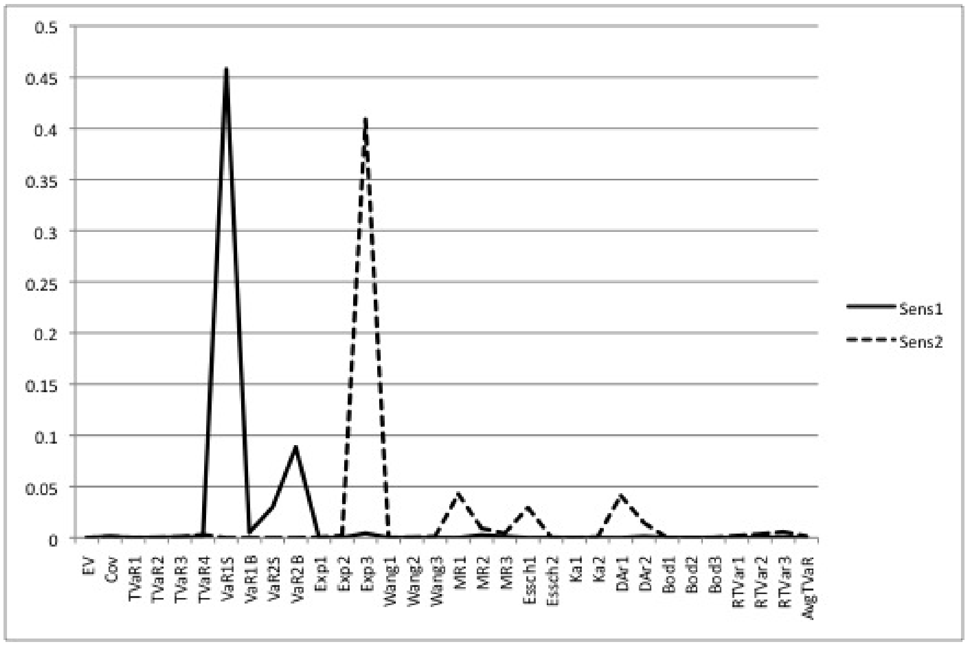
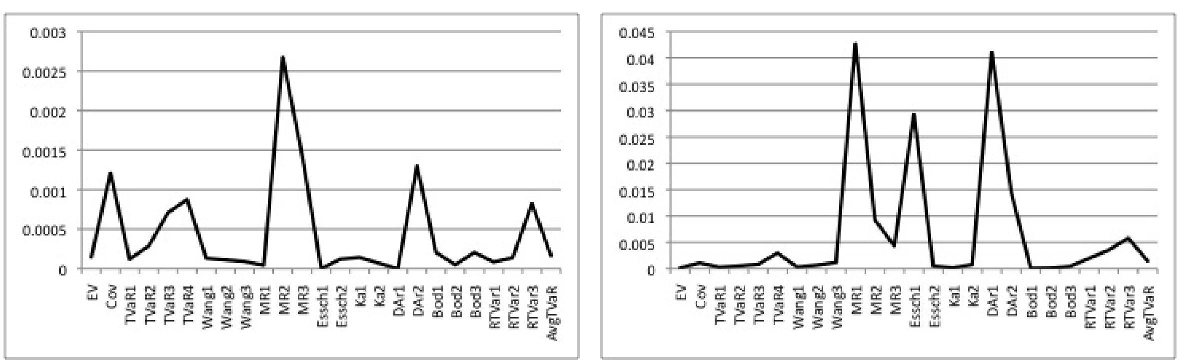


Figure 3.4: Stability of Allocations: Distance Between Allocations between Basic and Modified Portfolios for Different Methods



(a) Sensitivity Test 1

(b) Sensitivity Test 2

Figure 3.5: Stability of Allocations: Distance Between Allocations between Basic and Modified Portfolios for Different Methods

Chapter 4

Capital Allocation and Profit Maximization: A Multi-Period Model

In this chapter, we reconsider the the connection between the marginal cost of risk and the allocation of capital. As noted in Chapter 2, capital allocation can under some circumstances match up with the marginal cost of risk. However, demonstrations of this connection have generally focused on restricted settings, such as a single period. Here, we calculate the marginal cost of risk in the context of a multiperiod model of an insurer. The key question here is whether the marginal cost of risk can be reconciled *in theory* with capital allocation in a dynamic model of an insurance company.

We start off in Section 4.1 by laying out this model framework, then Section 4.2 derives the marginal cost of risk for the insurance company. Section 4.3 provides a discussion of the results, particularly their implications on capital allcations. Technical derivations and alternative model versions are collected in Appendix B. The next chapter then discusses implementation and calibration of the model for our example company.

4.1 Profit Maximization Problem in a Multi-Period Model

Formally, as in Section 2.2, we consider an insurance company with N business lines with corresponding loss realizations $L_t^{(i)}$, $i = 1, 2, \ldots, N$ each period $t = 1, 2, \ldots$ As before, these losses could be associated with certain perils, certain portfolios of contracts, or individual contracts/costumers. And again we consider the problem of allocating capital and/or costs to these N losses.

The first key difference is that we now consider a multi-period setup so that losses are indexed by line and period. We assume that for fixed i, $L_1^{(i)}$, $L_2^{(i)}, \ldots$ are non-negative, independent, and identically distributed (square-integrable) (iid) random variables on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. One may envision period-losses associated with a certain peril such as earth quakes or hurricanes in a certain region. We make the iid assumption for convenience of exposition and without much loss of generality. For instance, it is relatively straightforward to include a dynamic evolution due to (claims) inflation, e.g. by multiplying the losses with an inflation rate as in:

$$L_t^{(i)} = (1 + r_i^{\text{inf}})^t \times \tilde{L}_t^{(i)},$$

where $\left(\tilde{L}_t^{(i)}\right)_{t\geq 1}$ are iid.

For now, we abstract from investments, so that all the uncertainty is captured by the losses and we define the filtration $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$ that describes the information flow over time via $\mathcal{F}_t = \sigma(L_s^{(i)},\, i \in \{1, 2, \ldots, N\},\, s \leq t)$. However, generalizations towards a model with securities markets are possible (Bauer and Zanjani, 2013a).

Now at the beginning of every (underwriting) period t, the insurer chooses to underwrite certain portions of these risks and charges premium $p_t^{(i)}$, $1 \leq i \leq N$, in return. More precisely, as within the one-period model in Section 2.2, the underwriting decision corresponds to choosing an indemnity parameter $q_t^{(i)} \in \Phi_t^{(i)}$, where $\Phi_t^{(i)}$ are compact choice sets, so that the indemnity for loss i in period t is:

$$I_t^{(i)} = I_t^{(i)}(L_t^{(i)}, q_t^{(i)}),$$

where we require $I_t^{(i)}(0, q_t^{(i)}) = 0$, $i = \{1, 2, \ldots, N\}$. As before, for analytical convenience we focus on proportional insurance transactions, i.e. we assume:

$$I_t^{(i)} = I^{(i)}(L_t^{(i)}, q_t^{(i)}) = q_t^{(i)} \times L_t^{(i)}.$$

Again, generalizations are possible. We denote the aggregate period-t loss by $I_t = \sum_i I_t^{(i)}$.

The company has the possibility to raise or shed (i.e. pay dividends) capital R_t^b at the beginning of the period at cost $c_1(R_t^b)$, $c_1(x) = 0$ for $x \leq 0$. Moreover, it can raise capital R_t^e, $R_t^e \geq 0$, at the end of the every period—after losses have been realized—at a (higher) costs $c_2(R_t^e)$. Here we think of R_t^b as "regular" forms of capitalization whereas R_t^e is emergency capital. In particular, we assume

$$c_2(x) > c_1(x),\ x \geq 0, \tag{4.1}$$

i.e. raising capital under normal conditions is less costly than in distressed states. We also assume there exists a positive carrying cost for capital a_t within the company as a proportion τ of a_t. One may imagine τ to arise from cost-of-capital charges by the shareholders (sometimes referred to as the *average cost of capital*) and depreciation, whereas $c_1(\cdot)$ are costs arising from raising capital in the current period (which typically will be higher than τ).

Finally, the (constant) continuously compounded risk-free interest rate is denoted by r. Hence, the law of motion for the company's capital (budget constraint) is:

$$a_t = \left[a_{t-1} \times (1 - \tau) + R_t^b - c_1(R_t^b) + \sum_{j=1}^{N} p_t^{(j)}\right] e^r + R_t^e - c_2(R_t^e) - \sum_{j=1}^{N} I_t^{(j)} \tag{4.2}$$

for $a_{t-1} \geq 0$. We require that

$$R_t^b \geq -a_{t-1}(1-\tau), \tag{4.3}$$

the company cannot pay more in dividends than its capital (after capital costs have been deducted).

The company defaults if $a_t < 0$, which is equivalent to

$$\left[a_{t-1} \times (1-\tau) + R_t^b - c_1(R_t^b) + \sum_{j=1}^{N} p_t^{(j)}\right] e^r + R_t^e - c_2(R_t^e) < \sum_{j=1}^{N} I_t^{(j)}.$$

Due to limited liability, in this case the funds in the company are not sufficient to pay all the claims. We assume that the remaining assets in the firm are paid to claimants at the same rate per dollar of coverage, that is the recovery for policyholder i is

$$\min\left\{I_t^{(i)}, \frac{\left[a_{t-1} \times (1-\tau) + R_t^b - c_1(R_t^b) + \sum_{j=1}^{N} p_t^{(j)}\right] e^r + R_t^e - c_2(R_t^e)}{\sum_{j=1}^{N} I_t^{(j)}} \times I_t^{(i)}\right\}.$$

The premium the company is able to charge for providing insurance now depends on the riskiness of the coverage as well as the underwriting decision—that is, price is a function of demand as within an *inverse demand function*. Formally, this means that the premiums for line i, $p_t^{(i)}$, is a quantity known at time t (i.e., it is $\mathbf{F}$-predictable) given by a functional relationship:

$$\mathcal{P}_i\left(a_{t-1}, R_t^b, R_t^e, (p_t^{(j)})_{1\leq j\leq N}, (q_t^{(j)})_{1\leq j\leq N}\right) = 0,\ 1 \leq i \leq N.$$

One way to specify this functional relationship that is in line with the micro-foundations of insurance is to consider a set of (binding) *participation constraints* :

$$\gamma_i = \underbrace{\mathbb{E}_{t-1}\left[U_i\left(\left(\mathrm{w}_{t-1}^{(i)} - p_t^{(i)}\right) e^r - L_t^i + \min\left\{I_t^{(i)}, \frac{\left[a_{t-1}(1-\tau) + R_t^b - c_1(R_t^b) + \sum_j p_t^{(j)}\right] e^r + R_t^e - c_2(R_t^e)}{\sum_j I_t^{(j)}} I_t^{(i)}\right\}\right)\right]}_{=v_t^{(i)}(a_{t-1}, \mathrm{w}_{t-1}^{(i)}, q_t^{(1)}, \ldots, q_t^{(N)}, p_t^{(1)}, \ldots, p_t^{(N)})}, \tag{4.4}$$

where U_i is increasing and concave, and we call $\mathrm{w}_{t-1}^{(i)}$ "wealth" and γ_i "reservation utility." This is the approach taken in Bauer and Zanjani (2013a) in a simpler setting, and it ultimately relates capitalization and capital allocation to consumer concerns about company solvency. However, of course relating price and demand this way requires a great amount of supporting information for fixing the various inputs—so that relying on it for empirical applications is challenging.

Instead, for the purpose of this report, we rely on an alternative reduced-form specification that assumes premiums—as markups on discounted expected losses—are a function of the company's default probability. The underlying intuition is that consumers rely on insurance solvency ratings for making their insurance decisions, which in turn are closely related to default probabilities (see Section 5.1 for more details). More precisely, we set:

$$\begin{aligned} p_t^{(i)} &= e^{-r}\,\mathbb{E}_{t-1}\left[I_t^{(i)}\right] \\ &\times \exp\left\{\alpha - \beta\,\mathbb{P}_{t-1}\left(I_t > \left[a_{t-1}(1-\tau) + R_t^b - c_1(R_t^b) + \sum_{j=1}^{N} p_t^{(j)}\right] e^r + R_t^e - c_2(R_t^e)\right) - \gamma\mathbb{E}_{t-1}[I]\right\}, \end{aligned} \tag{4.5}$$

that is the premium charged is the expected indemnity multiplied by an exponential function of the default rate and the aggregate expected loss. Obviously, we expect both β and γ to have a positive sign, i.e. (i) the larger the default rate the smaller the premium loading and (ii) the more business the company writes, the smaller are the profit margins.

Therefore, all-in-all, the company solves:

$$\max_{\{p_t^{(j)}\},\{q_t^{(j)}\},\{R_t^b\},\{R_t^e\}} \left\{ \begin{array}{l} \mathbb{E}\Big[\sum_{t=1}^{\infty} 1_{\{a_1\geq 0,\dots,a_t\geq 0\}}\, e^{-rt}\left[e^r \sum_j p_t^{(j)} - \sum_j I_t^{(j)} - (\tau\, a_{t-1} + c_1(R_t^b))e^r - c_2(R_t^e)\right] \\ \quad -1_{\{a_1\geq 0,\dots,a_{t-1}\geq 0, a_t<0\}}\, e^{-rt}\left[(a_{t-1} + R_t^b)e^r + R_t^e\right]\Big] \end{array} \right\}, \tag{4.6}$$

subject to (4.2); (4.3); (4.5); $R_t^e \geq 0$; $\{p_t^{(j)}\}$, $\{q_t^{(j)}\}$, $\{R_t^b\}$ $\mathbf{F}$-predictable; and $\{R_t^e\}$ $\mathbf{F}$-adapted.

We obtain:

Lemma 4.1.1. *The objective function may be equivalently represented as:*

$$\max_{\{p_t^{(j)}\},\{q_t^{(j)}\},\{R_t^b\},\{R_t^e\}} \left\{ \mathbb{E}\left[\sum_{\{t\leq t^*: a_1\geq 0, a_2\geq 0,\dots,a_{t^*-1}\geq 0, a_{t^*}<0\}} e^{-rt}\left[-e^r R_t^b - R_t^e\right]\right] - a_0 \right\}. \tag{4.7}$$

Hence, the problem can be equivalently expressed as a dividend maximization problem conventional in the finance literature.

Denote the optimal value function, i.e. the solution to (4.6) or (4.7), by $V(a_0)$.[1] Then, under mild conditions, the value function is finite and—as the solution to a stationary infinite-horizon dynamic problem—satisfies the following Bellman equation:

Proposition 4.1.1 (Bellman Equation)**.** *Assume $r, \tau > 0$. Then the value function $V(\cdot)$ satisfies the following Bellman equation:*

$$V(a) = \max_{\{p^{(j)}\},\{q^{(j)}\},R^b,R^e} \left\{ \begin{array}{l} \mathbb{E}\Big[1_{\{(a(1-\tau)+R^b-c_1(R^b)+\sum p^{(j)})e^r+R^e-c_2(R^e)\geq\sum_j I^{(j)}\}} \\ \quad \times\Big(\sum_j p^{(j)} - e^{-r}\sum_j I^{(j)} - \tau\, a - c_1(R^b) - e^{-r}\, c_2(R^e) \\ \quad +e^{-r}\, V\left([a(1-\tau) + R^b - c_1(R^b) + \sum_j p^{(j)}]e^r + R^e - c_2(R^e) - \sum_j I^{(j)}\right)\Big) \\ \quad -1_{\{(a(1-\tau)+R^b-c_1(R^b)+\sum p^{(j)})e^r+R^e-c_2(R^e)<\sum_j I^{(j)}\}}\left(a + R^b + e^{-r} R^e\right)\Big] \end{array} \right\}.$$

[1]Note that the discounted expected value of dividends less capital raising will equate to $V(a_0) + a_0$, so that a locally decreasing $V(\cdot)$ does not necessarily imply that a less capitalized firm is worth less—solely that a slighlty less capitalized firm may be yielding higher profits.

(4.8)

subject to (4.4)*;* (4.3)*;* $\{p_t^{(j)}\}$*,* $\{q_t^{(j)}\}$*,* $R_t^b \in \mathbb{R}$*; and* $R_t^e \geq 0$ *is* $\sigma(L^{(j)}, j = 1, \ldots, N)$*-measurable.*

Note that since raising capital at the end of the period—which we interpret as raising capital in distressed states—is more costly than raising capital under "normal" conditions (cf. Eq. (4.1)), at the end of the period, it only makes sense to either raise exactly enough capital to save the company or not raise any capital at all: Raising more will not be optimal since it is possible to raise in the beginning of the next period at better conditions; raising less will not be optimal since the company will go bankrupt and the policyholders are the residual claimants. This yields:

Proposition 4.1.2. $R^e \in \{0, R_*^e\}$*, where* R_*^e *solves:*

$$\sum_j I^{(j)} - \left[a(1-\tau) + R^b - c_1(R^b) + \sum_j p^{(j)}\right] e^r = R_*^e - c_2(R_*^e). \tag{4.9}$$

More precisely,

- *for* $\left[a(1-\tau) + R^b - c_1(R^b) + \sum_j p^{(j)}\right] e^r \geq \sum_j I^{(j)}$*, we have* $R^e = 0$*;*
- *for* $\left[a(1-\tau) + R^b - c_1(R^b) + \sum_j p^{(j)}\right] e^r < \sum_j I^{(j)}$ *and* $V(0) \geq R_*^e$*, we have* $R^e = R_*^e$*;*
- *and for* $V(0) < R_*^e$*, we have* $R^e = 0$*.*

The latter assertion states that it only makes sense to save the company if the (stochastic) amount of capital to be raised at the end of the period is smaller than the value of the company, i.e. if the investment has a positive net present value. For a linear specification of end-of-period costs, this leads to the following simplification of the optimization problem:

Corollary 4.1.1. *For linear costs* $c_2(x) = \xi\, x,\ x \geq 0$*:*

$$R_*^e = \frac{1}{1-\xi}\left[\sum_j I^{(j)} - \left(a(1-\tau) + R^b - c_1(R^b) + \sum_j p^{(j)}\right) e^r\right]$$

In particular, the Bellman equation becomes:

$$V(a) = \tag{4.10}$$

$$\max_{\{p^{(j)}\},\{q^{(j)}\},R^b} \left\{ \begin{array}{l} \mathbb{E}\Big[1_{\{(a(1-\tau)+R^b-c_1(R^b)+\sum p^{(j)})e^r \geq \sum_j I^{(j)}\}} \times \Big(\sum_j p^{(j)} - e^{-r}\sum_j I^{(j)} - \tau\, a - c_1(R^b) \\ \qquad\qquad + e^{-r}\, V\left([a(1-\tau) + R^b - c_1(R^b) + \sum_j p^{(j)}]e^r - \sum_j I^{(j)} \right) \Big) \\ + 1_{\{(a(1-\tau)+R^b-c_1(R^b)+\sum p^{(j)})e^r < \sum_j I^{(j)} \leq (a(1-\tau)+R^b-c_1(R^b)+\sum p^{(j)})e^r + (1-\xi)\,V(0)\}} \times \\ \quad \left(\frac{1}{1-\xi}\left[[\sum_j p^{(j)} + a(1-\tau) + R^b - c_1(R^b)] - e^{-r}\sum_j I^{(j)} \right] + e^{-r}V(0) - [a + R^b] \right) \\ + 1_{\{\sum_j I^{(j)} > (a(1-\tau)+R^b-c_1(R^b)+\sum p^{(j)})e^r + (1-\xi)\,V(0)\}} \left(-(a + R^b) \right) \Big] \end{array} \right\}.$$

subject to

$$p^{(i)} = e^{-r}\,\mathbb{E}\left[I^{(i)}\right] \times \exp\left\{\alpha - \beta\,\mathbb{P}\left(I > \left[a(1-\tau) + R^b - c_1(R^b) + \sum_{j=1}^{N} p^{(j)}\right] e^r + R^e - c_2(R^e)\right) - \gamma\mathbb{E}[I]\right\}, \tag{4.11}$$

where $I = \sum_j I^{(j)}$.

In addition to the constraint arising from the relationship between risk and premium, we may be interested to consider an external solvency constraint imposed by regulation. Such a constraint will take the form:

$$s\left(\sum_j I^{(j)}\right) \leq \underbrace{a(1-\tau) + R^b - c_1(R^b) + \sum_{j=1}^{N} p^{(j)} + (1-\xi)\,V(0)}_{\text{Available Capital}}, \tag{4.12}$$

where s is a *monetary risk measure* (see Frittelli and Gianin (2002) for details on the properties of risk measures).[2]

4.2 The Marginal Cost of Risk

In what follows, we will generally assume a linear cost for end of period capital as in Corollary 4.1.1, and study the problem (4.10) subject to the participation constraint (4.11) and the regulatory constraint (4.12). For ease of presentation, we define

$$S = \left[a(1-\tau) + R^b - c_1(R^b) + \sum_j p^{(j)}\right] e^r \tag{4.13}$$

and

$$D = \left[a(1-\tau) + R^b - c_1(R^b) + \sum_j p^{(j)}\right] e^r + (1-\xi)V(0) \tag{4.14}$$

as the thresholds for I for *saving* the company and for letting it *default*, respectively.

Similarly to Bauer and Zanjani (2013a), we analyze the company's marginal cost to derive suitable capital allocations. More precisely, we know that at the optimum the marginal premium income will equal the company's marginal cost, and we consider a formulation that accounts for the cost of raising capital and keeps the default probability at the margin constant:

[2]This specification assumes that the company can put up the *present value of future profits* as a part if its capital, which is consistent with market-consistent embedded value principles (American Academy of Actuaries, 2011).

Proposition 4.2.1. *We have for the marginal cost for risk $i \in \{1, 2, \ldots, N\}$:*

$$
\begin{aligned}
& \frac{\mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}}\right] \exp\{\alpha - \beta \mathbb{P}(I \geq D)\}}{(1 - c_1'^b))} \qquad (4.15) \\
= \quad & \mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}} \, 1_{\{I \leq D\}}\right] + \gamma \, \mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}}\right] \exp\{\alpha - \beta \mathbb{P}(I > D) - \gamma \, \mathbb{E}[I]\} \\
& \quad + \mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}} \, V'(S - I) \, 1_{\{I \leq S\}}\right] + \mathbb{E}\left[\frac{\xi}{1 - \xi} \frac{\partial I^{(i)}}{\partial q^{(i)}} \, 1_{\{S < I \leq D\}}\right] \\
& + \mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}} \middle| I = D\right] \times \left\{\frac{1}{1 - c_1'^b)} \beta f_I(D) \mathbb{E}[I] \exp\{\alpha - \beta \, \mathbb{P}(I \geq D)\}\right\} \\
& + \frac{\partial \rho(I)}{\partial q^{(i)}} \times \left\{\mathbb{P}(I \geq D) + \frac{c_1'^b)}{1 - c_1'(R_b)} - \frac{\xi}{1 - \xi} \, \mathbb{P}(S \leq I \leq D)\right. \\
& \quad \left. - \mathbb{E}\left[V'\left(S - I\right) \, 1_{\{S \geq I\}}\right] - \frac{1}{1 - c_1'(R^b)} \, \beta \, f_I(D) \, \mathbb{E}[I] \, \exp\{\alpha - \beta \, \mathbb{P}(I \geq D)\}\right\}.
\end{aligned}
$$

The allocation formula (4.15) may appear opaque at first sight but we observe that the latter parts present a "weighted allocation" of the capital level D, where the weights relate to a the company's cost and are discussed in more detail below. The allocation parts arise from the external risk measure $(\frac{\partial \rho(I)}{\partial q^{(i)}})$ and Value-at-risk $(\mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}} \middle| I = D\right])$—which comes from the specification of the premium function that assumes policyholders assess company solvency via default probabilities.

As indicated above, D indeed is a pretty good notion of "capital:"

$$D = [a(1 - \tau) + R^b - c_1(R(b)) + \sum p^{(j)}]e^r + (1 - \xi)V(0)$$

The first part is the amount of available assets (previous assets a plus money raised plus premiums, compounded over the period) and the second part is the cost-adjusted "present value of future profits" for a zero-capital firm. This level "D" also is the default threshold, i.e. the company defaults when losses are greater than D whereas consumers get their full indemnity for aggregate loss realizations less than D.[3] Therefore, the marginal cost representation gives us the allocation of "the capital" D according to marginal cost—and it is indeed a weighted allocation echoing results from Bauer and Zanjani (2013a).

Of course the question arises regarding the significance of this result. As we pointed out in Section 2.1, the motivation for capital allocation approaches arises from the necessity of allocating capital costs (see also Venter (2004) or D'Arcy (2011)). For this purpose, it is important to understand what contributes to these costs. For this purpose, we consider the marginal cost representation in the absence of a regulatory constraint:

[3] The problem with this notion of capital is that upon default, the consumers actually only receive S. Hence, $(D - S)$ may be interpreted as an endogenous default cost/penalty.

Corollary 4.2.1. *We have for the marginal cost for risk* $i \in \{1, 2, \dots, N\}$*:*

$$\frac{\mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}}\right] \exp\{\alpha - \beta \mathbb{P}(I \geq D)\}}{(1 - c_1'^{b}))} \tag{4.16}$$

$$= \underbrace{\mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}}\, 1_{\{I \leq D\}}\right]}_{(i)} + \underbrace{\gamma\, \mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}}\right] \exp\{\alpha - \beta \mathbb{P}(I > D) - \gamma\, \mathbb{E}[I]\}}_{(ii)} + \underbrace{\mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}}\, V'(S - I)\, 1_{\{I \leq S\}}\right]}_{(iii)}$$

$$+ \underbrace{\mathbb{E}\left[\frac{\xi}{1-\xi} \frac{\partial I^{(i)}}{\partial q^{(i)}}\, 1_{\{S < I \leq D\}}\right]}_{(iv)}$$

$$+ \underbrace{\mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}} \middle| I = D\right] \times \left\{\mathbb{P}(I \geq D) + \tau^*\right\}}_{(v)},$$

where the "shadow cost of capital" τ^* *is defined as:*

$$\tau^* = \frac{c_1'^{b})}{1 - c_1'(R_b)} - \frac{\xi}{1-\xi}\, \mathbb{P}(S \leq I \leq D) - \mathbb{E}\Big[V'\Big(S - I\Big)\, 1_{\{S \geq I\}}\Big]\Big\} \tag{4.17}$$

Obviously, the marginal cost has various components. First, an additional dollar issued in some line will increase the actuarial value of the liability in solvent states (i). Moreover, the increased supply will yield a decrease in the price of insurance—i.e. there are "scale costs" (ii). Also, the higher exposure will lead to a change in the capitalization at the end of the period, which will affect the value of the company—that is, there may be a cost associated with the continuation value of the company (iii). Since a company with a larger exposure is more expensive to bail out, there further are costs associated with "saving" it (iv). Finally, it will be necessary to increase capital holdings, which yields capital costs (v). However, since the company can—and will—access company in various ways, the cost for an additional dollar of capital is not straightforward. In particular, it generally is lower since the cost of raising external since the company will seek the most effective way of raising capital in different states of the world. This yields the "shadow cost" of capital τ^*.

Adding up the marginal costs, we arrive at:

$$\begin{aligned}
&\frac{e^{-r}\, \mathbb{E}[I] \exp\{\alpha - \beta \mathbb{P}(I \geq D)\}}{(1 - c_1'^{b}))\, e^{-r}} \\
= \;& \mathbb{E}\left[I\, 1_{\{I \leq D\}}\right] + S\, \mathbb{P}(I > D) \\
&+ \gamma\, \mathbb{E}[I] \exp\{\alpha - \beta \mathbb{P}(I > D) - \gamma\, \mathbb{E}[I]\} \\
&+ (1 - \xi)\, V(0) \left[\frac{\mathbb{P}(I \geq D)}{1 - c_1'^{b})} + \left[\frac{c_1'^{b})}{1 - c_1'^{b})} - \frac{\xi}{1-\xi}\right] \mathbb{P}(S \leq I < D)\right. \\
&\qquad\qquad\qquad\qquad\qquad \left. + \mathbb{E}\left[\left(\frac{c_1'^{b})}{1 - c_1'^{b})} - V'(S - I)\right) 1_{\{S \geq I\}}\right]\right] \\
&- \mathbb{E}\left[(S - I)\, V'(S - I)\, 1_{\{S \geq I\}}\right] \\
&+ \mathbb{E}\left[\xi \frac{(I - S)}{1 - \xi}\, 1_{\{S \leq I < D\}}\right] + \left[\frac{a(1-\tau) + \sum_j p^{(j)}}{1 - c_1'^{b})} + \frac{R^b - c_1(R^b)}{1 - c_1'^{b})}\right] c_1'^{b})\, e^r.
\end{aligned}$$

Generally the interpretation is not straightforward. In particular, the marginal costs may not "add up" to total costs due to non-linearities in $V(\cdot)$ and $c_1(\cdot)$.

To obtain a correct representation of the costs of the company, all of these components have to be allocated—which implies that "allocating capital" may not be sufficient. In solvent states, these allocations are according to recoveries. In non-solvent states, allocations can be inferred from the marginal cost representation (4.15)/(4.16).

4.3 Discussion

The multi-period setting, featuring company options to bear external financing costs in order to raise additional funding, complicates the risk pricing problem considerably. Noteworthy is the fact that this setting no longer features a mapping of capital allocation as currently practiced to marginal cost. That is, allocating the capital of the firm is no longer sufficient to recapture all of the risk components of marginal cost. Risk has the potential now to consume not only the current capital of the firm, but also capital that has yet to be raised. Moreover, unlike the model in Bauer and Zanjani (2013a), the continuation value of the firm is no longer independent of its capital position since there are frictional costs of financing. The consequences of these changes is that the current capital of the firm is no longer the only cushion absorbing risk, so the marginal cost of risk can no longer be recovered solely by allocating the same.

We proceed in the next section to numerical analysis of the model, with the aim of comparing the marginal costs obtained to those implied by conventional allocation methods. Since we have specified the objectives and constraints facing the firm, we have gone through the "rigors" of the pricing project—with the consequence that, within the context of the model, we have the "right" answer on marginal cost. And, while one could of course make different choices in terms of model specification, the model addresses some key areas of interest that affect the marginal cost of risk, such as the continuation value of the firm and costly external financing, that should provide for revealing comparisons with static single period capital allocation models.

Chapter 5

Capital Allocation versus Marginal Cost in the Multi-Period Model

In this chapter, we calibrate and solve the model introduced in Section 4.1. We then calculate the true marginal cost of risk in the solved model to that obtained from conventional allocation methods. We find significant differences.

5.1 Calibration of the Multi-Period Model

The model as developed in the preceding sections requires calibration in several areas. It is necessary to specify costs of raising and holding capital. It is also necessary to specify how the insurance company is affected by changes in risk.

As a starting point for the costs of holding capital, Cummins and Phillips (2005) estimate the cost of equity capital for insurance companies using data from the 1997-2000 period. They use several methods to derive a variety of estimates, including a single factor CAPM and Fama-French three-factor cost of capital model (Fama and French, 1993). The estimates for property-casualty insurance fall in the neighborhood of 10% to 20%. Given that the risk-free interest rate used in the analysis was based on the 30-day T-Bill rate, which averaged about 5% over the sample period, the estimates suggest a risk premium for property-casualty insurance ranging from as little as 5% to as much as 15%. However, previous research has found unstable estimates of the cost of capital, suggesting that the risk premium may be considerably smaller; some specifications even suggest that the industry's "beta" may be zero or even negative (Cox and Rudd (1991), Cummins and Harrington (1985)). Given the range of results, we use τ ranging from 3% to 5% in the model. Calibrating the cost of raising capital is more difficult, as we are aware of no studies specific to the property-casualty industry. Hennessy and Whited (2007), however, analyze the cost of external financing average across industries by using the entire sample of Compustat firms. They find marginal equity flotation costs ranging from 5% for large firms to 11% for small firms, and we base our quadratic calibration of external financing costs on these figures, with the linear piece being set at 7.5%.

Changes in risk are known to affect insurance companies. Epermanis and Harrington (2006) focus on the property-casualty insurance industry in particular, documenting significant declines in premium growth following ratings downgrades. Sommer (1996) documented a significant connection between default risk and pricing in the property-casualty industry. The foregoing research suggests two possible ways to model the consequences of risk for a property-casualty insurer: Increases in risk could either produce involuntary drops in exposure volume or drops in price, or both. In our case, the price channel is most easily incorporated into our model setup, so we proceed by calibrating the influence of risk on price. Proceeding, however, requires a more precise definition of "risk" and an analysis of how risk so defined correlates with pricing.

On the question of how to define "risk," credit ratings are a tempting solution. While credit ratings are widely accepted proxies for market assessments of a company's risk level, using them requires us to map credit ratings to some quantitative measure of risk amenable to numerical analysis. The path we choose here is to map credit ratings to default risk levels, which is a feasible exercise given the validation studies provided by rating companies that document the historical connection between default risk and the various letter ratings.

The question of how to connect risk with pricing is an empirical one, requiring an analysis of the historical relation between default risk inferred from credit ratings and insurance prices. Since the data used for our numerical analysis is drawn from a reinsurance company, we focus on empirical analysis of reinsurers, and specifically those identified in the Reinsurance Association of America's annual review of underwriting and operating results for the years 2008-2012. These reviews yield 30 companies for the analysis, and we collected all available ratings for that set of 30 companies from Moody's, S&P, and A.M. Best.

To calculate the default rate, we use a multi-stage procedure. We start by collecting 1) Moody's, S&P, and A.M. Best ratings from the 2008-2012 period for the sample of insurance companies, 2) the joint distribution of Moody's and S&P ratings for corporate debt as reported in Table 1 of Cantor, Packer, and Cole (1997), and 3) one year default rates by rating as reported in Tables 34 and 35 of Moody's *Annual Default Study: Corporate Default and Recovery Rates, 1920-2012* and Tables 9 and 24 of S&P's *2012 Annual Corporate Default Study and Rating Transitions.* We then fit smoothed default rates for Moody's by choosing default rates for the AA1, AA2, AA3, A1, A2, and A3 categories (AAA, BAA1, and other historical default rates are held at their historic values)[1] and perform a similar procedure for S&P ratings. We then calculate an average one year default rate for A++, A+, A, and A- A.M. Best ratings by calculating an average "Moody's" default rate based

[1]Fit is assessed by evaluating 8 measures: 1) the weighted average default rate in the Aa category (using the modifier distribution in Cantor (Packer) for the weights), 2) the weighted average default rate in the A category, and "fuzzy" default rates for Aa1, Aa2, Aa3, A1, A2, and A3 categories, where the fuzzy rate is calculated by applying the distribution of S&P ratings for each modified category to the default rates (for example, if S&P rated 20% of Moody's Aa1 as AAA, 50% as AA+ and 30% as AA, we would calculate the "fuzzy" default rate for Aa1 as 20%*Aaa default rate + 50%*Aa1 default rate + 30%*Aa2 default rate. We calculate squared errors between fitted averages and averages using the actual empirical data, and select fitted values to minimize the straight sum of squared errors over the eight measures.

on the our sample distribution of Moody's ratings for each A.M. Best rating, calculating an average"S&P" default rate in a similar manner, and then averaging the two. This yields one-year default rates for each A.M. Best rating in the 2008-2012 sample as shown in Table 5.1.[2]

A++	0.006%
A+	0.044%
A	0.072%
A-	0.095%

Table 5.1: Fitted One-Year Default Rates for A.M. Best Ratings

Finally, we identify the relation between price and default risk by fitting the model

$$\ln p_{it} = \alpha + \alpha_t - \beta d_{it} - \gamma E_{it} + e_{it}$$

where p_{it} is calculated as the ratio of net premiums earned to the sum of loss and loss adjustment expenses incurred and company i in year t, d_{it} is the default rate corresponding to the letter rating of company i in year t, E_{it} is the expected loss of company i in year t, and e_{it} is an error term. The expected loss is calculated by applying the average net loss and loss adjustment expense ratio over the sample period for each firm to that year's net premium earned.

We use NAIC data for the period 2002 to 2010 for the sample of companies identified above for the analysis. The results of the regression are presented in Table 5.2.

Variable	Coefficient	Std. Error	t-value
Intercept (α)	.65897	0.0614	10.73
Default rate (β)	3.92958	0.5090	-7.72
Expected Loss (γ)	1.48 E-10	2.24 E-11	-6.57

Year dummies are omitted. Observations: 288. Adj. $R^2 = 26\%$

Table 5.2: Premium Parametrizations

[2]It is worth noting that these are somewhat lower than suggested by A.M. Best's own review of one-year impairment rates, which indicated 0.06% for the A++/A+ category and 0.17% for the A/A- category (see Exhibit 2 of *Best's Impairment Rate and Rating Transition Study – 1977 to 2011*). In the empirical analysis that followed, it was also necessary to assign a default rate for the B++ rating, which did not occur in the 2008-2012 sample but did surface when performing the analysis over a longer time period. We used 0.20% for this default rate, which is roughly consistent with the default rates for Baa or BBB ratings.

Parameter	1 ("base case")	2 ("profitable company")	3 ("empty company")
τ	3.00%	5.00%	5.00%
$c_1^{(1)}$	7.50%	7.50%	7.50%
$c_1^{(2)}$	1.00E-010	5.00E-011	1.00E-010
ξ	50.00%	75.00%	20.00%
r	3.00%	6.00%	3.00%
α	0.3156	0.9730	0.9730
β	392.96	550.20	550.20
γ	1.48E-010	1.61E-010	1.61E-010

Table 5.3: Parametrizations

5.2 Implementation of the Multi-Period Model

We use three sets of parameters based on the calibrations above. The sets are described in Table 5.3. We vary the cost of holding capital τ from 3% to 5%; the cost of raising capital is represented by a quadratic cost function with the linear coefficient $c_1^{(1)}$ fixed at 7.5%; the cost of raising capital in distressed circumstances, ξ, varies from 20% to 75%; the interest rate r varies from 3% to 6%; and the parameters α, β, and γ specify the underwriting margin. For the latter parameters, we use the regression results from the previous section, with the alpha intercept being adjusted for the average of the unreported year dummy coefficients, and we also use an alternative, more generous specification based on previous unreported analysis that omits loss adjustment expenses.

Using the loss distributions described in Section 3.1, we solve the optimization problem by value iteration relying on the corresponding Bellman equation (4.10) on a discretized grid for the capital level a. That is, we commence with an arbitrary value function (constant at zero in our case), and then iteratively solve the one-period optimization problem (4.10) by using the optimized value function from the previous step on the right hand side. Standard results on dynamic programming guarantee the convergence of this procedure (see e.g. Bertsekas (1995)).

For solving the one-period problem in each step, we rely on the following basic algorithm. Details on some of the steps, on the implementation, and evidence on the convergence is provided below.

Algorithm 5.2.1. *For a given (discretizted) end-of-period value function V^{end}, and for capital levels $a_k = ADEL \times k$, $k = 0, 1, 2, \ldots, AGRID$:*

1. *Given capital level a_k, optimize over $q^{(1)}, q^{(2)}, q^{(3)}, q^{(4)}$.*
2. *Given capital level a_k and a portfolio $(q^{(1)}, q^{(2)}, q^{(3)}, q^{(4)})$, optimize over $R^{(b)}$.*

3. *Given* a_k*, portfolio* $(q^{(1)}, q^{(2)}, q^{(3)}, q^{(4)})$*, and raising decision* $R^{(b)}$*, determine the premium levels by evaluating Equation* (4.11)*.*

4. *Given* a_k*, portfolio* $(q^{(1)}, q^{(2)}, q^{(3)}, q^{(4)})$*,* $R^{(b)}$*, and premiums* $(p^{(1)}, p^{(2)}, p^{(3)}, p^{(4)})$ *evaluate* $V^{beg}(a_k)$ *based on the given end-of-period function* V^{end} *by interpolating in between the grid and extrapolating off the grid.*

Discretization

We rely on an equidistant grid of size of 26 (AGRID = 25) with different increments depending on the parameters (ADEL = 250, 000, 000 for the base case and ADEL = 750, 000, 000 for the (2) "profitable company" and (2) "empty company" cases). We experimented with larger grids with finer intercepts but 26 points proved to be a suitable compromise between accuracy and run time of the program.

Optimization

For carrying out the numerical optimization of the portfolio values $q^{(i)}$, $i = 1, 2, 3, 4$, we rely on the so-called *downhill simplex method* proposed by Nelder and Mead (1965) as available within most numerical software packages. For the starting values, we rely on the optimized values from the previous step, with occasional manual adjustments during the early iteration in order to smooth out the portfolio profiles.

For the optimization of the optimal raising decision R^b, in order to not get stuck in a local maximum, we first calculate the value function based on sixty different values across the range of possible values $[-a \times (1 - \tau), \infty)$. We then use the optimum of these as the starting value in the Nelder-Mead method to derive the optimized value.

Calculation of the Premium Levels

The primary difficulty in evaluating the optimal premium level is that premiums enter the constraint (4.11) on both sides of the equation as the default rate itself depends on the premium, and this dependence is discontinuous (given our discrete loss distributions). We use the following approach: Starting from a zero default rate, we calculate the minimal amount necessary to attain the given default rate; we then check whether this amount is incentive-compatible, i.e. if the policyholders would be willing to pay it given the default probability; if so, we calculate a smoothed version of the premium level using (4.11) by deriving the (hypothetical) default rate considering how much the policyholders are willing to pay over to the minimal amount at that level relative to the amount necessary to decrease the default rate based on the our discretized distribution; if not, we move to the next possible default rate given our discrete loss distribution and check again.

Interpolation and Extrapolation

For arguments in between grid points, we use linear interpolation. For values off the grid ($a >$ AGRID $\times$ ADEL), supported by the general shape of the value functions across iterations, we use either linear or quadratic extrapolation. More precisely, in case fitting a quadratic regression in a to the five greatest grid values does not yield a significant quadratic coefficient—i.e., if the value function appears linear in this region—we use linear extrapolation starting from the largest grid point. Otherwise, we use a quadratic extrapolation fitted over the entire range starting from the largest grid point.

Implementation and Run Time

We implement the algorithm in *matlab*, and perform the computations on a numerical server with Dual Six Core Intel Xeon Processors and 12 Gigabytes of RAM running Ubuntu. While we do not make direct use of parallel computing, the multi-processor environment allows us to perform the calculations for different parametrizations and program specification simultaneously.

The one-period evaluations take in between several hours (up to twelve) and slightly less than two hours, depending on the optimization. In particular, since the starting values for the optimization are very close to the optimized values as the algorithm converges, the optimization procedure runs a lot faster. The final values shown in the following sections rely on 320 iterations of the value function, which altogether took several months of calculations. However, deriving sensitivities with respect to the cost parameters for the given premiums specification is faster since we can use the optimized value function as the initial value function, which considerably reduces the number of necessary iterations.

Convergence

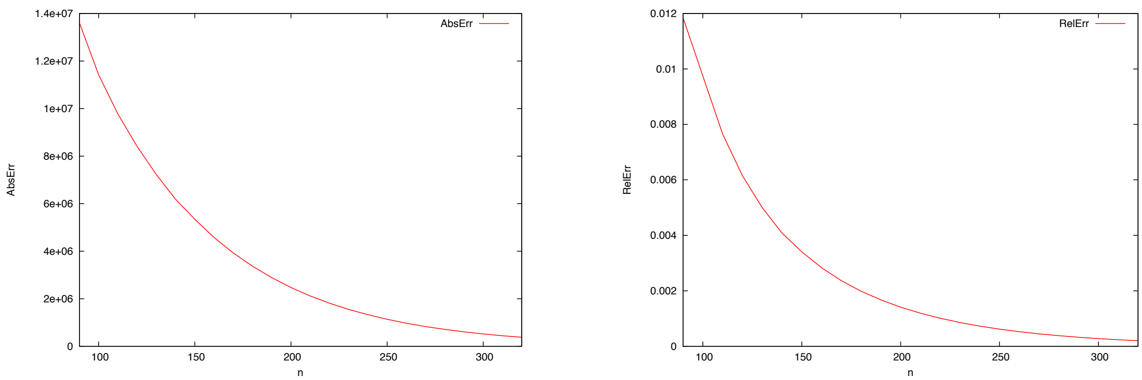
We assess convergence by calculating absolute and relative errors in the value and the policy functions from one iteration to the next. These errors are directly proportional to error bounds for the algorithm, where the proportionality coefficients depend on the interest and the default rate (an upper bound is given by $\bar{c} = \frac{e^{-r}}{1-e^{-r}}$, see e.g. Proposition 3.1 in (Bertsekas, 1995, Chap. 1))). More precisely, we define the absolute and relative errors for the value function by:

$$
\begin{aligned}
\text{AbsErr}_n &= \max_k \left\{ \| V_n(a_k) - V_{n-1}(a_k) \| \right\}, \\
\text{RelErr}_n &= \max_k \left\{ \frac{\| V_n(a_k) - V_{n-1}(a_k) \|}{V_n(a_k)} \right\},
\end{aligned}
$$

where V_n denotes the value function after iteration n. Similarly, we define absolute and relative errors for the policy function by:

$$\begin{aligned} \text{AbsErr}(q^{(i)})_n &= \max_k \left\| q_n^{(i)}(a_k) - q_{n-1}^{(i)}(a_k) \right\|, \; i = 1, 2, 3, 4, \\ \text{RelErr}(q^{(i)})_n &= \max_k \frac{\left\| q_n^{(i)}(a_k) - q_{n-1}^{(i)}(a_k) \right\|}{q_n^{(i)}(a_k)}, \; i = 1, 2, 3, 4, \end{aligned}$$

where $q_n^{(i)}$ denotes the (optimized) exposure to line i after iteration n.



(a) Absolute Error in V

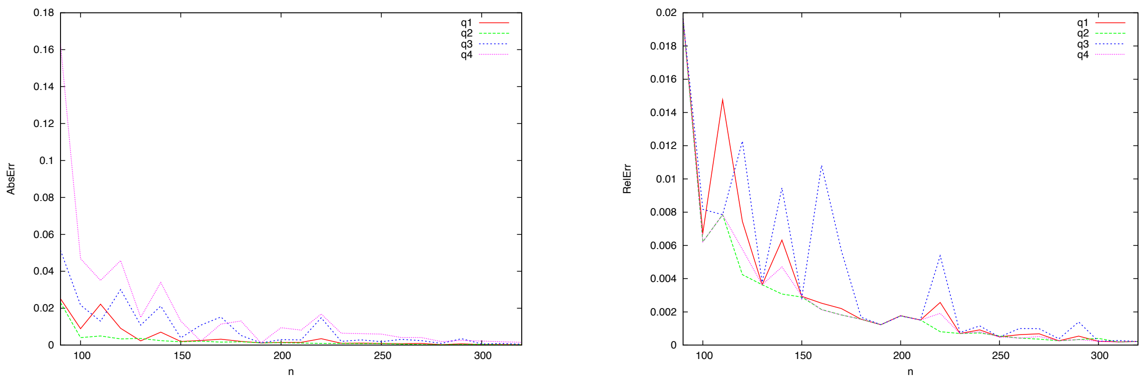
(b) Relative Error in V

Figure 5.1: Absolute and relative error in the value function for a company with carrying cost $\tau = 3\%$, raising costs $c^{(1)} = 7.5\%$, $c^{(2)} = 1.00\text{E-}10$, and $\xi = 50\%$, interest rate $r = 3\%$, and premium parameters $\alpha = 0.315624284$, $\beta = 392.958$, and $\gamma = 1.48\text{E-}10$.

Figure 5.1 show the errors for the value function using the base case parameters and different values of n for between 90 and 320 (in increments of 10). Results (and error bounds) for the remaining parametrizations are even smaller. After 320 iterations, the absolute error in the value function is $379,230$, which is only a very small fraction of the value function ranging from $1,813,454,921$ to $1,955,844,603$ (about 0.02%). In particular, considering the rather conservative error bound above, these results imply that the error in V amounts to less than one percent. Similarly, Figure 5.2 shows the absolute and relative errors for the portfolio functions. Again, we observe that relative changes from one iteration to the next after 320 iterations are maximally around 0.02%.

5.3 Discussion of Solutions

The solutions vary considerably across the parameterizations. While the value function in the base case ranges from approximately 1.8 billion to 2 billion for the considered capital



(a) Absolute Error in $q^{(i)}$, $i = 1, 2, 3, 4$

(b) Relative Error in $q^{(i)}$, $i = 1, 2, 3, 4$

Figure 5.2: Absolute and relative error in the optimal portfolio weights for a company with carrying cost $\tau = 3\%$, raising costs $c^{(1)} = 7.5\%$, $c^{(2)} = 1.00\text{E-}10$, and $\xi = 50\%$, interest rate $r = 3\%$, and premium parameters $\alpha = 0.315624284$, $\beta = 392.958$, and $\gamma = 1.48\text{E-}10$.

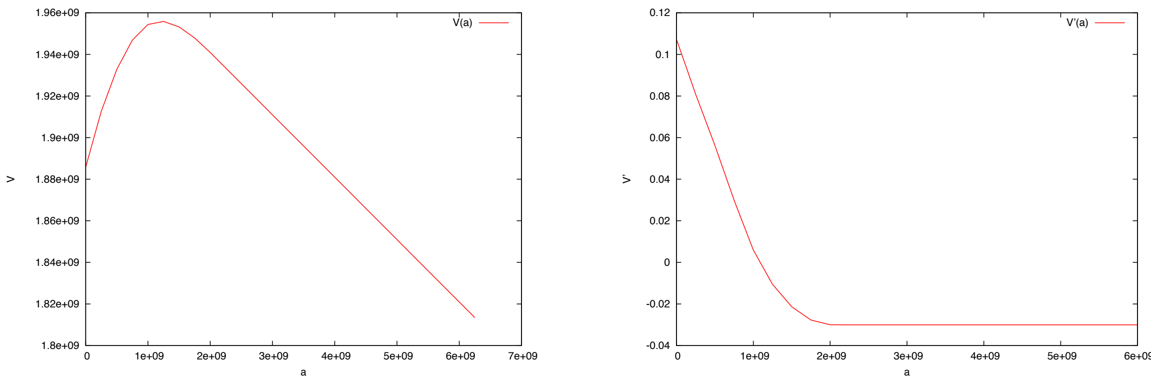
levels, the ranges for the "profitable company" are in between 21.7 billion to 22.4 billion and even around 56 to 57 billion for our "empty company." Nonetheless the basic shape of the solution is similar across the first two cases, whereas the "empty company case" yields a qualitatively different form (hence the name). In what follows, we discuss the solutions in the different cases in more detail, where we particularly emphasize the economic intuition behind the results.

5.3.1 Base Case Solution

Various aspects of the "base case" solution are depicted in Figures 5.3, 5.4 and 5.5. Table 5.4 presents detailed results at three key capital levels.

Figure 5.3 displays the value function and its derivative. We observe that the value function is "hump-shaped" and concave—that the derivative V' is decreasing in capital. However, for high capital levels, the derivative is approaching a constant level of $-\tau = -3\%$ and the value function is essentially linear.

The optimal level of capitalization here is 1 billion. If the company has less than 1 billion, it raises capital as can be seen from Figure 5.4, where the optimal raising decision for the company is displayed. However, the high and convex cost of raising external financing prevents the company from moving immediately to the optimal level. The adjustment can take time: Since internally generated funds are cheaper than funds raised from investors, the optimal policy trades off the advantages associated with higher levels of capitalization against the costs of getting there. As capitalization increases, there is a region around the optimal level where the company neither raises nor sheds capital. In this region, additional



(a) Value Function $V(a)$ (b) Derivative $V'(a)$

Figure 5.3: Value function V and its derivative V' for a company with carrying cost $\tau = 3\%$, raising costs $c^{(1)} = 7.5\%$, $c^{(2)} = 1.00\text{E-}10$, and $\xi = 50\%$, interest rate $r = 3\%$, and premium parameters $\alpha = 0.315624284$, $\beta = 392.958$, and $\gamma = 1.48\text{E-}10$.

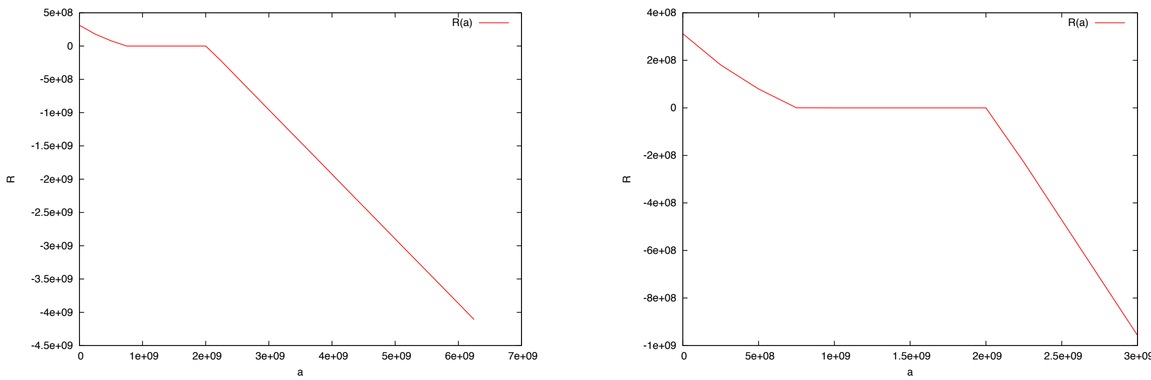
capital may bring a benefit, but it is below the marginal cost associated with raising an additional dollar, which is $c_1^{(1)} = 7.5\%$. The benefit of capital may also be less than its carrying cost of $\tau = 3\%$, but since this cost is sunk in the context of the model, capital may be retained in excess of its optimal level. For extremely high levels of capital, however, the firm optimally sheds capital through dividends to immediately return to a maximal level at which point the marginal benefit of holding an additional unit of capital (aside from the sunk carrying cost) is zero. The transition is immediate, as excess capital incurs an unnecessary carrying cost and shedding capital is costless in the model. This is also the reason that the slope of the value function approaches $-\tau$ in this region.

Figure 5.5 reveals how the optimal portfolio varies with different levels of capitalization. As capital is expanded, more risk can be supported, and the portfolio exposures grow in each of the lines until capitalization reaches its maximal level. After this point, the optimal portfolio remains constant: Even though larger amounts of risk could in principle be supported by larger amounts of capital, it is, as noted above, preferable to immediately shed any capital beyond a certain point and, concurrently, choose the value maximizing portfolio. Note that the firm here has an optimal scale because of the γ parameter in the premium function. As the firm gets larger in scale, margins shrink because of γ.

Table 5.4 reveals that firm rarely exercises its default option (measured by $\mathbb{P}(I \geq D)$, which is evidently 0.002% even at low levels of capitalization). The firm does experience financial distress more often at low levels of capitalization. For example, the probability of facing claims that exceed immediate financial resources, given by $\mathbb{P}(I > S)$, is 4.54% when initial capital is zero but 0.45% when capital is at the optimal level and 0.13% when capitalization is at its maximal point. In all of these cases, the firm usually resorts to

	zero capital	optimal capital	high capital
a	0	1,000,000,000	4,000,000,000
$V(a)$	1,885,787,820	1,954,359,481	1,880,954,936
$R(a)$	311,998,061	0	-1,926,420,812
$q_1(a)$	0.78	1.23	1.86
$q_2(a)$	0.72	1.13	1.71
$q_3(a)$	1.60	2.51	3.80
$q_4(a)$	5.06	7.96	12.06
S	550,597,000	1,406,761,416	2,615,202,661
D	1,493,490,910	2,349,655,327	3,558,096,571
$\mathbb{E}[I]$	199,297,482	313,561,933	474,841,815
$\sum p^{(i)}/\mathbb{E}[i]$	1.32	1.30	1.27
$\mathbb{P}(I > a)$	100.00%	2.66%	0.002%
$\mathbb{P}(I > S)$	4.54%	0.45%	0.13%
$\mathbb{P}(I > D)$	0.002%	0.002%	0.002%
$c_1'(R_b)$	13.74%	4.65%	0.00%
$\frac{\xi}{1-\xi}\,\mathbb{P}(S < I < D)$	4.54%	0.45%	0.12%
$\mathbb{E}[V'\,\mathbf{1}_{\{I<S\}}]$	8.03%	1.09%	-2.66%
τ^*	3.36%	3.34%	2.53%

Table 5.4: Results for a company with carrying cost $\tau = 3\%$, raising costs $c^{(1)} = 7.5\%$, $c^{(2)} =$ 1.00E-10, and $\xi = 50\%$, interest rate $r = 3\%$, and premium parameters $\alpha = 0.315624284$, $\beta = 392.958$, and $\gamma =$ 1.48E-10.



(a) Raising decisions $R(a)$ (b) Raising decisions $R(a)$ (lim. range)

Figure 5.4: Optimal raising decision R for a company with carrying cost $\tau = 3\%$, raising costs $c^{(1)} = 7.5\%$, $c^{(2)} = 1.00\text{E-}10$, and $\xi = 50\%$, interest rate $r = 3\%$, and premium parameters $\alpha = 0.315624284$, $\beta = 392.958$, and $\gamma = 1.48\text{E-}10$.

emergency financing when claims exceed its cash, at a per unit cost of $\xi = 50\%$, to remedy the deficit. Because of the high cost of emergency financing, however, it restrains its risk taking when undercapitalized and also raises capital before underwriting to reduce the probability of having to experience financial distress.

The bottom lines of the table show the various cost parameters at the optimized value. Here, the marginal cost of raising capital is significantly greater than 7.5% for $a = 0$ due to the quadratic adjustment, whereas clearly the marginal cost is zero in the shedding region ($a = 4$ billion). As indicated above, around the optimal capitalization level of 1 billion neither raising or shedding is optimal—so that technically the marginal cost is undefined due to the non-differentiability of the cost function c_1 at zero. To determine the correct "shadow cost" of raising capital, we use an indirect methods: We use the aggregated marginal cost condition from Corollary 4.2.1 to back out the value of $c_1'(0)$ that yields the left- and right-hand side to match up.[3] The cost of emergency raising in this case is exactly the probability of using this option (as $\xi = 50\%$), which—as indicated—increases in the capital level. Finally, the expected cost in terms of impact on the value function ($-\mathbb{E}[V' \boldsymbol{1}_{\{I<S\}}]$) is negative for low capital levels since the value function is increasing in this region, whereas it is positive and approaching τ for high capital levels. Combining the different cost components, we obtain a "shadow cost" of capital τ^* as defined (4.16) that is decreasing in a, though the level is not too different across capitalizations. In particular, it is noteworthy that τ^* is considerably below the cost of raising capital. The next section

[3] In the differentiable regions ($a = 0$ and 4 bn), the aggregated marginal cost condition further validate our results—despite discretization and approximation errors, the deviation between the left- and right-hand side is maximally about 0.025% of the left-hand side.

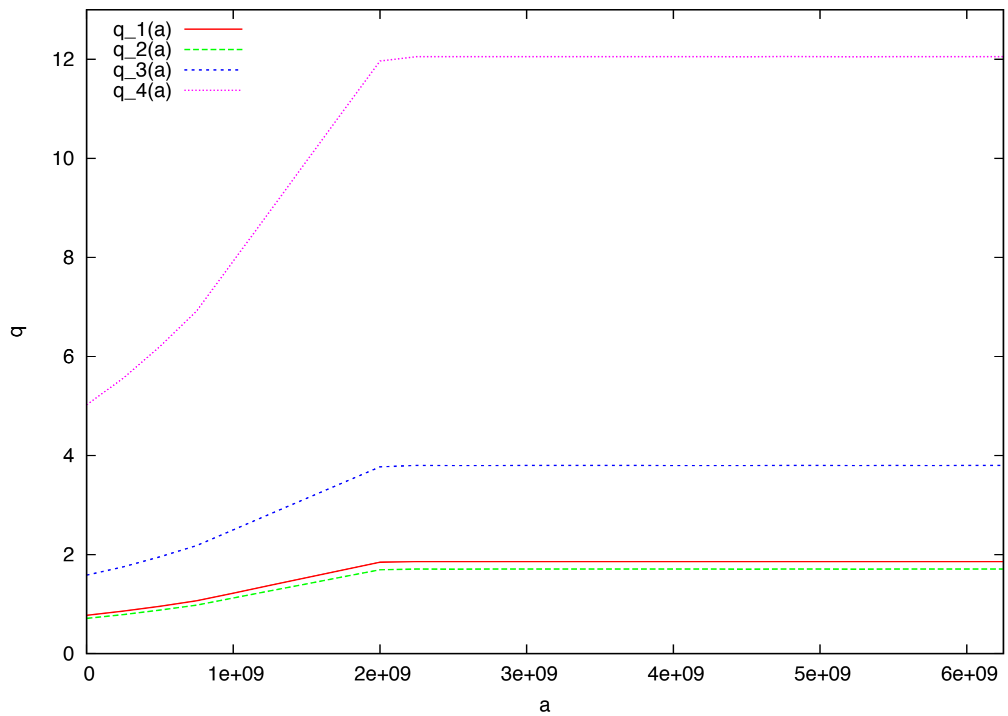


Figure 5.5: Optimal portfolio weights q_1, q_2, q_3, and q_4 for a company with carrying cost $\tau = 3\%$, raising costs $c^{(1)} = 7.5\%$, $c^{(2)} = 1.00\text{E-}10$, and $\xi = 50\%$, interest rate $r = 3\%$, and premium parameters $\alpha = 0.315624284$, $\beta = 392.958$, and $\gamma = 1.48\text{E-}10$.

provides a more detailed discussion of the marginal cost of risk.

5.3.2 Profitable Company

The results for the profitable company are similar in flavor to the "base case" presented above, except that the company is now much more valuable—despite the increases in the carrying cost of capital and in the cost of emergency financing—because of the more attractive premium function. The corresponding results are collected in Appendix A. More precisely, Figure A.1 displays the value function and its derivative, Figure A.2 displays the optimal raising decision, and Figure A.3 displays the optimal exposure to the different lines as a function of capital.

Again, there is an interior optimum for capitalization, and the company optimally adjusts toward that point when undercapitalized. If overcapitalized, it optimally sheds to

	zero capital	optimal capital	high capital
a	0	3,000,000,000	12,000,000,000
$V(a)$	22,164,966,957	22,404,142,801	22,018,805,587
$R(a)$	1,106,927,845	0	-6,102,498,331
$q_1(a)$	4.81	6.14	7.82
$q_2(a)$	4.42	5.64	7.18
$q_3(a)$	9.83	12.56	15.98
$q_4(a)$	31.19	39.85	50.69
S	3,659,208,135	6,215,949,417	9,412,766,805
D	9,200,449,874	11,757,191,157	14,954,008,545
$\mathbb{E}[I]$	1,227,901,222	1,569,126,466	1,995,776,907
$\sum p^{(i)}/\mathbb{E}[i]$	2.15	2.03	1.90
$\mathbb{P}(I > a)$	1.00%	10.70%	0.07%
$\mathbb{P}(I > S)$	3.65%	0.91%	0.34%
$\mathbb{P}(I > D)$	0.002%	0.002%	0.002%
$c_1'(R_b)$	18.57%	5.97%	0.00%
$\frac{\xi}{1-\xi}\,\mathbb{P}(S < I < D)$	10.94%	2.72%	1.00%
$\mathbb{E}[V'\,\mathbf{1}_{\{I<S\}}]$	2.93%	-2.99%	-4.58%
τ^*	8.94%	6.62%	3.58%

Table 5.5: Results for a company with carrying cost $\tau = 5\%$, raising costs $c^{(1)} = 7.5\%$, $c^{(2)} = 5.00\text{E-}11$, and $\xi = 75\%$, interest rate $r = 6\%$, and premium parameters $\alpha = 0.973046$, $\beta = 550.203$, and $\gamma = 1.61\text{E-}10$.

a point where the net marginal benefit associated with holding a dollar of capital (aside from the current period carrying cost which is a sunk cost) is zero. There is thus a range where the company neither raises nor sheds capital, and the risk portfolio gradually expands with capitalization until it reaches the point where the firm is optimally shedding additional capital on a dollar-for-dollar basis.

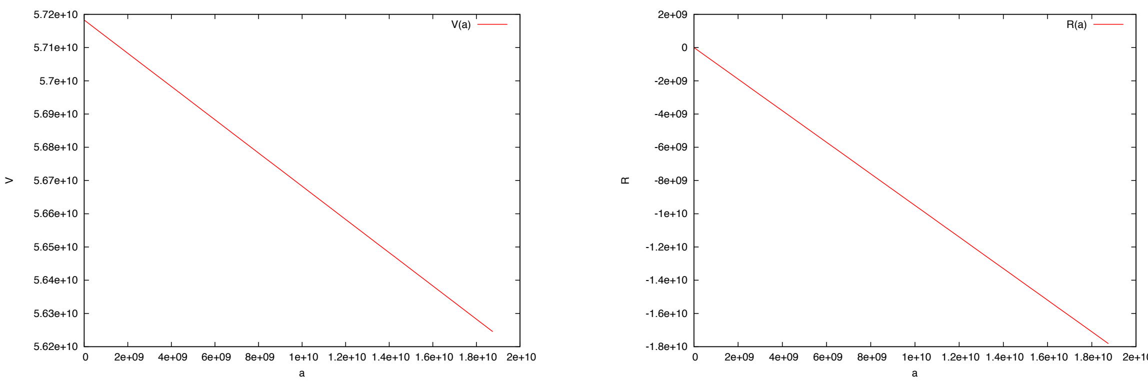
As before, Table 5.5 again presents detailed results at three key capital levels. Although parameters have changed, the company again rarely exercises the option to default, which has a probability of occurrence of 0.002% even at low levels of capitalization. In most circumstances, the firm chooses to raise emergency financing when claims exceed cash resources, which happens as much as 3.65% of the time (at zero capitalization).

In contrast to the base case, the "shadow cost" of capital τ^* now is considerably higher than before. To some extent, this originates from the different cost parameters. In particular, the cost of raising emergency capital now is $\xi = 75\%$ and the carrying cost $\tau = 5\%$. However, in addition to higher costs, another aspect is that given the more profitable premium function, it now is optimal to write more business requiring a higher level of capital—which in turn leads to higher capital costs. Essentially, the marginal pricing condition (4.16) requires marginal cost to equal marginal return/profit—and the point where the two sides align now is at a higher level.

5.3.3 Empty Company

Figure 5.6 presents the value function and the optimal raising decision for the "empty company." Figure 5.7 plots the corresponding optimal exposures to the different business lines.

We call this case the "empty company" because it is optimal to run the company without any capital. This can be seen from Figure 5.6, which shows that the total continuation value of the company is decreasing in capital and that the optimal policy is to shed any and all accumulated capital through dividends. The optimal portfolio is thus, as can be seen in Figure 5.7, always the same—corresponding to the portfolio chosen when $a = 0$. Again, note that there is an optimal scale in this case, because greater size is associated with a compression in margins.



(a) Value function $V(a)$

(b) Optimal raising decision R^b

Figure 5.6: Value function V and optimal raising decision R^b for a company with carrying cost $\tau = 5\%$, raising costs $c^{(1)} = 7.5\%$, $c^{(2)} = 1.00\text{E-}10$, and $\xi = 20\%$, interest rate $r = 3\%$, and premium parameters $\alpha = 0.973046$, $\beta = 550.203$, and $\gamma = 1.61\text{E-}10$.

However, even though the company is always empty, it never defaults. This extreme result is produced by two key drivers—the premium function and the cost of emergency financing. As with the "profitable company," the premium function is extremely profitable in expectation. Because of these high margins, staying in business is extremely valuable. Usually, the premiums collected are sufficient to cover losses. When they are not, which happens about 12% of the time, the company resorts to emergency financing. This happens because, in contrast to the "profitable company," emergency financing is relatively cheap

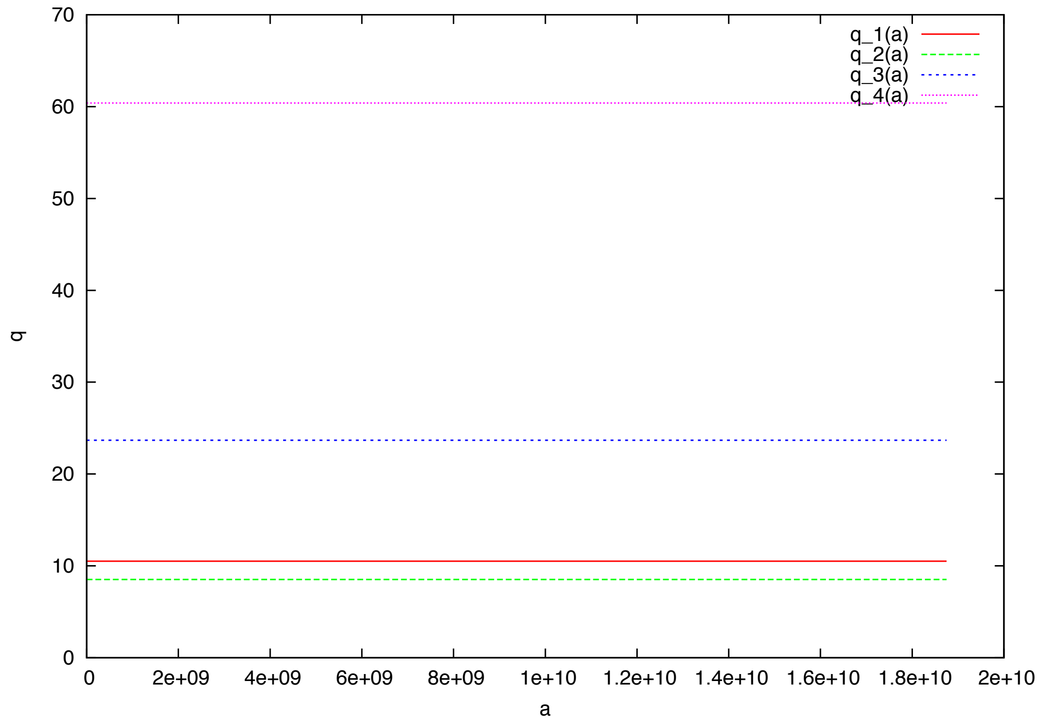


Figure 5.7: Optimal portfolio weights q_1, q_2, q_3, and q_4 for a company with carrying cost $\tau = 5\%$, raising costs $c^{(1)} = 7.5\%$, $c^{(2)} = 1.00\text{E-}10$, and $\xi = 20\%$, interest rate $r = 3\%$, and premium parameters $\alpha = 0.973046$, $\beta = 550.203$, and $\gamma = 1.61\text{E-}10$.

at 20% (versus 75% in the "profitable company" case). Thus, it makes sense for the company to forego the certain cost of holding capital—the primary benefit of which is to lessen the probability of having to resort to emergency financing—and instead just endure the emergency cost whenever it has to be incurred. In numbers, the cost of holding capital at $a = 0$ is $\tau \times \mathbb{P}(I \leq S) = 4.38\%$, whereas the cost of raising emergency funds is $\frac{\xi}{1-\xi}\mathbb{P}(I > S) = 3.08\%$.

5.4 The Marginal Cost of Risk and Capital Allocation

We now contemplate the marginal cost of risk and the allocation of capital within the three example setups. The solutions presented above suggest some immediate difficulties in applying traditional capital allocation methods to price risk. For example, consider the extremely unrealistic (but pedagogically important) example of the "empty company," where it is optimal to hold zero capital. Zero capital would imply zero risk penalties under typical allocation methods. However, risk clearly extends beyond actuarial values, as the insurer, even though it shuns default, may have to raise emergency financing to dig out of a hole. This example thus raises a key question: If we are to use an risk allocation approach for pricing, *what* exactly we should be allocating? Clearly, the traditional accounting capital, which is isomorphic with risk in single period models featuring risk measure constraints, is no longer adequate: After all, in the case of the "empty company," it is zero.

Equation (4.16) suggests that we need a broader conception of what capital is and in fact indicates that the correct quantity to allocate is D, which represents all financial resources currently held by the firm (capital and premiums) *and* the maximum amount of emergency financing that it is willing to raise in case of distress. The correct allocation of D, as well as allocations obtained from a variety of current allocation approaches, are presented for the various capitalization levels of the "base case" in Tables 5.6 through 5.8. Corresponding results for the "profitable company" are presented in Tables A.3, A.4, and A.5 in Appendix A.

If we focus on percentage allocations to line, allocation using traditional methods comes close to the correct allocation in a number of circumstances. Value-at-Risk allocations centered on the point of default, for example, are generally very close to the correct allocations as suggested by the theory. Allocations according to the expected value ("ExpVal") also tend to be close in terms of the relative risk weightings assigned to the lines. TVaR allocations are often, though not always, close—depending on the specification and on the threshold.

However, the problems suggested by Equation (4.16) go much deeper than relative weightings. Even if we identify the correct quantity to allocate, Equation (4.16) shows that the marginal cost of risk goes far beyond that obtained from a simple allocation of D in two respects. First, calculating the cost of "capital" when allocating D is not straightforward: The theoretical analysis indicates that the key quantity is $\mathbb{P}(I \geq D) + \tau^*$, where τ^* is defined as a marginal cost of raising capital, net of the benefits it provides, as defined by Equation (4.17). Second, the marginal cost of risk involves terms connected to the scale of the company, the continuation value of the company, and the costs associated with emergency capital raises that lie outside any allocation of capital, whether broadly defined or otherwise: These additional terms are not difficult to allocate to line in theory (as their allocation follows from actuarial values or conditional actuarial values), but 1) they are not ones typically considered in insurance pricing practice and 2) they are not embedded in the risk penalty emerging from a traditional allocation.

Tables 5.9, 5.10, and 5.11 show the cost allocation decomposition for the various scenarios in the three cases.[4]

The component corresponding to the allocation of D—identified as "Capital Cost (v)" in the tables—varies considerably in terms of its importance in the total cost picture. Restricting our attention to parts (ii) through (v), which are the cost components other

[4] In the tables we are taking the product of marginal cost and total exposure quantity, so it is worth noting that the sum total will not add up to total costs because of nonlinearities in the cost function.

	Line 1	Line 2	Line 3	Line 4	Aggregate
Correct Allocation	149,175,009	730,716,509	232,234,592	381,467,263	1,493,593,373
	9.99%	48.92%	15.55%	25.54%	100.00%
$\text{VaR}_{\mathbb{P}(I>D)}$ (bell)	148,255,950	658,888,004	258,908,109	426,676,536	1,492,728,598
	9.93%	44.14%	17.34%	28.58%	100.00%
$\text{VaR}_{\mathbb{P}(I>D)}$ (simple)	153,267,910	655,303,241	263,136,489	421,593,655	1,493,301,295
	10.26%	43.88%	17.62%	28.23%	100.00%
$\text{VaR}_{\mathbb{P}(I>S)}$ (bell)	60,284,530	282,096,680	78,152,633	129,837,379	550,371,222
	10.95%	51.26%	14.20%	23.59%	100.00%
$\text{VaR}_{\mathbb{P}(I>a)}$ (bell)	7,689,462	34,310,013	9,235,134	20,716,784	71,951,393
	10.69%	47.68%	12.84%	28.79%	100.00%
$\text{TVaR}_{\mathbb{P}(I>D)}$	6,420,436	1,424,420,512	86,571,987	365,933,109	1,883,346,043
	0.34%	75.63%	4.60%	19.43%	100.00%
$\text{TVaR}_{\mathbb{P}(I>S)}$	53,954,170	337,608,266	97,876,124	214,796,846	704,235,407
	7.66%	47.94%	13.90%	30.50%	100.00%
$\text{TVaR}_{\mathbb{P}(I>a)}$	18,220,905	96,769,504	29,767,618	54,539,455	199,297,482
	9.14%	48.56%	14.94%	27.37%	100.00%
CovWBeta ($\beta = 2$)	43,787,166	247,492,845	72,036,476	139,593,820	502,910,307
	8.71%	49.21%	14.32%	27.76%	100.00%
MyersRead (D)	-47,443,233	1,138,355,651	-1,425,462	204,706,472	1,294,193,428
	-3.67%	87.96%	-0.11%	15.82%	100.00%
MyersRead (S)	21,686,773	166,239,162	45,160,653	118,212,931	351,299,518
	6.17%	47.32%	12.86%	33.65%	100.00%
MyersRead (a)	-18,220,905	-96,769,504	-29,767,618	-54,539,455	-199,297,482
	9.14%	48.56%	14.94%	27.37%	100.00%
$\text{RTVaR}_{\mathbb{P}(I>S)}$ ($\beta = 2$)	75,562,556	510,094,422	152,856,860	264,660,684	1,003,174,523
	7.53%	50.85%	15.24%	26.38%	100.00%
$\text{RTVaR}_{\mathbb{P}(I>a)}$ ($\beta = 2$)	37,431,008	257,760,951	81,138,377	144,113,111	520,443,447
	7.19%	49.53%	15.59%	27.69%	100.00%
AvgTVaR	26,198,504	619,599,427	71,405,243	211,756,470	928,959,644
	2.82%	66.70%	7.69%	22.80%	100.00%
ExpVal	18,220,905	96,769,504	29,767,618	54,539,455	199,297,482
	9.14%	48.56%	14.94%	27.37%	100.00%

Table 5.6: Capital allocations in the base case, $a = 0$.

	Line 1	Line 2	Line 3	Line 4	Aggregate
Correct Allocation	217,683,368	1,135,677,070	349,208,277	646,896,471	2,349,465,186
	9.27%	48.34%	14.86%	27.53%	100.00%
$\text{VaR}_{\mathbb{P}(I>D)}$ (bell)	189,930,440	1,120,941,130	358,864,552	673,023,611	2,342,759,733
	8.11%	47.85%	15.32%	28.73%	100.00%
$\text{VaR}_{\mathbb{P}(I>D)}$ (simple)	318,656,623	790,497,728	482,903,474	757,407,881	2,349,465,707
	13.56%	33.65%	20.55%	32.24%	100.00%
$\text{VaR}_{\mathbb{P}(I>S)}$ (bell)	104,913,687	689,427,368	212,102,988	397,041,508	1,403,485,552
	7.48%	49.12%	15.11%	28.29%	100.00%
$\text{VaR}_{\mathbb{P}(I>a)}$ (bell)	64,610,209	419,263,617	138,669,493	371,631,693	994,175,011
	6.50%	42.17%	13.95%	37.38%	100.00%
$\text{TVaR}_{\mathbb{P}(I>D)}$	10,101,520	2,241,089,988	136,207,073	575,735,990	2,963,134,571
	0.34%	75.63%	4.60%	19.43%	100.00%
$\text{TVaR}_{\mathbb{P}(I>S)}$	135,605,382	843,389,578	271,352,094	402,104,339	1,652,451,393
	8.21%	51.04%	16.42%	24.33%	100.00%
$\text{TVaR}_{\mathbb{P}(I>a)}$	98,906,139	609,566,595	174,086,637	347,485,834	1,230,045,205
	8.04%	49.56%	14.15%	28.25%	100.00%
CovWBeta ($\beta = 2$)	68,892,108	389,388,848	113,337,863	219,628,132	791,246,950
	8.71%	49.21%	14.32%	27.76%	100.00%
MyersRead (D)	-74,653,968	1,790,962,760	-2,258,578	322,043,179	2,036,093,393
	-3.67%	87.96%	-0.11%	15.82%	100.00%
MyersRead (S)	84,475,324	571,843,375	187,820,551	249,060,233	1,093,199,483
	7.73%	52.31%	17.18%	22.78%	100.00%
MyersRead (a)	49,206,418	345,616,751	92,891,844	198,723,054	686,438,067
	7.17%	50.35%	13.53%	28.95%	100.00%
$\text{RTVaR}_{\mathbb{P}(I>S)}$ ($\beta = 2$)	169,603,332	1,114,766,034	357,856,176	480,557,523	2,122,783,065
	7.99%	52.51%	16.86%	22.64%	100.00%
$\text{RTVaR}_{\mathbb{P}(I>a)}$ ($\beta = 2$)	129,130,587	862,858,690	254,910,745	488,415,718	1,735,315,741
	7.44%	49.72%	14.69%	28.15%	100.00%
AvgTVaR	81,537,680	1,231,348,720	193,881,935	441,775,388	1,948,543,723
	4.18%	63.19%	9.95%	22.67%	100.00%
ExpVal	28,667,653	152,250,803	46,834,551	85,808,926	313,561,933
	9.14%	48.56%	14.94%	27.37%	100.00%

Table 5.7: Capital allocations in the base case, $a = 1,000,000,000$.

	Line 1	Line 2	Line 3	Line 4	Aggregate
Correct Allocation	328,403,144	1,724,946,618	527,156,732	980,870,637	3,561,377,132
	9.22%	48.43%	14.80%	27.54%	100.00%
$\text{VaR}_{\mathbb{P}(I>D)}$ (bell)	319,534,978	1,622,723,777	581,996,230	1,027,979,717	3,552,234,701
	9.00%	45.68%	16.38%	28.94%	100.00%
$\text{VaR}_{\mathbb{P}(I>D)}$ (simple)	365,172,764	1,561,311,149	626,943,235	1,004,479,808	3,557,906,956
	10.26%	43.88%	17.62%	28.23%	100.00%
$\text{VaR}_{\mathbb{P}(I>S)}$ (bell)	71,944,639	1,695,239,957	439,712,073	403,382,905	2,610,279,573
	2.76%	64.94%	16.85%	15.45%	100.00%
$\text{VaR}_{\mathbb{P}(I>a)}$ (bell)	364,712,276	1,559,342,958	626,152,829	1,003,213,664	3,553,421,727
	10.26%	43.88%	17.62%	28.23%	100.00%
$\text{TVaR}_{\mathbb{P}(I>D)}$	15,297,190	3,393,793,111	206,264,519	871,864,211	4,487,219,031
	0.34%	75.63%	4.60%	19.43%	100.00%
$\text{TVaR}_{\mathbb{P}(I>S)}$	244,649,099	1,547,523,123	558,781,234	638,042,037	2,988,995,494
	8.18%	51.77%	18.69%	21.35%	100.00%
$\text{TVaR}_{\mathbb{P}(I>a)}$	15,297,190	3,393,793,111	206,264,519	871,864,211	4,487,219,031
	0.34%	75.63%	4.60%	19.43%	100.00%
CovWBeta ($\beta = 2$)	104,326,342	589,671,032	171,632,529	332,593,179	1,198,223,083
	8.71%	49.21%	14.32%	27.76%	100.00%
MyersRead (D)	-113,061,175	2,712,093,947	-3,435,431	487,657,416	3,083,254,757
	-3.67%	87.96%	-0.11%	15.82%	100.00%
MyersRead (S)	167,062,113	1,135,465,965	432,026,838	405,805,930	2,140,360,847
	7.81%	53.05%	20.18%	18.96%	100.00%
MyersRead (a)	-72,659,861	2,926,661,514	62,568,475	608,588,057	3,525,158,185
	-2.06%	83.02%	1.77%	17.26%	100.00%
$\text{RTVaR}_{\mathbb{P}(I>S)}$ ($\beta = 2$)	296,132,765	1,958,484,860	689,777,132	756,846,526	3,701,241,284
	8.00%	52.91%	18.64%	20.45%	100.00%
$\text{RTVaR}_{\mathbb{P}(I>a)}$ ($\beta = 2$)	61,066,762	3,777,367,806	328,659,365	1,085,280,442	5,252,374,375
	1.16%	71.92%	6.26%	20.66%	100.00%
AvgTVaR	91,747,826	2,778,369,782	323,770,091	793,923,486	3,987,811,185
	2.30%	69.67%	8.12%	19.91%	100.00%
ExpVal	43,412,728	230,560,901	70,923,674	129,944,511	474,841,815
	9.14%	48.56%	14.94%	27.37%	100.00%

Table 5.8: Capital allocations in the base case, $a = 4,000,000,000$.

$a = 0$	Line 1	Line 2	Line 3	Line 4	Aggregate
	Line 1	Line 2	Line 3	Line 4	Aggregate
Solvent payments, (i)	23,345,530	135,002,000	18,657,049	10,772,967	199,259,815
($\mathbb{E}[L^{(i)} \mathbf{1}_{\{I<D\}}]$)	9.14%	48.55%	14.94%	27.37%	100.00%
Scale effect, (ii)	1,054,415	6,099,197	842,700	486,629	9,001,325
($\frac{\gamma}{1-c_1'(R_b)} \mathbb{E}[L^{(i)}] \sum_k p^{(k)}$)	9.14%	48.56%	14.94%	27.37%	100.00%
Continuation value, (iii)	1,787,494	10,069,585	1,395,397	781,737	14,794,219
($\mathbb{E}[L^{(i)} \mathbf{1}_{\{I<S\}} V']$)	9.43%	48.77%	15.05%	26.75%	100.00%
Raising cost, (iv)	3,136,921	21,340,216	2,782,890	1,924,195	31,920,536
($\frac{\xi}{1-\xi} \mathbb{E}[L^{(i)} \mathbf{1}_{\{S<I<D\}}]$)	7.67%	47.91%	13.91%	30.51%	100.00%
Capital cost, (v)	6,423,322	34,269,301	4,891,903	2,532,602	50,194,848
($\mathbb{E}[L^{(i)} \mid I = D] \times [\mathbb{P}(I > D) + \tau^*]$)	9.99%	48.92%	15.55%	25.54%	100.00%
Cost, (iii)-(v)	11,347,737	65,679,102	9,070,189	5,238,533	96,909,604
	9.14%	48.57%	14.93%	27.36%	100.00%
Non payments, (ii)-(v)	12,402,152	71,778,299	9,912,889	5,725,162	105,910,928
	9.14%	48.56%	14.93%	27.36%	100.00%

$a = 1,250,000,000$	Line 1	Line 2	Line 3	Line 4	Aggregate
	Line 1	Line 2	Line 3	Line 4	Aggregate
Solvent payments, (i)	23,345,530	135,002,000	18,657,049	10,772,967	313,502,671
($\mathbb{E}[L^{(i)} \mathbf{1}_{\{I<D\}}]$)	9.14%	48.55%	14.94%	27.37%	100.00%
Scale effect, (ii)	1,475,632	8,535,701	1,179,341	681,028	19,819,580
($\frac{\gamma}{1-c_1'(R_b)} \mathbb{E}[L^{(i)}] \sum_k p^{(k)}$)	9.14%	48.56%	14.94%	27.37%	100.00%
Continuation value, (iii)	557,566	3,411,166	442,838	281,845	7,886,781
($\mathbb{E}[L^{(i)} \mathbf{1}_{\{I<S\}} V']$)	8.68%	48.76%	14.09%	28.46%	100.00%
Raising cost, (iv)	501,193	3,356,439	489,699	227,776	7,442,867
($\frac{\xi}{1-\xi} \mathbb{E}[L^{(i)} \mathbf{1}_{\{S<I<D\}}]$)	8.27%	50.84%	16.52%	24.37%	100.00%
Capital cost, (v)	5,917,572	33,625,361	4,643,976	2,711,437	78,428,268
($\mathbb{E}[L^{(i)} \mid I = D] \times [\mathbb{P}(I > D) + \tau^*]$)	9.27%	48.34%	14.86%	27.53%	100.00%
Cost, (iii)-(v)	6,976,331	40,392,966	5,576,513	3,221,059	93,757,915
	9.14%	48.57%	14.93%	27.36%	100.00%
Non payments, (ii)-(v)	8,451,962	48,928,667	6,755,854	3,902,086	113,577,496
	9.14%	48.57%	14.93%	27.36%	100.00%

$a = 4,000,000,000$	Line 1	Line 2	Line 3	Line 4	Aggregate
Solvent payments, (i)	23,345,530	135,002,000	18,657,049	10,772,967	474,752,070
($\mathbb{E}[L^{(i)} \mathbf{1}_{\{I<D\}}]$)	9.14%	48.55%	14.94%	27.37%	100.00%
Scale effect, (ii)	2,080,451	12,034,241	1,662,719	960,161	42,315,511
($\frac{\gamma}{1-c_1'(R_b)} \mathbb{E}[L^{(i)}] \sum_k p^{(k)}$)	9.14%	48.56%	14.94%	27.37%	100.00%
Continuation value, (iii)	-467,409	-2,524,447	-361,943	-198,677	-8,951,208
($\mathbb{E}[L^{(i)} \mathbf{1}_{\{I<S\}} V']$)	9.71%	48.15%	15.37%	26.77%	100.00%
Raising cost, (iv)	165,605	1,102,308	184,135	65,213	3,676,390
($\frac{\xi}{1-\xi} \mathbb{E}[L^{(i)} \mathbf{1}_{\{S<I<D\}}]$)	8.38%	51.19%	19.04%	21.39%	100.00%
Capital cost, (v)	4,479,576	25,627,009	3,517,682	2,062,943	90,335,366
($\mathbb{E}[L^{(i)} \mid I = D] \times [\mathbb{P}(I > D) + \tau^*]$)	9.22%	48.43%	14.80%	27.54%	100.00%
Cost, (iii)-(v)	4,177,771	24,204,869	3,339,874	1,929,479	85,060,548
	9.13%	48.58%	14.93%	27.36%	100.00%
Non payments, (ii)-(v)	6,258,222	36,239,111	5,002,593	2,889,640	127,376,059
	9.14%	48.57%	14.93%	27.36%	100.00%

Table 5.9: Cost allocation in the base case.

$a = 0$	Line 1	Line 2	Line 3	Line 4	Aggregate
Solvent payments, (i)	23,345,530	135,002,000	18,657,049	10,772,967	1,227,669,151
($\mathbb{E}[L^{(i)}\, \mathbf{1}_{\{I<D\}}]$)	9.14%	48.55%	14.94%	27.37%	100.00%
Scale effect, (ii)	12,171,966	70,408,002	9,727,968	5,617,558	640,202,514
($\frac{\gamma}{1-c_1'(R_b)}\, \mathbb{E}[L^{(i)}] \sum_k p^{(k)}$)	9.14%	48.56%	14.94%	27.37%	100.00%
Continuation value, (iii)	1,029,167	5,837,049	782,958	436,750	52,036,839
($\mathbb{E}[L^{(i)}\, \mathbf{1}_{\{I<S\}}\, V']$)	9.51%	49.52%	14.79%	26.18%	100.00%
Raising cost, (iv)	7,837,319	53,467,331	6,897,003	4,951,601	495,968,512
($\frac{\xi}{1-\xi}\, \mathbb{E}[L^{(i)}\, \mathbf{1}_{\{S<I<D\}}]$)	7.60%	47.60%	13.67%	31.14%	100.00%
Capital cost, (v)	17,186,342	91,435,424	13,142,696	6,636,818	822,504,916
($\mathbb{E}[L^{(i)} \vert I = D] \times [\mathbb{P}(I > D) + \tau^*]$)	10.05%	49.08%	15.71%	25.17%	100.00%
Cost, (iii)-(v)	26,052,827	150,739,804	20,822,658	12,025,168	1,370,510,267
	9.14%	48.56%	14.93%	27.36%	100.00%
Non payments, (ii)-(v)	38,224,794	221,147,806	30,550,626	17,642,726	2,010,712,781
	9.14%	48.56%	14.94%	27.36%	100.00%

$a = 3,000,000,000$	Line 1	Line 2	Line 3	Line 4	Aggregate
Solvent payments, (i)	23,345,530	135,002,000	18,657,049	10,772,967	1,568,829,904
($\mathbb{E}[L^{(i)}\, \mathbf{1}_{\{I<D\}}]$)	9.14%	48.55%	14.94%	27.37%	100.00%
Scale effect, (ii)	12,749,807	73,750,484	10,189,785	5,884,241	856,948,543
($\frac{\gamma}{1-c_1'(R_b)}\, \mathbb{E}[L^{(i)}] \sum_k p^{(k)}$)	9.14%	48.56%	14.94%	27.37%	100.00%
Continuation value, (iii)	-235,099	-1,116,997	-202,680	-67,987	-13,002,123
($\mathbb{E}[L^{(i)}\, \mathbf{1}_{\{I<S\}}\, V']$)	11.11%	48.47%	19.58%	20.84%	100.00%
Raising cost, (iv)	2,579,077	18,401,683	2,557,559	1,239,091	201,178,046
($\frac{\xi}{1-\xi}\, \mathbb{E}[L^{(i)}\, \mathbf{1}_{\{S<I<D\}}]$)	7.88%	51.61%	15.97%	24.55%	100.00%
Capital cost, (v)	12,029,113	65,894,390	9,133,206	5,463,678	778,163,393
($\mathbb{E}[L^{(i)} \vert I = D] \times [\mathbb{P}(I > D) + \tau^*]$)	9.50%	47.78%	14.74%	27.98%	100.00%
Cost, (iii)-(v)	14,373,091	83,179,077	11,488,085	6,634,782	966,339,316
	9.14%	48.56%	14.93%	27.36%	100.00%
Non payments, (ii)-(v)	27,122,898	156,929,561	21,677,870	12,519,023	1,823,287,859
	9.14%	48.56%	14.93%	27.36%	100.00%

$a = 12,000,000,000$	Line 1	Line 2	Line 3	Line 4	Aggregate
	Line 1	Line 2	Line 3	Line 4	Aggregate
Solvent payments, (i)	23,345,530	135,002,000	18,657,049	10,772,967	1,995,399,708
($\mathbb{E}[L^{(i)}\, \mathbf{1}_{\{I<D\}}]$)	9.14%	48.55%	14.94%	27.37%	100.00%
Scale effect, (ii)	14,236,612	82,350,817	11,378,055	6,570,425	1,217,059,553
($\frac{\gamma}{1-c_1'(R_b)}\, \mathbb{E}[L^{(i)}] \sum_k p^{(k)}$)	9.14%	48.56%	14.94%	27.37%	100.00%
Continuation value, (iii)	-861,553	-4,764,429	-676,234	-373,605	-70,665,323
($\mathbb{E}[L^{(i)}\, \mathbf{1}_{\{I<S\}}\, V']$)	9.53%	48.38%	15.29%	26.80%	100.00%
Raising cost, (iv)	1,063,664	8,031,264	1,171,071	494,784	109,736,215
($\frac{\xi}{1-\xi}\, \mathbb{E}[L^{(i)}\, \mathbf{1}_{\{S<I<D\}}]$)	7.58%	52.52%	17.05%	22.86%	100.00%
Capital cost, (v)	6,522,382	35,669,393	4,880,405	2,983,652	536,155,774
($\mathbb{E}[L^{(i)} \vert I = D] \times [\mathbb{P}(I > D) + \tau^*]$)	9.51%	47.74%	14.54%	28.21%	100.00%
Cost, (iii)-(v)	6,724,493	38,936,228	5,375,242	3,104,832	575,226,667
	9.14%	48.57%	14.93%	27.36%	100.00%
Non payments, (ii)-(v)	20,961,105	121,287,045	16,753,298	9,675,257	1,792,286,220
	9.14%	48.56%	14.93%	27.36%	100.00%

Table 5.10: Cost allocation in the profitable company.

$a = 0$	Line 1	Line 2	Line 3	Line 4	Aggregate
Solvent payments, (i)	23,345,695	135,041,756	18,658,134	10,774,413	2,487,582,817
($\mathbb{E}[L^{(i)}\, 1_{\{I<D\}}]$)	9.86%	46.23%	17.75%	26.16%	100.00%
Scale effect, (ii)	17,919,327	103,653,262	14,321,322	8,270,057	1,909,380,339
($\frac{\gamma}{1-c_1'(R_b)}\, \mathbb{E}[L^{(i)}] \sum_k p^{(k)}$)	9.86%	46.23%	17.75%	26.16%	100.00%
Continuation value, (iii)	-782,453	-4,526,226	-625,396	-361,120	-83,376,677
($\mathbb{E}[L^{(i)}\, 1_{\{I<S\}}\, V']$)	9.86%	46.23%	17.75%	26.16%	100.00%
Raising cost, (iv)	1,924,160	11,129,311	1,537,555	888,004	205,012,321
($\frac{\xi}{1-\xi}\, \mathbb{E}[L^{(i)}\, 1_{\{S<I<D\}}]$)	9.86%	46.23%	17.75%	26.16%	100.00%
Capital cost, (v)	na	na	na	na	na
($\mathbb{E}[L^{(i)} \vert I = D] \times [\mathbb{P}(I > D) + \tau^*]$)	na	na	na	na	na
Cost, (iii)-(v)	1,141,707	6,603,085	912,159	526,885	121,635,644
	9.86%	46.23%	17.75%	26.16%	100.00%
Non payments, (ii)-(v)	19,061,034	110,256,347	15,233,481	8,796,942	2,031,015,983
	9.86%	46.23%	17.75%	26.16%	100.00%

Table 5.11: Cost allocation in the empty company.

than claims payments, we see that the capital cost component share of non-payment costs ranges from more than 90% (in the “base case” at an initial capitalization of $a = 0$) to less than a third (in the “profitable” case with $a = 12,000,000,000$).

It is therefore evident that correct risk pricing entails an appreciation of *all* of the components of the marginal cost of risk, one of which can be obtained from an allocation of a broadly defined measure of capital. However, existing allocation methods will misprice risk if the other marginal cost components are overlooked or if capital is defined too narrowly. To get a sense of this, we calculate risk-adjusted returns on capital (RAROC) for the “base” case and the “profitable company” case under VaR, TVaR, and Myers-Read, and we vary both the definition of capital and the types of costs being considered. We compare the results to correct RAROC obtained from the model. We calculate RAROC as the marginal premium received minus costs (with the types of costs considered varying from a complete set to partial sets) divided by allocated capital. The results are presented in Tables 5.12 and 5.13.

Since the insurance portfolio has been optimized, true RAROC has been equated across all lines. If all costs are properly accounted for and an appropriately broad definition of capital (D) is used, VaR-based allocation yields RAROCs that are quite close to the true RAROCs. The results with TVaR and Myers-Read, however, do not tend to give accurate results. In particular, they allocate too little capital to lines 1 and 3 (with Myers-Read, the amounts are actually negative), while overpenalizing line 2. The misstatements are significant. For example, although the correct figure never exceeds 10 % in our scenarios, RAROC estimated under TVaR for line 1 routinely exceeds 50% and sometimes goes beyond 100%.

Things get worse if cost components are overlooked, or if an incorrect notion of capital

	Allocating	Cost considered	Line 1	Line 2	Line 3	Line 4
$a = 0$						
Correct Allocation	D	yes	3.36%	3.36%	3.36%	3.36%
VaR Allocation	D	yes	3.38%	3.73%	3.01%	3.00%
TVaR Allocation	D	yes	98.46%	2.17%	11.37%	4.42%
MyersRead	D	yes	-9.16%	1.87%	-474.45%	5.43%
VaR Allocation	D	act. only	6.53%	7.81%	6.11%	6.79%
VaR Allocation	S	act. only	16.06%	18.23%	20.24%	22.32%
VaR Allocation	a	act. only	na	na	na	na
VaR Allocation	D	act. and scale	5.97%	7.14%	5.59%	6.21%
VaR Allocation	S	act. and scale	14.69%	16.68%	18.52%	20.42%
VaR Allocation	a	act. and scale	na	na	na	na
TVaR Allocation	D	act. only	190.10%	4.55%	23.04%	9.99%
TVaR Allocation	S	act. only	22.95%	19.49%	20.67%	17.26%
TVaR Allocation	a	act. only	na	na	na	na
TVaR Allocation	D	act. and scale	173.94%	4.17%	21.08%	9.14%
TVaR Allocation	S	act. and scale	21.00%	17.83%	18.91%	15.79%
TVaR Allocation	a	act. and scale	na	na	na	na
$a = 1,000,000,000$						
Correct Allocation	D	yes	3.34%	3.34%	3.34%	3.34%
VaR Allocation	D	yes	3.83%	3.38%	3.25%	3.21%
TVaR Allocation	D	yes	90.72%	2.13%	10.79%	4.73%
MyersRead	D	yes	-8.43%	1.83%	-447.25%	5.81%
VaR Allocation	D	act. only	5.46%	4.92%	4.73%	4.62%
VaR Allocation	S	act. only	9.89%	8.00%	8.00%	7.83%
VaR Allocation	a	act. only	16.06%	13.16%	12.23%	8.36%
VaR Allocation	D	act. and scale	4.51%	4.06%	3.90%	3.81%
VaR Allocation	S	act. and scale	8.17%	6.61%	6.60%	6.46%
VaR Allocation	a	act. and scale	13.26%	10.86%	10.09%	6.90%
TVaR Allocation	D	act. only	129.58%	3.10%	15.70%	6.81%
TVaR Allocation	S	act. only	8.99%	7.68%	7.34%	9.08%
TVaR Allocation	a	act. only	12.91%	11.13%	11.98%	11.00%
TVaR Allocation	D	act. and scale	106.96%	2.56%	12.96%	5.62%
TVaR Allocation	S	act. and scale	7.42%	6.34%	6.06%	7.49%
TVaR Allocation	a	act. and scale	10.65%	9.19%	9.89%	9.08%
$a = 4,000,000,000$						
	Allocating	Cost considered	Line 1	Line 2	Line 3	Line 4
Correct Allocation	D	yes	2.54%	2.54%	2.54%	2.54%
VaR Allocation	D	yes	2.61%	2.70%	2.30%	2.42%
TVaR Allocation	D	yes	68.61%	1.62%	8.17%	3.60%
MyersRead	D	yes	-6.38%	1.40%	-337.28%	4.42%
VaR Allocation	D	act. only	3.64%	3.81%	3.27%	3.39%
VaR Allocation	S	act. only	16.18%	3.65%	4.32%	8.64%
VaR Allocation	a	act. only	3.19%	3.97%	3.04%	3.47%
VaR Allocation	D	act. and scale	2.43%	2.55%	2.18%	2.26%
VaR Allocation	S	act. and scale	10.80%	2.44%	2.89%	5.77%
VaR Allocation	a	act. and scale	2.13%	2.65%	2.03%	2.32%
TVaR Allocation	D	act. only	95.85%	2.30%	11.62%	5.04%
TVaR Allocation	S	act. only	5.44%	4.57%	3.89%	6.24%
TVaR Allocation	a	act. only	85.34%	2.05%	10.34%	4.48%
TVaR Allocation	D	act. and scale	63.99%	1.53%	7.76%	3.36%
TVaR Allocation	S	act. and scale	3.63%	3.05%	2.60%	4.17%
TVaR Allocation	a	act. and scale	56.97%	1.37%	6.90%	2.99%

Table 5.12: RAROC calculations, base case.

	Allocating	Cost considered	Line 1	Line 2	Line 3	Line 4
$a = 0$						
Correct Allocation	D	yes	8.94%	8.94%	8.94%	8.94%
VaR Allocation	D	yes	9.23%	9.90%	8.19%	7.84%
TVaR Allocation	D	yes	263.48%	5.80%	30.55%	11.58%
MyersRead	D	yes	-24.49%	4.99%	-1249.82%	14.22%
VaR Allocation	D	act. only	20.53%	23.93%	19.03%	20.83%
VaR Allocation	S	act. only	83.91%	61.61%	69.60%	38.51%
VaR Allocation	a	act. only	na	na	na	na
VaR Allocation	D	act. and scale	13.99%	16.31%	12.97%	14.20%
VaR Allocation	S	act. and scale	57.19%	41.99%	47.44%	26.25%
VaR Allocation	a	act. and scale	na	na	na	na
TVaR Allocation	D	act. only	586.01%	14.03%	71.00%	30.78%
TVaR Allocation	S	act. only	66.20%	56.01%	60.09%	48.32%
TVaR Allocation	a	act. only	na	na	na	na
TVaR Allocation	D	act. and scale	399.40%	9.56%	48.39%	20.98%
TVaR Allocation	S	act. and scale	45.12%	38.18%	40.96%	32.93%
TVaR Allocation	a	act. and scale	na	na	na	na
$3,000,000,000$						
Correct Allocation	D	yes	6.62%	6.62%	6.62%	6.62%
VaR Allocation	D	yes	7.21%	6.85%	6.08%	6.42%
TVaR Allocation	D	yes	184.42%	4.18%	21.23%	9.53%
MyersRead	D	yes	-17.14%	3.59%	-868.54%	11.71%
VaR Allocation	D	act. only	16.25%	16.32%	14.42%	14.70%
VaR Allocation	S	act. only	33.65%	27.35%	31.83%	30.88%
VaR Allocation	a	act. only	58.55%	59.79%	63.05%	62.84%
VaR Allocation	D	act. and scale	8.61%	8.65%	7.64%	7.79%
VaR Allocation	S	act. and scale	17.83%	14.50%	16.87%	16.37%
VaR Allocation	a	act. and scale	31.03%	31.69%	33.42%	33.31%
TVaR Allocation	D	act. only	415.83%	9.96%	50.38%	21.84%
TVaR Allocation	S	act. only	34.18%	27.54%	27.52%	32.73%
TVaR Allocation	a	act. only	64.22%	59.89%	64.24%	59.52%
TVaR Allocation	D	act. and scale	220.36%	5.28%	26.70%	11.58%
TVaR Allocation	S	act. and scale	18.11%	14.60%	14.58%	17.35%
TVaR Allocation	a	act. and scale	34.03%	31.74%	34.04%	31.55%
$a = 4,000,000,000$						
Correct Allocation	D	yes	3.58%	3.58%	3.58%	3.58%
VaR Allocation	D	yes	3.75%	3.77%	3.16%	3.50%
TVaR Allocation	D	yes	99.82%	2.26%	11.32%	5.20%
MyersRead	D	yes	-9.29%	1.95%	-464.11%	6.39%
VaR Allocation	D	act. only	12.05%	12.82%	10.86%	11.35%
VaR Allocation	S	act. only	19.48%	18.18%	20.07%	20.06%
VaR Allocation	a	act. only	11.41%	15.32%	11.92%	18.85%
VaR Allocation	D	act. and scale	3.87%	4.12%	3.48%	3.64%
VaR Allocation	S	act. and scale	6.25%	5.84%	6.44%	6.44%
VaR Allocation	a	act. and scale	3.66%	4.92%	3.83%	6.05%
TVaR Allocation	D	act. only	320.79%	7.68%	38.87%	16.85%
TVaR Allocation	S	act. only	23.20%	17.53%	16.80%	22.83%
TVaR Allocation	a	act. only	15.96%	14.51%	10.59%	20.03%
TVaR Allocation	D	act. and scale	102.91%	2.47%	12.47%	5.41%
TVaR Allocation	S	act. and scale	7.44%	5.63%	5.39%	7.33%
TVaR Allocation	a	act. and scale	5.12%	4.66%	3.40%	6.43%

Table 5.13: RAROC calculations, profitable company case.

is used in the allocation process. We consider such errors in the table by reporting what happens if return in the RAROC numerator is calculated only by referencing actuarial (i) or actuarial and scale costs (i) and (ii), or if a more narrow definition of capital—such as a or S—is used when constructing the numerator. In our scenarios, where the additional cost components are positive, both of these errors tend to work in the same direction to inflate estimates of RAROC. In a number of cases, the incorrectly calculated RAROC appears to indicate high levels of profitability across the board.

Chapter 6

Conclusion

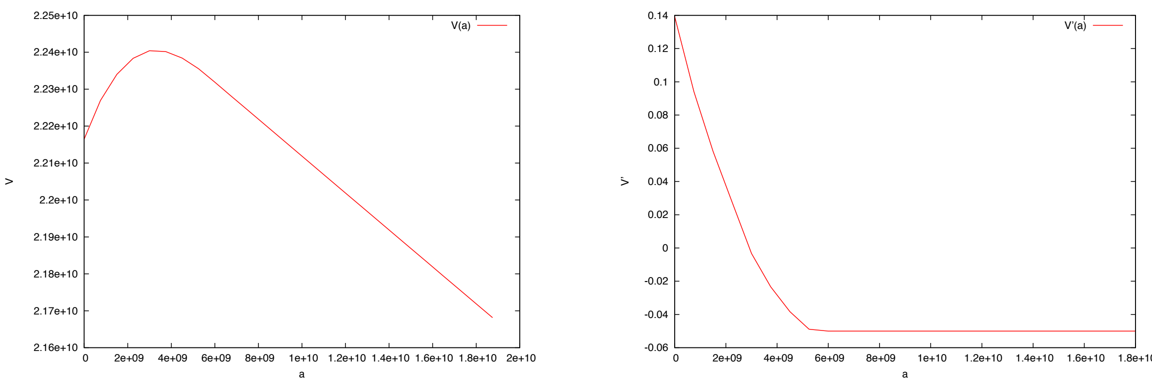
The model presented in Chapter 4 represents a step toward greater sophistication in firm valuation and risk pricing, but only a step. Other nuances—such as regulatory frictions and rating agency requirements—would obviously merit consideration in a richer model. Moreover, calibration of any model would obviously have to be tailored to the unique circumstances of each firm.

For example, different model specifications could favor different risk measures. Our setup was a relatively favorable one for VaR rather than TVaR, and part of this was rooted in how we specified the premium function. More realistic specifications would undoubtedly point the way to risk measures more complicated than either VaR or TVaR. But a practitioner faces deeper problems: Even if armed with the right risk measure and capital allocation technique, one can get highly distorted results unless the right threshold is chosen and the right costs are considered in setting hurdle rates.

These caveats are addressed only with a deeper understanding of what creates value at the level of the firm. As our models move in this direction, it is evident that greater sophisitication is bound to lead to more complication in pricing risk and measuring performance in insurance. Allocation is valid as a pricing guide only if great care is taken in its implementation and interpretation.

Appendix A

Additional Tables and Figures



(a) Value function $V(a)$

(b) Derivative $V'(a)$

Figure A.1: Value function V and its derivative V' for a company with carrying cost $\tau = 5\%$, raising costs $c^{(1)} = 7.5\%$, $c^{(2)} = 5.00\text{E-}11$, and $\xi = 75\%$, interest rate $r = 6\%$, and premium parameters $\alpha = 0.973046$, $\beta = 550.203$, and $\gamma = 1.61\text{E-}10$.

Allocation	**Line 1**	**Line 2**	**Line 3**	**Line 4**	**Sum**	**RiskMeas**
ExpVal	23,329,288	135,081,949	18,649,808	10,796,053	187,857,098	187,857,098
	12.42%	71.91%	9.93%	5.75%	100%	
CovWBeta	61,319,372	393,205,296	38,138,703	13,273,421	505,936,792	505,936,792
$\beta = 2$	12.12%	77.72%	7.54%	2.62%	100%	
CovWBeta/RMK	62,421,630	399,587,622	39,019,865	13,783,511	514,812,628	514,812,628
	12.13%	77.62%	7.58%	2.68%	100%	
$\text{TVaR}_{75\%}$	51,175,628	301,536,000	34,962,680	13,584,693	401,259,001	401,259,001
	12.75%	75.15%	8.71%	3.39%	100%	
$\text{TVaR}_{90\%}$	73,065,626	462,257,299	42,467,260	12,320,392	590,110,577	590,110,577
	12.38%	78.33%	7.20%	2.09%	100%	
$\text{TVaR}_{95\%}$	85,919,276	595,845,552	44,645,301	11,854,265	738,264,394	738,264,394
	11.64%	80.71%	6.05%	1.61%	100%	
$\text{TVaR}_{99\%}$	105,723,179	870,554,590	58,366,883	11,061,400	1,045,706,051	1,045,706,051
	10.11%	83.25%	5.58%	1.06%	100%	
$\text{VaR}_{95\%}$ (simple)	9,790,141	506,240,268	7,441,716	4,978,026	528,450,152	528,450,152
	1.85%	95.80%	1.41%	0.94%	100%	
$\text{VaR}_{95\%}$ (bell)	62,650,462	406,650,419	50,921,868	8,262,860	528,485,609	528,485,609
	11.85%	76.95%	9.64%	1.56%	100%	
$\text{VaR}_{99\%}$ (simple)	9,542,646	832,379,600	19,010,865	5,915,580	866,848,691	866,848,691
	1.10%	96.02%	2.19%	0.68%	100%	
$\text{VaR}_{99\%}$ (bell)	95,848,173	726,083,642	30,395,988	14,471,275	866,799,078	866,799,078
	11.06%	83.77%	3.51%	1.67%	100%	
Exponential Alloc.	27,278,567	163,314,603	20,467,020	10,785,671	221,845,861	221,845,861
$c = 0.1$	12.30%	73.62%	9.23%	4.86%	100%	
Exponential Alloc'	36,470,857	241,091,476	22,984,852	8,563,043	309,110,227	309,110,227
$c = 0.25$	11.80%	78.00%	7.44%	2.77%	100%	
Exponential Alloc.	-3,645,886,606	23,549,065,241	-4,852,942,072	-3,243,983,432	11,806,253,131	11,806,253,131
$c = 1$	-30.88%	199.46%	-41.10%	-27.48%	100%	
Wang	27,318,178	161,115,903	20,879,150	11,242,753	220,555,984	220,555,984
$\lambda = 0.25$	12.39%	73.05%	9.47%	5.10%	100%	
Wang	32,716,629	197,421,588	23,801,554	11,779,604	265,719,375	265,719,375
$\lambda = 0.5$	12.31%	74.30%	8.96%	4.43%	100%	
Wang	39,056,077	242,148,639	27,017,016	12,168,327	320,390,060	320,390,060
$\lambda = 0.75$	12.19%	75.58%	8.43%	3.80%	100%	
MyersRead, $a =$	120,916,916	1,006,167,989	50,620,099	-7,918,138	1,169,786,866	1,169,786,866
1,357,643,965	10.34%	86.01%	4.33%	-0.68%	100%	
MyersRead, $a =$	66,203,197	755,703,821	37,250,753	-8,601,447	850,556,325	850,556,325
1,038,413,423	7.78%	88.85%	4.38%	-1.01%	100%	
MyersRead, $a =$	60,182,247	606,862,111	21,960,730	-10,013,495	678,991,592	678,991,592
867,358,356	8.86%	89.38%	3.23%	-1.47%	100%	
Esscher, $t =$	8,226,240	1,987,777,539	54,262,714	72,291,051	2,122,557,544	2,122,557,544
1.E-07	0.39%	93.65%	2.56%	3.41%	100%	
Esscher, t=	27,344,590	163,247,072	20,684,092	11,047,543	222,323,297	222,323,297
1.E-09	12.30%	73.43%	9.30%	4.97%	100%	
Kamps, t=	26,634,429	155,054,747	20,715,324	11,428,757	213,833,257	213,833,257
1.E-08	12.46%	72.51%	9.69%	5.34%	100%	
Kamps, t=	40,174,490	249,023,141	27,429,888	12,085,777	328,713,296	328,713,296
1.E-11	12.22%	75.76%	8.34%	3.68%	100%	
D'Arcy, $a =$	56,585,273	417,182,061	25,843,237	5,606,089	505,216,661	505,216,661
1,357,643,965	11.20%	82.57%	5.12%	1.11%	100%	
D'Arcy, $a =$	106,249,613	882,347,465	62,762,070	11,376,773	1,062,735,922	1,062,735,922
1,038,413,423	10.00%	83.03%	5.91%	1.07%	100%	
Bodoff,	47,746,418	273,299,373	36,568,471	18,636,590	376,250,851	376,250,851
$\text{VaR}_{90\%}$	12.69%	72.64%	9.72%	4.95%	100%	
Bodoff,	66,858,358	393,031,893	46,939,467	21,620,434	528,450,152	528,450,152
$\text{VaR}_{95\%}$	12.65%	74.37%	8.88%	4.09%	100%	
Bodoff,	104,399,510	670,161,589	66,169,820	26,117,771	866,848,691	866,848,691
$\text{VaR}_{99\%}$	12.04%	77.31%	7.63%	3.01%	100%	
RTVaR	93,578,301	654,516,995	50,094,493	11,340,089	809,529,877	809,529,877
$\alpha = 75\%,\ \beta = 2$	11.56%	80.85%	6.19%	1.40%	100%	
RTVaR	107,311,793	824,901,649	53,570,493	11,367,332	997,151,267	997,151,267
$\alpha = 90\%,\ \beta = 2$	10.76%	82.73%	5.37%	1.14%	100%	
RTVaR	113,722,457	936,385,190	61,465,579	11,542,358	1,123,115,583	1,123,115,583
$\alpha = 95\%,\ \beta = 2$	10.13%	83.37%	5.47%	1.03%	100%	
AvgTVaR	78,970,927	557,548,360	45,110,531	12,205,188	693,835,006	693,835,006
	11.38%	80.36%	6.50%	1.76%	100%	

Table A.1: Agg 3 Allocations when Removing 1000 Arbitrary Samples

Allocation	**Line 1**	**Line 2**	**Line 3**	**Line 4**	**Sum**	**RiskMeas**
ExpVal	23,320,023	135,041,756	18,658,134	10,774,413	187,794,326	187,807,891
	12.42%	71.91%	9.94%	5.74%	100%	
CovWBeta	61,811,437	398,728,060	39,014,221	13,826,768	513,380,486	513,362,750
$\beta = 2$	12.04%	77.67%	7.60%	2.69%	100%	
CovWBeta/RMK	61,811,437	398,708,430	39,006,992	13,822,327	513,349,186	513,362,750
	12.04%	77.67%	7.60%	2.69%	100%	
$\text{TVaR}_{75\%}$	51,109,439	301,524,393	34,972,585	13,579,677	401,186,094	401,186,094
	12.74%	75.16%	8.72%	3.38%	100%	
$\text{TVaR}_{90\%}$	72,718,843	462,574,223	42,463,907	12,369,992	590,126,965	590,126,965
	12.32%	78.39%	7.20%	2.10%	100%	
$\text{TVaR}_{95\%}$	85,094,349	596,621,170	44,585,766	12,018,216	738,319,501	738,319,501
	11.53%	80.81%	6.04%	1.63%	100%	
$\text{TVaR}_{99\%}$	103,726,159	870,456,639	58,481,885	11,487,420	1,044,152,103	1,044,152,103
	9.93%	83.36%	5.60%	1.10%	100%	
$\text{VaR}_{95\%}$ (simple)	9,776,274	330,811,906	173,984,700	14,118,944	528,691,824	528,691,824
	1.85%	62.57%	32.91%	2.67%	100%	
$\text{VaR}_{95\%}$ (bell)	62,367,597	404,259,189	51,869,109	10,157,414	528,653,309	528,653,309
	11.80%	76.47%	9.81%	1.92%	100%	
$\text{VaR}_{99\%}$ (simple)	7,170,815	816,870,497	39,256,266	4,060,778	867,358,356	867,358,356
	0.83%	94.18%	4.53%	0.47%	100%	
$\text{VaR}_{99\%}$ (bell)	40,819,623	780,226,792	29,835,131	16,807,649	867,689,194	867,689,194
	4.70%	89.92%	3.44%	1.94%	100%	
Exponential Alloc.	27,743,921	166,623,092	20,911,877	11,001,079	226,279,969	226,276,726
$c = 0.1$	12.26%	73.64%	9.24%	4.86%	100%	
Exponential Alloc'	36,585,576	245,779,923	23,664,375	8,855,200	314,885,074	314,871,167
$c = 0.25$	11.62%	78.05%	7.52%	2.81%	100%	
Exponential Alloc.	-1,577,802,041	13,966,106,590	-1,766,629,859	-2,185,234,909	8,436,439,780	8,432,460,413
$c = 1$	-18.70%	165.55%	-20.94%	-25.90%	100%	
Wang	27,829,070	164,381,902	21,321,736	11,456,790	224,989,498	224,989,498
$\lambda = 0.25$	12.37%	73.06%	9.48%	5.09%	100%	
Wang	33,249,531	201,449,328	24,314,253	12,015,497	271,028,610	271,028,610
$\lambda = 0.5$	12.27%	74.33%	8.97%	4.43%	100%	
Wang	39,529,281	247,125,378	27,618,048	12,433,274	326,705,981	326,705,981
$\lambda = 0.75$	12.10%	75.64%	8.45%	3.81%	100%	
MyersRead, $a =$	80,648,896	1,034,900,995	57,824,571	-3,512,528	1,169,861,934	1,169,836,074
1,357,643,965	6.89%	88.46%	4.94%	-0.30%	100%	
MyersRead, $a =$	57,940,077	761,392,555	38,892,125	-7,593,136	850,631,621	850,605,533
1,038,413,423	6.81%	89.51%	4.57%	-0.89%	100%	
MyersRead, $a =$	58,453,734	608,292,759	22,259,838	-9,429,533	679,576,799	679,550,465
867,358,356	8.60%	89.51%	3.28%	-1.39%	100%	
Esscher, $t =$	7,199,337	1,554,281,231	82,553,803	47,811,835	1,691,846,205	1,691,846,205
1.E-07	0.43%	91.87%	4.88%	2.83%	100%	
Esscher, t=	27,242,017	163,132,734	20,702,112	11,036,495	222,113,358	222,113,358
1.E-09	12.26%	73.45%	9.32%	4.97%	100%	
Kamps, t=	26,620,099	155,031,979	20,729,329	11,408,823	213,790,229	213,790,229
1.E-08	12.45%	72.52%	9.70%	5.34%	100%	
Kamps, t=	39,962,084	249,005,484	27,461,013	12,096,360	328,524,942	328,524,942
1.E-11	12.16%	75.80%	8.36%	3.68%	100%	
D'Arcy, $a =$	35,948,479	385,874,795	26,670,954	5,927,468	454,421,695	454,421,695
1,357,643,965	7.91%	84.92%	5.87%	1.30%	100%	
D'Arcy, $a =$	90,130,123	865,840,580	63,717,184	11,941,086	1,031,628,973	1,031,628,973
1,038,413,423	8.74%	83.93%	6.18%	1.16%	100%	
Bodoff,	47,749,632	273,103,335	36,575,583	18,593,413	376,021,964	376,021,964
$\text{VaR}_{90\%}$	12.70%	72.63%	9.73%	4.94%	100%	
Bodoff,	66,837,235	393,278,916	46,978,020	21,597,652	528,691,824	528,691,824
$\text{VaR}_{95\%}$	12.64%	74.39%	8.89%	4.09%	100%	
Bodoff,	104,288,359	670,654,110	66,224,381	26,191,506	867,358,356	867,358,356
$\text{VaR}_{99\%}$	12.02%	77.32%	7.64%	3.02%	100%	
RTVaR	92,191,642	654,907,352	50,243,952	11,538,735	808,881,681	808,881,681
$\alpha = 75\%,\ \beta = 2$	11.40%	80.96%	6.21%	1.43%	100%	
RTVaR	104,309,477	825,609,089	53,918,083	11,770,668	995,607,317	995,607,317
$\alpha = 90\%,\ \beta = 2$	10.48%	82.93%	5.42%	1.18%	100%	
RTVaR	108,694,586	936,445,113	62,250,592	12,110,754	1,119,501,045	1,119,501,045
$\alpha = 95\%,\ \beta = 2$	9.71%	83.65%	5.56%	1.08%	100%	
AvgTVaR	78,162,198	557,794,106	45,126,036	12,363,826	693,446,166	693,446,166
	11.27%	80.44%	6.51%	1.78%	100%	

Table A.2: Agg 3 Allocations when Replacing Five Worst Cases by Sixth Worst

	Line 1	Line 2	Line 3	Line 4	Aggregate
Correct Allocation	924,492,199	4,515,871,816	1,445,163,636	2,315,441,323	9,200,968,974
	10.05%	49.08%	15.71%	25.17%	100.00%
$\text{VaR}_{\mathbb{P}(I>D)}$ (bell)	895,312,658	4,079,524,656	1,578,307,057	2,641,594,891	9,194,739,261
	9.74%	44.37%	17.17%	28.73%	100.00%
$\text{VaR}_{\mathbb{P}(I>D)}$ (simple)	944,306,229	4,037,420,059	1,621,222,774	2,597,500,779	9,200,449,840
	10.26%	43.88%	17.62%	28.23%	100.00%
$\text{VaR}_{\mathbb{P}(I>S)}$ (bell)	219,067,031	1,584,762,125	431,475,923	1,428,645,156	3,663,950,235
	5.98%	43.25%	11.78%	38.99%	100.00%
$\text{VaR}_{\mathbb{P}(I>a)}$ (bell)	47,144,333	207,868,587	49,844,712	126,645,156	431,502,789
	10.93%	48.17%	11.55%	29.35%	100.00%
$\text{TVaR}_{\mathbb{P}(I>D)}$	39,557,255	8,776,065,161	533,382,797	2,254,567,934	11,603,573,147
	0.34%	75.63%	4.60%	19.43%	100.00%
$\text{TVaR}_{\mathbb{P}(I>S)}$	344,384,440	2,161,783,307	619,764,663	1,412,305,219	4,538,237,629
	7.59%	47.63%	13.66%	31.12%	100.00%
$\text{TVaR}_{\mathbb{P}(I>a)}$	112,261,684	596,211,208	183,402,691	336,025,639	1,227,901,222
	9.14%	48.56%	14.94%	27.37%	100.00%
CovWBeta ($\beta = 2$)	269,779,195	1,524,839,975	443,827,368	860,058,147	3,098,504,684
	8.71%	49.21%	14.32%	27.76%	100.00%
MyersRead (D)	-292,411,570	7,013,010,071	-8,956,970	1,260,907,121	7,972,548,652
	-3.67%	87.96%	-0.11%	15.82%	100.00%
MyersRead (S)	151,756,901	1,138,756,629	305,067,719	835,725,664	2,431,306,913
	6.24%	46.84%	12.55%	34.37%	100.00%
MyersRead (a)	-112,261,684	-596,211,208	-183,402,691	-336,025,639	-1,227,901,222
	9.14%	48.56%	14.94%	27.37%	100.00%
$\text{RTVaR}_{\mathbb{P}(I>S)}$ ($\beta = 2$)	477,516,896	3,224,495,989	958,509,100	1,719,523,692	6,380,045,677
	7.48%	50.54%	15.02%	26.95%	100.00%
$\text{RTVaR}_{\mathbb{P}(I>a)}$ ($\beta = 2$)	230,617,968	1,588,103,291	499,905,525	887,902,166	3,206,528,950
	7.19%	49.53%	15.59%	27.69%	100.00%
AvgTVaR	165,401,126	3,844,686,559	445,516,717	1,334,299,597	5,789,903,999
	2.86%	66.40%	7.69%	23.05%	100.00%
ExpVal	112,261,684	596,211,208	183,402,691	336,025,639	1,227,901,222
	9.14%	48.56%	14.94%	27.37%	100.00%

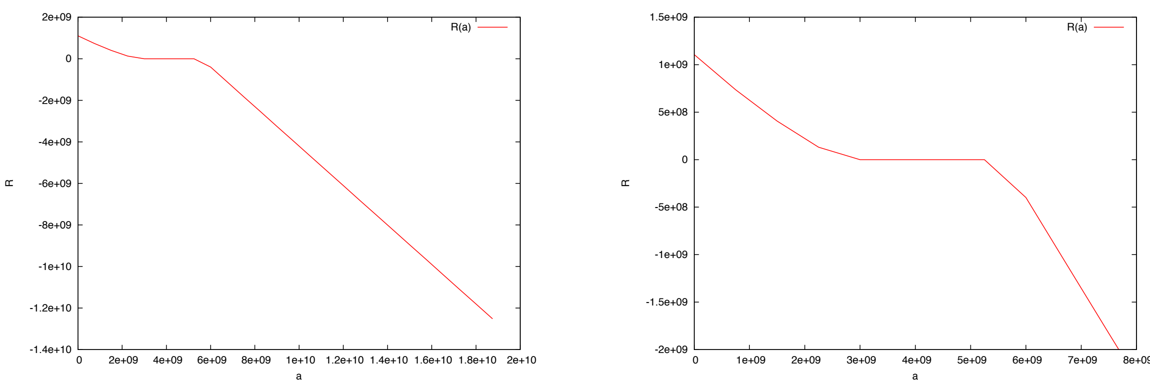
Table A.3: Capital allocations for the profitable company, $a = 0$.

	Line 1	Line 2	Line 3	Line 4	Aggregate
Correct Allocation	1,116,826,342	5,617,043,707	1,733,357,009	3,289,964,064	11,757,191,122
	9.50%	47.78%	14.74%	27.98%	100.00%
$\text{VaR}_{\mathbb{P}(I>D)}$ (bell)	1,025,637,266	5,426,604,978	1,888,133,348	3,393,934,505	11,734,310,096
	8.74%	46.25%	16.09%	28.92%	100.00%
$\text{VaR}_{\mathbb{P}(I>D)}$ (simple)	1,206,722,389	5,159,391,128	2,071,749,351	3,319,328,254	11,757,191,122
	10.26%	43.88%	17.62%	28.23%	100.00%
$\text{VaR}_{\mathbb{P}(I>S)}$ (bell)	495,331,163	3,237,355,077	855,486,486	1,615,680,969	6,203,853,695
	7.98%	52.18%	13.79%	26.04%	100.00%
$\text{VaR}_{\mathbb{P}(I>a)}$ (bell)	284,658,353	1,480,892,132	431,850,078	793,933,117	2,991,333,679
	9.52%	49.51%	14.44%	26.54%	100.00%
$\text{TVaR}_{\mathbb{P}(I>D)}$	50,549,942	11,214,872,882	681,606,181	2,881,096,747	14,828,125,752
	0.34%	75.63%	4.60%	19.43%	100.00%
$\text{TVaR}_{\mathbb{P}(I>S)}$	581,915,040	3,836,036,482	1,180,872,652	1,819,227,730	7,418,051,904
	7.84%	51.71%	15.92%	24.52%	100.00%
$\text{TVaR}_{\mathbb{P}(I>a)}$	380,859,753	2,169,705,613	622,108,862	1,230,170,351	4,402,844,579
	8.65%	49.28%	14.13%	27.94%	100.00%
CovWBeta ($\beta = 2$)	344,748,964	1,948,582,443	567,163,918	1,099,062,349	3,959,557,673
	8.71%	49.21%	14.32%	27.76%	100.00%
MyersRead (D)	-373,670,720	8,961,876,988	-11,446,051	1,611,304,474	10,188,064,690
	-3.67%	87.96%	-0.11%	15.82%	100.00%
MyersRead (S)	328,553,594	2,490,457,756	766,954,303	1,060,857,297	4,646,822,951
	7.07%	53.59%	16.50%	22.83%	100.00%
MyersRead (a)	109,145,328	726,655,947	178,207,125	416,865,134	1,430,873,534
	7.63%	50.78%	12.45%	29.13%	100.00%
$\text{RTVaR}_{\mathbb{P}(I>S)}$ ($\beta = 2$)	752,044,083	5,194,069,658	1,613,751,819	2,211,820,091	9,771,685,651
	7.70%	53.15%	16.51%	22.63%	100.00%
$\text{RTVaR}_{\mathbb{P}(I>a)}$ ($\beta = 2$)	532,106,428	3,437,237,635	1,026,565,641	1,935,409,544	6,931,319,248
	7.68%	49.59%	14.81%	27.92%	100.00%
AvgTVaR	337,774,912	5,740,204,992	828,195,898	1,976,831,609	8,883,007,412
	3.80%	64.62%	9.32%	22.25%	100.00%
ExpVal	143,458,428	761,894,173	234,369,028	429,404,837	1,569,126,466
	9.14%	48.56%	14.94%	27.37%	100.00%

Table A.4: Capital allocations for the profitable company, $a = 3,000,000,000$.

	Line 1	Line 2	Line 3	Line 4	Aggregate
Correct Allocation	1,424,358,388	7,151,812,046	2,178,620,986	4,225,861,527	14,980,652,946
	9.51%	47.74%	14.54%	28.21%	100.00%
$\mathrm{VaR}_{\mathbb{P}(I>D)}$ (bell)	1,359,255,983	6,787,374,942	2,464,638,899	4,321,093,427	14,932,363,249
	9.10%	45.45%	16.51%	28.94%	100.00%
$\mathrm{VaR}_{\mathbb{P}(I>D)}$ (simple)	1,534,834,017	6,562,245,866	2,635,064,543	4,221,864,085	14,954,008,511
	10.26%	43.88%	17.62%	28.23%	100.00%
$\mathrm{VaR}_{\mathbb{P}(I>S)}$ (bell)	841,023,894	4,787,835,746	1,333,940,496	2,445,339,478	9,408,139,614
	8.94%	50.89%	14.18%	25.99%	100.00%
$\mathrm{VaR}_{\mathbb{P}(I>a)}$ (bell)	1,435,335,446	5,681,611,221	2,244,813,624	2,601,250,385	11,963,010,676
	12.00%	47.49%	18.76%	21.74%	100.00%
$\mathrm{TVaR}_{\mathbb{P}(I>D)}$	64,294,630	14,264,232,228	866,937,054	3,664,476,049	18,859,939,961
	0.34%	75.63%	4.60%	19.43%	100.00%
$\mathrm{TVaR}_{\mathbb{P}(I>S)}$	825,123,192	5,802,377,598	1,861,291,677	2,509,998,369	10,998,790,835
	7.50%	52.75%	16.92%	22.82%	100.00%
$\mathrm{TVaR}_{\mathbb{P}(I>a)}$	1,151,683,772	6,729,175,361	2,836,620,159	2,747,092,977	13,464,572,269
	8.55%	49.98%	21.07%	20.40%	100.00%
CovWBeta ($\beta = 2$)	438,487,296	2,478,408,161	721,377,579	1,397,900,871	5,036,173,907
	8.71%	49.21%	14.32%	27.76%	100.00%
MyersRead (D)	-475,272,970	11,398,639,633	-14,558,270	2,049,423,245	12,958,231,638
	-3.67%	87.96%	-0.11%	15.82%	100.00%
MyersRead (S)	497,654,664	4,063,223,157	1,326,304,076	1,529,808,002	7,416,989,898
	6.71%	54.78%	17.88%	20.63%	100.00%
MyersRead (a)	835,319,052	5,048,992,195	2,319,772,925	1,800,138,920	10,004,223,093
	8.35%	50.47%	23.19%	17.99%	100.00%
$\mathrm{RTVaR}_{\mathbb{P}(I>S)}$ ($\beta = 2$)	1,041,510,859	7,529,664,282	2,411,872,059	3,009,337,830	13,992,385,030
	7.44%	53.81%	17.24%	21.51%	100.00%
$\mathrm{RTVaR}_{\mathbb{P}(I>a)}$ ($\beta = 2$)	1,344,054,896	8,341,353,342	3,351,050,015	3,644,088,931	16,680,547,184
	8.06%	50.01%	20.09%	21.85%	100.00%
AvgTVaR	680,367,198	8,931,928,396	1,854,949,630	2,973,855,798	14,441,101,022
	4.71%	61.85%	12.84%	20.59%	100.00%
ExpVal	182,465,228	969,055,604	298,094,707	546,161,368	1,995,776,907
	9.14%	48.56%	14.94%	27.37%	100.00%

Table A.5: Capital allocations for the profitable company, $a = 12,000,000,000$.



(a) Raising decisions $R(a)$ (b) Raising decisions $R(a)$ (lim. range)

Figure A.2: Optimal raising decision R for a company with carrying cost $\tau = 5\%$, raising costs $c^{(1)} = 7.5\%$, $c^{(2)} = 5.00\text{E-}11$, and $\xi = 75\%$, interest rate $r = 6\%$, and premium parameters $\alpha = 0.973046$, $\beta = 550.203$, and $\gamma = 1.61\text{E-}10$.

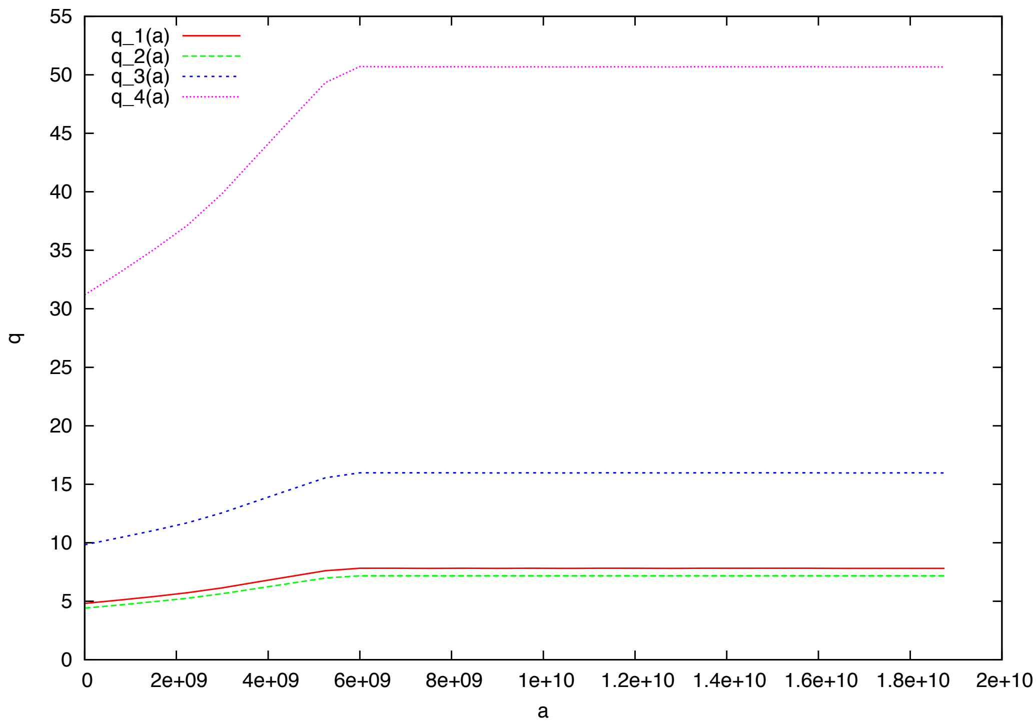


Figure A.3: Optimal portfolio weights q_1, q_2, q_3, and q_4 for a company with carrying cost $\tau = 5\%$, raising costs $c^{(1)} = 7.5\%$, $c^{(2)} = 5.00\text{E-}11$, and $\xi = 75\%$, interest rate $r = 6\%$, and premium parameters $\alpha = 0.973046$, $\beta = 550.203$, and $\gamma = 1.61\text{E-}10$.

Appendix B

Technical Appendix

B.1 Proofs

Proof of Lemma 4.1.1. With the budget constraint (4.2):

$$e^{-rt}\, a_t - e^{-r(t-1)}\, a_{t-1} - e^{-rt}[e^r R_t^b + R_t^e] \;=\; e^{-rt}\left[e^r \sum_j p_t^{(j)} - \sum_j I_t^{(j)} - (\tau\, a_{t-1} + c_1(R_t^b))\, e^r - c_2(R_t^e)\right].$$

Hence, the sum in (4.6) can be written as:

$$\begin{aligned}
&\sum_{t=1}^{\infty} \mathbf{1}_{\{a_1 \geq 0,\ldots,a_t \geq 0\}}\, e^{-rt}\left[e^r \sum_j p_t^{(j)} - \sum_j I_t^{(j)} - (\tau\, a_{t-1} + c_1(R_t^b))e^r - c_2(R_t^e)\right] \\
&\quad - \mathbf{1}_{\{a_1 \geq 0,\ldots,a_{t-1} \geq 0, a_t < 0\}}\, e^{-rt}\left[(a_{t-1} + R_t^b)e^r + R_t^e\right] \\
&= \sum_{\{t<t^*: a_1 \geq 0, a_2 \geq 0, \ldots, a_{t^*-1} \geq 0, a_{t^*} < 0\}} \left[e^{-rt}\, a_t - e^{-r(t-1)}\, a_{t-1}\right] - e^{-rt}[e^r R_t^b + R_t^e] \\
&\quad - e^{-rt^*}\left[(a_{t^*-1} + R_{t^*}^b)e^r + R_{t^*}^e\right] \\
&= \left[\sum_{t \leq t^*} e^{-rt}[-e^r R_t^b - R_t^e]\right] + e^{-r(t^*-1)}\, a_{t^*-1} - a_0 - e^{-r(t^*-1)}\, a_{t^*-1} \\
&= \left[\sum_{t \leq t^*} e^{-rt}[-e^r R_t^b - R_t^e]\right] - a_0,
\end{aligned}$$

which completes the proof. □

Proof of Proposition 4.1.1. Notice that our per-period profit function in (4.6) is bounded from above, so the Bellman equation follows from classical infinite-horizon dynamic programming results (see e.g. Proposition 1.1 in Bertsekas (1995, Chap. 3)). □

Proof of Proposition 4.1.2. Let

$$a' = \left[a(1-\tau) + R^b - c_1(R^b) + \sum_j p^{(j)} \right] e^r + R^e - c_2(R^e) - \sum_j I^{(j)};$$

then, conditional on $a' < 0$, the objective function is decreasing in R^e so that zero is the optimal choice. Conditional on $a' > 0$, if $R^e > 0$, on the other hand, decreasing R^e by a (small) $\varepsilon > 0$ and increasing R^b in the beginning of the next period will be dominant (since $c_2 > c_1$), so $R_e > 0$ cannot be optimal. Finally, if $a' = 0$ and $R_e > 0$, then $R^e = R^e_*$.

Moreover,

$$\begin{aligned}
-\left(a + R^b\right) &< \sum_j p^{(j)} - e^{-r} \sum_j I^{(j)} - \tau\, a - c_1(R^b) - e^{-r}\, c_2(R^e_*) \\
&\quad + e^{-r}\, V \left([a(1-\tau) + R^b - c_1(R^b) + \sum_j p^{(j)}]e^r + R^e_* - c_2(R^e_*) - \sum_j I^{(j)} \right) \\
\Leftrightarrow V(0) &> -\left(\sum_j p^{(j)} + (1-\tau)a + R^b - c_1(R^b) \right) e^r + \sum_j I^{(j)} + c_2(R^e_*) \\
\Leftrightarrow V(0) &> R^e_*,
\end{aligned}$$

which proves the last assertion. □

Proof of Proposition 4.2.1. The first order conditions from the Bellman equation (4.10) are:

$$\begin{aligned}
[q_i] \qquad & -e^{-r}\mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}} \left(1 + V' \left([a(1-\tau) + R^b - c_1(R^b) + \sum_j p^{(j)}]e^r - I \right) \right) \mathbf{1}_{\{S \geq I\}} \right] \\
& - \frac{e^{-r}}{1-\xi} \mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}} \mathbf{1}_{\{S < I \leq D\}} \right] \\
& - \sum_{k \neq i} \lambda_k \left(\beta\, \mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}} | I = D \right] f_I(D) + \gamma\, \mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}} \right] \right) e^{-r}\, \mathbb{E}[I^{(k)}] \exp\{\alpha - \beta \mathbb{P}(I > D) - \gamma\, \mathbb{E}[I]\} \\
& + \lambda_i \left(e^{-r}\mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}} \right] - \beta\, \mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}} | I = D \right] f_I(D)\, \mathbb{E}[I^{(i)}]\, e^{-r} - \gamma\, \mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}} \right] \mathbb{E}[I^{(i)}]\, e^{-r} \right) \\
& \qquad \times \exp\{\alpha - \beta \mathbb{P}(I > D) - \gamma\, \mathbb{E}[I]\} - \zeta\, \frac{\partial \rho}{\partial q^{(i)}}\, e^{-r} = 0, \\
[p_i] \qquad & \mathbb{E}\left[\left(1 + V' \left([a(1-\tau) + R^b - c_1(R^b) + \sum_j p^{(j)}]e^r - I \right) \right) \mathbf{1}_{\{S \geq I\}} \right] + \frac{1}{1-\xi} \mathbb{E}\left[\mathbf{1}_{\{S < I \leq D\}} \right] \\
& - \lambda_i + \sum_k \lambda_k\, \mathbb{E}[I^{(k)}]\, \beta\, f_I(D) \exp\{\alpha - \beta \mathbb{P}(I > D) - \gamma\, \mathbb{E}[I]\} \\
& + \zeta = 0,
\end{aligned}$$

$$[R_b] \qquad \mathbb{E}\left[\left(V'\left([a(1-\tau)+R^b-c_1(R^b)+\sum_j p^{(j)}]e^r-I\right)(1-c_1'(R_b))-c_1'(R_b)\right)\boldsymbol{1}_{\{S\geq I\}}\right]$$
$$+\mathbb{E}\left[\left(\frac{1}{1-\xi}-\frac{c_1'(R_b)}{1-\xi}-1\right)\boldsymbol{1}_{\{S<I\leq D\}}\right]-\mathbb{P}(I>D)$$
$$+\sum_k \lambda_k\,\mathbb{E}[I^{(k)}]\,\beta\,f_I(D)\,(1-c_1'(R^b))\exp\{\alpha-\beta\mathbb{P}(I>D)-\gamma\,\mathbb{E}[I]\}$$
$$+\zeta(1-c_1'(R_b))=0.$$

From $[p_i]$ and $[R^b]$, we obtain:

$$\begin{aligned}
\lambda_i &= \sum_k \lambda_k\,\mathbb{E}[I^{(k)}]\,\beta\,f_I(D)\,\exp\{\alpha-\beta\mathbb{P}(I>D)-\gamma\,\mathbb{E}[I]\}+\frac{1}{1-\xi}\mathbb{E}\left[\boldsymbol{1}_{\{S<I\leq D\}}\right]+\zeta \\
&\quad +\mathbb{E}\left[\left(1+V'\left([a(1-\tau)+R^b-c_1(R^b)+\sum_j p^{(j)}]e^r-I\right)\right)\boldsymbol{1}_{\{S\geq I\}}\right] \\
&= \frac{1}{1-c_1'(R^b)}\Bigg[\zeta(1-c_1'(R_b))+\sum_k \lambda_k\,\mathbb{E}[I^{(k)}]\,\beta\,f_I(D)\,(1-c_1'(R^b))\exp\{\alpha-\beta\mathbb{P}(I>D)-\gamma\,\mathbb{E}[I]\} \\
&\qquad +(1-c_1'(R^b))\,\mathbb{E}\left[V'\left([a(1-\tau)+R^b-c_1(R^b)+\sum_j p^{(j)}]e^r-I\right)\boldsymbol{1}_{\{S\geq I\}}\right] \\
&\qquad +\frac{1-c_1'(R^b)}{1-\xi}\mathbb{P}(S<I\leq D)+(1-c_1'(R^b))\,\mathbb{P}(S\geq I)\Bigg] \\
&= \frac{1}{1-c_1'(R^b)}.
\end{aligned}$$

Then, we can write $[R^b]$ as:

$$\begin{aligned}
\zeta &= \frac{\mathbb{P}(I>D)}{1-c_1'(R^b)}+\frac{c_1'(R^b)}{1-c_1'(R^b)}\mathbb{P}(I\leq D)-\frac{\xi}{1-\xi}\,\mathbb{P}(S<I\leq D) \\
&\quad -\mathbb{E}\left[V'\left([a(1-\tau)+R^b-c_1(R^b)+\sum_j p^{(j)}]e^r-I\right)\boldsymbol{1}_{\{S\geq I\}}\right] \\
&\quad -\frac{1}{1-c_1'(R^b)}\sum_k \mathbb{E}[I^{(k)}]\,\beta\,f_I(D)\exp\{\alpha-\beta\mathbb{P}(I>D)-\gamma\,\mathbb{E}[I]\}
\end{aligned} \tag{B.1}$$

Then we obtain with $[q_i]$:

$$\begin{aligned}
&\frac{\mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}}\right]\ \exp\left\{\alpha - \beta \mathbb{P}(I > D) - \gamma\, \mathbb{E}[I]\right\}}{1 - c_1'(R^b)} \\
= \;& \mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}}\, \boldsymbol{1}_{\{D \geq I\}}\right] + \mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}}\ \ V'\left(S - I\right) \boldsymbol{1}_{\{S \geq I\}}\right] + \frac{\xi}{1-\xi}\mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}}\boldsymbol{1}_{\{S < I \leq D\}}\right] \\
& + \frac{1}{1 - c_1'(R^b)}\gamma\, \mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}}\right]\ \mathbb{E}[I]\ \exp\left\{\alpha - \beta \mathbb{P}(I > D) - \gamma\, \mathbb{E}[I]\right\} \\
& + \left[\frac{1}{1 - c_1'(R^b)}\beta\, f_I(D)\, E[I]\ \exp\left\{\alpha - \beta \mathbb{P}(I > D) - \gamma\, \mathbb{E}[I]\right\}\right] \mathbb{E}\left[\frac{\partial I^{(i)}}{\partial q^{(i)}} | I = D\right] \\
& + \Bigg[\mathbb{P}(I > D) + \frac{c_1'(R^b)}{1 - c_1'(R^b)} - \frac{\xi}{1-\xi}\,\mathbb{P}(S < I \leq D) - \mathbb{E}\left[V'\left(S - I\right)\ \boldsymbol{1}_{\{S \geq I\}}\right] \\
& \qquad \frac{1}{1 - c_1'(R^b)}\beta\, f_I(D)\, E[I]\ \exp\left\{\alpha - \beta \mathbb{P}(I > D) - \gamma\, \mathbb{E}[I]\right\}\Bigg] \frac{\partial \rho}{\partial q^{(i)}},
\end{aligned}$$

which is exactly the expression in Proposition 4.2.1. □

Proof of Corollary 4.2.1. In the absence of a regulatory constraint, $\zeta =$ so that Equation (B.1) yields:

$$\begin{aligned}
&\frac{1}{1 - c_1'(R^b)} \sum_k \mathbb{E}[I^{(k)}]\, \beta\, f_I(D) \exp\left\{\alpha - \beta \mathbb{P}(I > D) - \gamma\, \mathbb{E}[I]\right\} \\
= \;& \mathbb{P}(I > D) + \frac{c_1'(R^b)}{1 - c_1'(R^b)} - \frac{\xi}{1-\xi}\,\mathbb{P}(S < I \leq D) - \mathbb{E}\left[V'\left(S - I\right)\ \boldsymbol{1}_{\{S \geq I\}}\right],
\end{aligned}$$

which together with Proposition 4.2.1 yields the claim. □

Appendix C

Supporting Material

C.1 Proposal

Capital Allocation And Insurer's Objective

Question raised: *How do the line level marginal costs implied by various capital allocation methods proposed in the actuarial literature compare with those obtained from profit-maximizing models of the firm?*

It is well known that different allocation methods can yield different results, even given the same data (e.g., Venter (2002)), and we know from Gründl and Schmeiser (2007) that popular capital allocation methods may yield pricing results that are suboptimal from the standpoint of firm profitability, a result which stems from the fact that the risk measures on which the methods rely are not tailored to the task of maximizing profits. We show in our recent paper Bauer and Zanjani (2013a) that risk measures can be engineered to yield capital allocation results consistent with profit maximization, although this result has been derived in a simplified one-period model for the firm. Moreover, thus far, we have not considered the practical applications and implications of this approach.

For this research project, we will consider multi-period models of a profit-maximizing insurance company, where the company faces various costs or constraints in raising capital between periods. An extreme and simple example is the case where the company cannot raise funds from external financiers and simply defaults whenever liabilities exceed assets. First analyses show that in this case, the optimal capital allocation rule for a profit maximizing insurer consists of three components: (1) An allocation rule driven by regulatory constraints that adheres to the Euler allocation based on an exogenously specified risk measure (assuming that the regulatory constraint is in the form of a risk measure); (2) an allocation rule that derives from the firm's value as a going concern and takes the form of the gradient of Value-at-Risk (VaR); and (3) an "internal policyholder" allocation rule which depends on how that risk associated with a certain line of business affects the firm's other lines or, more precisely, the recoveries of the corresponding beneficiaries in the case of default. In the case of a complete and frictionless market for risk, the latter allocation rule collapses to the one derived by Ibragimov, Jaffee and Walden (2010), who extend the results from Myers and Read (2001) to a more realistic pro-rata sharing rule in the case of default. A more realistic multi-period model would feature the company being able to raise funds from outside investors at significant cost. Preliminary analyses indicate that the second component of the capital allocation—the piece derived from the firm's value as a "going concern"—appears to evolve into a form corresponding to the gradient of Tail-Value-at-Risk (TVaR), possibly with a spectral weighting function associated with the marginal cost of raising capital.

Thus, gradients of popular risk measures such as VaR or (spectral) TVaR, as well as recent approaches proposed in the academic literature (such as the Ibragimov et al. allocation formula) do arise endogenously within the framework as parts of the theoretically correct capital allocation. However, the correct allocation becomes more complex as the models become more complex, and it contains components that do not directly align with predominant allocation methods (such as the second component described above—the "internal policyholder" allocation). Consequently, it is not clear if existing methods are able to produce close approximations to correct capital allocations, or if there exist alternative, practicable methods that are able to yield accurate answers. Part of this uncertainty can be attributed to the multi-faceted nature of capital allocation: Since there are different "pieces" to the puzzle, it is not clear which pieces are of the greatest relative importance in practice, and the answer may well depend on the specifics of the institutional environment.

After studying in more detail the theoretical implications of the multi-period framework outlined above, we propose to investigate these questions in the context of a model for a representative P&C insurer featuring realistic loss distributions.[1] Specifically, by relying on numerical techniques, we will first solve the company's optimization problem and then derive the "optimal" allocation according to marginal costs. In order to give nuanced answers, we intend to study different institutional circumstances—for example, circumstances with uninsured policyholders (where the effects of risk on financial strength may have profound influences on consumer demand) versus circumstances with insured policyholders (where the greater issue with risk may concern its affects on the firm's value as a "going concern"). Subsequently, we will compare the ensuing allocations to allocations yielded by popular allocation methods from the actuarial literature in order to evaluate their validity in different situations.[2]

[1] This "representative P&C insurer" is to be created in collaboration with our industry advisors (see below), where we intend to build on related efforts such as the models proposed in response to the CAS Committee on Dynamic Financial Analysis 2001 Call for Papers (e.g., Burkett, McIntyre and Sonlin (2001)), previous related studies (e.g., Ruhm and Mango (2003) or Vaughn (2007)), and recent industry investigations based on example companies In particular, we intend to use available data to estimate/calibrate the corresponding model.

[2] For examples, general versions of the gradient method are derived in Denault (2001) and Kalkbrener (2005). Other methods include Dhaene et al. (2008) and Sherris (2006).

Stability of allocations

Question Raised: *How stable are line level allocations obtained from the various methods when confronted with representative and/or actual insurance data?*

As early as Myers and Read (2001), it was anticipated that stability of allocations could be an issue in property-casualty insurance. Blackboard examples of this can be extreme (e.g., Zanjani (2010)), and the general problem stems from the sensitivity of most allocation methods to portfolio composition. Small deviations from a target portfolio could, in theory, produce significant changes in capital allocations and such instability could engender economically irrational decision-making to the extent that portfolio composition is determined in a decentralized manner by independent line managers.

This is a crucial way in which capital allocations can distort company portfolios in the presence of asymmetric information within the company: Ex ante allocations determine the costs that individual line managers "see" for the purposes of making underwriting decisions in the course of building the portfolio, but the correct allocations will only be known ex post when the portfolio is complete. In theory, small changes in the ultimate portfolio could yield drastically different allocations. However, the key question is whether this instability exists in practice. Given realistic loss distributions, how stable are the results produced by different capital allocation methods?

We propose to investigate this question in the context of a "representative P&C insurer" as outlined above

Outline

We will address these in the manner described below:

- **Step 1: Identify capital allocation methods.** We will review current and past literature to identify reasonable capital allocation methods for evaluation.

- **Step 2: Obtain data for multi-period models of multi-line insurance companies.** We will obtain (or fit) suitable loss distributions for various lines of property-casualty insurance. We will use historic industry data for information on prevailing market prices and expense ratios in various lines of insurance. We will survey the literature for evidence on suitable assumptions for the cost of insurance company capital and the cost of external financing such as equity flotations.

- **Step 3: Build and estimate models.** We will build structural models of multi-period/multi-line insurance companies and numerically calculate optimal portfolios under various assumptions from ranges derived from our Step 2 data. Infer true marginal costs and optimal capital allocations from optimized portfolios.

- **Step 4: Calculate capital allocations.** Using the optimized portfolios from Step 3, we will apply the various methods identified in Step 1 to obtain capital allocations, and these will be compared to optimal allocations identified in Step 3. We will develop summary measures of the correspondence between the method-based allocations and the optimal allocations.

- **Step 5: Study robustness of optimal allocations and of method-based allocations.** We will test the sensitivity of optimal allocations and of method-based allocations to changes in underlying assumptions and to changes in the optimal portfolios. We will develop summary measures of sensitivity.

- **Step 6: Summary and communication** We will fully report the results of the research in a final report.

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