LARGEST CLAIMS REINSURANCE (LCR). A QUICK METHOD TO CALCULATE LCR-RISK RATES FROM EXCESS OF LOSS RISK RATES

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Let us denote by $E(x)$ the pure risk premium of an unlimited excess cover with the retention x and by $H(x)$ and $m(x)$ the corresponding expected frequency and severity.

We thus have $E(x) = H(x) \cdot m(x)$.

$H(x)$ is a non-increasing function of x and for practical purposes we can assume that it is decreasing; $H'(x) < 0$. The equation $H(x) = n$ has then only one solution x_n, where n is a fixed integer.

Let E_n denote the risk premium for a reinsurance covering the n largest claims from the bottom.

Let us define $E_n' = nx_n + E(x_n) = n(x_n + m(x_n))$. Intuitively we feel that E_n' is a good approximation for E_n.

We shall first show that when the claims size distribution is Pareto and the number of claims is Poisson distributed, E_n' is a good approximation for E_n, being slightly on the safe side. We further include a proof given by G. Ottaviani that the inequality $E_n < E_n'$ always holds.

In the Pareto case we have

$$H(x) = t(1 - F(x)) = t \cdot x^{-\alpha}$$

where the Poisson parameter t stands for the expected number of claims in excess of 1 (equal to a suitably chosen monetary unit) and

$$m(x) = \frac{x}{\alpha - 1}.$$

The retention x_n over which we expect n claims should satisfy

$$n = H(x_n) = t \cdot x_n^{-\alpha}$$

which gives

$$t = n \cdot x_n^{\alpha}$$

or

$$x_n = \left(\frac{t}{n}\right)^{1/\alpha}.$$

According to B. Berliner [2] we have, when the number of claims is Poisson distributed

$$E_n = t^{1/\alpha} \sum_{i=1}^{n} \frac{1}{\Gamma(i)} \cdot \Gamma_t \left(i - \frac{1}{\alpha}\right)$$

where

$$\Gamma_t(n) = \int_0^t e^{-u} \cdot u^{n-1}\, du.$$

Replacing the incomplete Gamma function Γ_t by $\Gamma_\infty = \Gamma$ we arrive at

$$\bar{E}_n = t^{1/\alpha} \cdot \frac{\alpha}{\alpha - 1} \cdot \frac{1}{\Gamma(n)} \cdot \Gamma\left(n + 1 - \frac{1}{\alpha}\right)$$

which formula was given by H. Ammeter already in 1964 [1]. Obviously $E_n < \bar{E}_n$

In all cases when t is large compared to n, we have

$$\frac{E_n}{\bar{E}_n}\,(n, \alpha; t) \quad \text{very close to } 1.$$

If in a practical situation t is too small we can always increase t by decreasing the monetary unit, in other words by enlarging to the left the range of the Pareto distribution.

Inserting $t = nx_n^\alpha$, as deduced above, in $\bar{E}_n$, we obtain

$$\bar{E}_n = n^{1/\alpha} \cdot x_n \cdot \frac{\alpha}{\alpha - 1} \cdot \frac{1}{\Gamma(n)}\, \Gamma\left(n + 1 - \frac{1}{\alpha}\right).$$

However

$$E'_n = n(x_n + m(x_n)) = n \cdot x_n \cdot \frac{\alpha}{\alpha - 1} = x_n \cdot \frac{\alpha}{\alpha - 1}\, \frac{\Gamma(n+1)}{\Gamma(n)}.$$

Thus we have

$$\frac{\bar{E}_n}{E'_n} = \frac{n^{1/\alpha} \cdot \Gamma\left(n + 1 - \frac{1}{\alpha}\right)}{\Gamma(n+1)}$$

Tabulation of

$$\frac{\bar{E}_n}{E'_n}$$

n	$\alpha = 2$	$\alpha = 2.5$	$\alpha = 3$
1	0.886	0.894	0.903
2	0.940	0.943	0.948
3	0.959	0.961	0.964
4	0.969	0.971	0 973
5	0.975	0.976	0.978
.			
.			
10	0.988	0.988	0.989

The figures illustrate

that the approximation is good,
that the approximation is on the safe side,
and *that* the approximation is rather invariant to variations of the parameter alpha within the given interval.

The safety margin in the approximation—E'_n replacing $\bar{E}_n$—is roughly of the form constant/n.

This is illustrated below for alpha = 2.5

n	$\frac{\bar{E}_n}{E'_n}$	$n \cdot \frac{E'_n - \bar{E}_n}{E'_n}$
1	0.894	0.11
2	0.943	0.11
3	0.961	0.12
4	0.971	0.12
5	0.976	0.12
.		
.		
10	0.988	0.12

We have thus shown that in the Pareto case

$$\frac{E_n}{E'_n} \sim 1$$

and

$$E_n < \bar{E}_n < E'_n = nx_n + E(x_n) = nx_n + n\,\frac{x_n}{\alpha - 1} =$$

$$= nx_n \cdot \frac{\alpha}{\alpha - 1} = \alpha \cdot E(x_n).$$

Thus

$$\frac{E_n}{E(x_n)} \sim \alpha.$$

This means that the LCR risk premium is approximately equal to alpha times the risk premium of an XL cover with a retention chosen in such a way that the expected number of claims is equal to the number of LCR-claims protected.

In the Poisson-Pareto case E'_n gives a handy and fairly good approximation of E_n. The reader is invited to examine other claims size distributions $F(x)$ which are of importance in the practice.

Most such distributions will for all $x > x_0$ have $m''(x) < 0$. We believe that $m''(x) < 0$ will guarantee that E'_n will be a good approximation of E_n with $E'_n > E_n$.

We now give a proof by G. Ottaviani that the inequality $E_n < E'_n$ is valid for any n and for arbitrary distribution functions of the number of claims and of the claim size.

We do not even need the condition of section 2 that the equation $H(x) = n$ has only one solution since the proof will be valid for any X_n, such that $H(x_n) = n$.

Let s denote the total number of claims which occur and $N = \min(s, n)$. We thus allow for the possibility that less than n claims occur.

Let X_n be the set consisting of the N largest claims.

Let

$$\nu(X_n) = E(N)$$

$$\nu(X_n) \leq n \tag{1}$$

Let $\mu(X_n) = E_n/\nu(X_n)$ be the expected value of a claim in the set X_n.

Analoguesly we denote by X'_n the set consisting of all claims exceeding x_n, the expected number of claims exceeding x_n by $\nu(X'_n)$ and the expected value of a claim in the set X'_n by $\mu(X'_n)$.

We thus have

$$\nu(X'_n) = n \tag{2}$$

and

$$\mu(X'_n) = x_n + m(x_n).$$

Let

$$Y_n = X_n \frown X'_n$$
$$Z_n = (X_n \smile X'_n) - X'_n$$
$$Z'_n = (X_n \smile X'_n) - X_n$$

$\nu(Y_n)$, $\mu(Y_n)$, $\nu(Z_n)$, $\mu(Z_n)$, $\nu(Z'_n)$, $\mu(Z'_n)$ are defined analoguesly to $\nu(X_n)$ and $\mu(X_n)$. From the above definition it follows directly that

$$\mu(Z_i) < x_n \text{ and} \tag{3}$$

$$\mu(Z'_i) \geq x_n. \tag{4}$$

Thus

$$E_n = \nu(X_i) \cdot \mu(X_i) = \nu(Y_i)\,\mu(Y_i) + \nu(Z_i)\,\mu(Z_i) \tag{5}$$

and

$$E'_n = \nu(X'_i) \cdot \mu(X'_i) = \nu(Y_i)\,\mu(Y_i) + \nu(Z'_i)\,\mu(Z'_i). \tag{6}$$

From (1) and (2) it follows that

$$\nu(Y_i) + \nu(Z_i) = \nu(X_i) \leq n = \nu(X'_i) = \nu(Y_i) + \nu(Z'_i).$$

Thus

$$\nu(Z_i) \leq \nu(Z'_i). \tag{7}$$

From (3) and (4) it follows that

$$\mu(Z_i) < \mu(Z'_i) \tag{8}$$

and from (7) and (8)

$$\nu(Z_i) \cdot \mu(Z_i) < \nu(Z'_i)\,\mu(Z'_i). \tag{9}$$

Adding $\nu(Y_i) \cdot \mu(Y_i)$ to both sides of (9) and using (5) and (6) leads to

$$E_n < E'_n \qquad \text{q.e.d.}$$

REFERENCES

[1] AMMETER H., The Rating of "Largest Claim" Reinsurance Covers. Quarterly Letter from the Algemeene Reinsurance Companies Jubilee Number 2, July 1964

[2] BERLINER B., Correlations between Excess of Loss Reinsurance Covers and Reinsurance of the n Largest Claims. The ASTIN Bulletin Vol. VI, Part III, May 1972.