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A NUMERICAL ILLUSTRATION OF OPTIMAL SEMILINEAR CREDIBILITY*

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INTRODUCTION

The homogeneous (in time) model of credibility theory is defined by a sequence $\Theta, X_1, X_2, \ldots$ of random variables, where for $\Theta = \theta$ fixed, the variables $X_1, X_2, \ldots$ are independent and equidistributed. The structure variable Θ may be interpreted as the parameter of a contract chosen at random in a fixed portfolio, the variable X_k as the total cost (or number) of the claims of the kth year of that contract.

Bühlmann's linear credibility premium of the year $t+1$ may be written in the form

$$f(X_1) + \ldots + f(X_t), \tag{1}$$

where f is a linear function. In optimal semilinear credibility, we look for an optimal f, not necessarily linear, such that (1) is closest to X_{t+1} in the least squares sense. In the first section we prove that this optimal f, denoted by f^*, is solution of an integral equation of *Fredholm* type, which reduces to a system of linear equations in the case of a finite portfolio. That is a portfolio in which Θ and X_k can assume only a finite number of values.

In the second section we see that the structure of such a portfolio is closely connected with the decomposition of a quadratic form in a sum of squares of linear forms.

In the last section we calculate numerically the optimal premium for a concrete portfolio in automobile insurance. We limit ourselves to the consideration of the number of claims. The optimal premium is compared with the usual linear premium. The difference is far from negligible.

As basic statistics we need the probabilities

$$p_{ij} = P(X_1 = i,\ X_2 = j)$$

In the third section we give a simple general solution to the subsidiary problem of adjusting the matrix p_{ij} of such probabilities.

1. THE FUNDAMENTAL RESULT

1.1. *Hypotheses. Notations. Definitions*

We consider a sequence $\Theta, X_1, X_2, \ldots$ of random variables such that for $\Theta = \theta$ fixed, the variables $X_1, X_2, \ldots$ are conditionally independent and equidistributed.

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All variables considered are supposed to have finite second order moments. The *risk premium* of each year is defined by

$$m_\Theta = E(X_1 \mid \Theta).$$

Here, and also hereafter in similar situations, the index 1 could be replaced by another one. The variables $X_1, X_2, \ldots$ are exchangeable in the sense of *De Finetti*. More generally, for each function f of one variable, we denote by f_Θ the random variable

$$f_\Theta = E(f(X_1) \mid \Theta)$$

Hereafter t will be a fixed positive integer. It is the number of years that we have already observed our portfolio. We have to make forecasts for the year $t+1$. Since t is fixed, the dependence on t is not always indicated in our notations.

1.2. *Lemma*

(I) For each couple f, g of functions of one variable:

$$E(f(X_1)\,g(X_2)) = E(f_\Theta g(X_2)) = E(f(X_1)g_\Theta) = E(f_\Theta\, g_\Theta) \tag{2}$$

(II) For each function f of one variable and each function φ of t variables:

$$E(\varphi(X_1, \ldots, X_t)\, f(X_{t+1})) = E(\varphi(X_1, \ldots, X_t)\, f_\Theta) \tag{3}$$

(III) For each function f of one variable:

$$E(f(X_{t+1}) \mid X_1, X_2, \ldots, X_t) = E(f_\Theta \mid X_1, \ldots, X_t) \tag{4}$$

Demonstration.

(i) Using the conditional independence of X_1, X_2 for fixed Θ:

$$E(f(X_1)\,g(X_2)) = EE(f(X_1)\,g(X_2) \mid \Theta) =$$
$$E(E(f(X_1) \mid \Theta)\, E(g(X_2) \mid \Theta)) = E(f_\Theta g_\Theta)$$

Also

$$E(f_\Theta g(X_2)) = EE(f_\Theta g(X_2) \mid \Theta) = E(f_\Theta E(g(X_2) \mid \Theta)) = E(f_\Theta g_\Theta)$$

and similarly

$$E(f(X_1)\, g_\Theta) = E(f_\Theta g_\Theta)$$

(ii) Writing

$$\varphi_\Theta = E(\varphi(X_1, \ldots, X_t) \mid \Theta),$$

we have in a similar way the more general result

$$E(\varphi(X_1, \ldots, X_t)\, f(X_{t+1})) = E(\varphi_\Theta f_\Theta) = E(\varphi(X_1, \ldots, X_t)\, f_\Theta)$$

(iii) From the conditional independence of $X_1, X_2, \ldots, X_{t+1}$, for fixed Θ, it follows that

$$f_\Theta = E(f(X_{t+1}) \mid \Theta) = E(f(X_{t+1}) \mid \Theta, X_1, \ldots, X_t)$$

Then, by applying the operator $E(. \mid X_1, \ldots, X_t)$ and using a general property of conditional expectations:

$$E(f_\Theta \mid X_1, \ldots, X_t) = E(E(f(X_{t+1}) \mid \Theta, X_1, \ldots, X_t) \mid X_1, \ldots, X_t) = E(f(X_{t+1}) \mid X_1, \ldots, X_t)$$

1.3. *Theorem*

Let f^* be a solution of

(5) $$E(X_2 \mid X_1) = f^*(X_1) + (t-1)\, E(f^*(X_2) \mid X_1)$$

Then, for every function f:

(6) $$E(m_\Theta - f^*(X_1) - \ldots - f^*(X_t))^2 \leqslant E(m_\Theta - f(X_1) - \ldots - f(X_t))^2$$

The mean square error in the approximation of m_Θ by $f^*(X_1) + \ldots + f^*(X_t)$ is given by

(7) $$E(m_\Theta - f^*(X_1) - \ldots - f^*(X_t))^2 = E(X_1 X_2) - t\, E(X_1 f^*(X_2))$$

If g^* also satisfies

(8) $$E(X_2 \mid X_1) = g^*(X_1) + (t-1)\, E(g^*(X_2) \mid X_1),$$

then

(9) $$f^*(X_1) = g^*(X_1) \qquad \text{a.e.}$$

Demonstration.

Multiplying (5) by $f(X_1)$ and taking the mean value, we have

(10) $$E(f(X_1)\, X_2) = E(f(X_1)\, f^*(X_1)) + (t-1)\, E(f(X_1)\, f^*(X_2))$$

In particular, for $f = f^*$, we have

(11) $$E(f^*(X_1)\, X_2) = E(f^*(X_1))^2 + (t-1)\, E(f^*(X_1)\, f^*(X_2))$$

Using (2), we have for every f:

$$E(m_\Theta - f(X_1) - \ldots - f(X_t))^2 =$$
$$E(m_\Theta^2) - 2t\, E(m_\Theta f(X_1)) + E(f(X_1) + \ldots + f(X_t))^2 =$$
$$E(m_\Theta^2) - 2t\, E(m_\Theta f(X_1)) + t\, E f^2(X_1) + t(t-1)\, E(f(X_1)\, f(X_2)) =$$

(12) $$E(X_1 X_2) - 2tE(f(X_1)\, X_2) + t\, E f^2(X_1) + t(t-1)\, E(f(X_1)\, f(X_2))$$

Taking $f=f^*$ and using (11), we have

$$E(m_\Theta - f^*(X_1) - \ldots - f^*(X_t))^2 =$$
$$E(X_1X_2) - 2t\,E(f^*(X_1)\,X_2) + t[E(f^*(X_1))^2 + (t-1)\,E(f^*(X_1)\,f^*(X_2))]$$
$$= E(X_1X_2) - 2t\,E(f^*(X_1)\,X_2) + t\,E(f^*(X_1)\,X_2) =$$
$$E(X_1X_2) - t\,E(f^*(X_1)\,X_2) \tag{13}$$

Since X_1 and X_2 are exchangeable, this proves (7). Neglecting a factor t, using (12) and (13), the difference between the second and the first member of (6) equals

$$d = E(f^*(X_1)\,X_2) - 2E(f(X_1)\,X_2) + E\,f^2(X_1) + (t-1)\,E(f(X_1)\,f(X_2))$$

Replacing the first two terms by their expression given by (10) and (11) and using (2), we have

$$\begin{aligned} d = {} & E(f^*(X_1))^2 && + (t-1)\,E(f^*(X_1)\,f^*(X_2)) \\ & - 2\,E(f(X_1)\,f^*(X_1)) && - 2(t-1)\,E(f(X_1)\,f^*(X_2)) \\ & + E(f(X_1))^2 && + (t-1)\,E(f(X_1)\,f(X_2)) = \end{aligned}$$
$$E(f^*(X_1) - f(X_1))^2 + (t-1)\,[E(f^*_\Theta)^2 - 2E(f_\Theta f^*_\Theta) + E(f_\Theta)^2] =$$
$$E(f^*(X_1) - f(X_1))^2 + (t-1)\,E(f^*_\Theta - f_\Theta)^2 \geqslant 0$$

This proves (6) and it only remains to show that (9) is true. Writing $h^* = f^* - g^*$, we have from (5) and (8):

$$0 = h^*(X_1) + (t-1)\,E(h^*(X_2) \mid X_1)$$

Multiplying this last relation by $h^*(X_1)$ and taking the mean value, we have

$$0 = E(h^*(X_1))^2 + (t-1)\,E(h^*(X_1)\,h^*(X_2))$$

or, by (2):

$$0 = E(h(^*X_1))^2 + (t-1)\,E(h^*_\Theta)^2$$

This implies

$$E(h^*(X_1))^2 = 0$$

and thus (9).

1.4. *Corollary*

Let f^* be solution of (5). Then, for each f:

$$E(X_{t+1} - f^*(X_1) - \ldots - f^*(X_t))^2 \leqslant E(X_{t+1} - f(X_1) - \ldots - f(X_t))^2 \tag{14}$$

Demonstration.

Using (3) it easily follows that for every function φ of t variables we have

$$E(X_{t+1} - \varphi(X_1, \ldots, X_t))^2 = E(X_{t+1} - m_\Theta)^2 + E(m_\Theta - \varphi(X_1, \ldots, X_t))^2$$

The difference between the members of (14) then is the same as that between the members of (6).

1.5. *Remark. Notation. Definition*

In DE VYLDER (1976), the fundamental relation (5) is derived in a geometrical way. In that paper the existence of f^* is proved.

The *optimal semilinear credibility premium* of the year $t+1$ is defined and denoted by

$$E^*(X_{t+1} \mid X_1, \ldots, X_t) = f^*(X_1) + \ldots + f^*(X_t), \tag{15}$$

where f^* is solution of (5).

1.6. *Theorem*

$$E\, E^*(X_{t+1} \mid X_1, \ldots, X_t) = E(X_{t+1}) \tag{16}$$

Demonstration.

Follows from (5) and (15) by taking the mean values.

1.7. *Determination of the Optimal Premium*

If the variables X_1 and X_2 have a joint density $p(x, y)$, then equation (5) becomes

$$\int y\, p(x, y)\, dy = f^*(x) \int p(x, y)\, dy + (t-1) \int f^*(y)\, p(x, y)\, dy \tag{17}$$

This is an integral equation of *Fredholm* type for the unknown function f^*.

If X_1 can only assume, with probability one, a finite number of values, say $0, 1, 2, \ldots, n$, then (5) becomes the linear system

$$\sum_{j=0}^{n} j p_{ij} = f_i^* \sum_{j=0}^{n} p_{ij} + (t-1) \sum_{j=0}^{n} f_j^* p_{ij} \quad (i = 0, \ldots, n), \tag{18}$$

where

$$p_{ij} = P(X_1 = i, X_2 = j), \tag{19}$$

$$f_i^* = f^*(i). \tag{20}$$

Equations (17) and (18) may serve as well for theoretical investigations as for the numerical computation of the optimal premium. Only the joint distribution of X_1 and X_2 is needed.

1.8. *The Linear Credibility Premium*

We shall denote the usual linear credibility premium of the year $t+1$ by

$$\hat{E}(X_{t+1} \mid X_1, \ldots, X_t) = (1-Z)\, E(X_1) + \frac{Z}{t} (X_1 + \ldots + X_t), \tag{21}$$

where

$$Z = \frac{t\, cov(X_1, X_2)}{var\, X_1 + (t-1)\, cov(X_1, X_2)} \tag{22}$$

The mean square error in the approximation of m_Θ by this premium equals

$$(1 - Z)\, cov(X_1, X_2). \tag{23}$$

By what precedes, it is never less than the mean square error in the approximation of m_Θ by the optimal premium, given by (7).

2. FINITE PORTFOLIOS AND QUADRATIC FORMS

2.1. *Hypotheses. Definition*

From now on we assume that the range of values of X_1 is a finite set of numbers say 0, 1, 2, ..., n.

We use the notation (19) for p_{ij} and set

$$p_i = P(X_1 = i) = \sum_{j=0}^{n} p_{ij} \quad (i = 0, 1, \ldots, n)$$

We denote by Q_p the quadratic form in the variables $x_0, x_1, \ldots, x_n$:

$$Q_p = \sum_{i,j=0}^{n} p_{ij}\, x_i x_j \tag{24}$$

(In the notation Q_p, p is of course not a numerical index, but a fixed symbol related to the notation p_{ij}.)

If Θ also can only assume a finite number of distinct values, say $\theta_0, \theta_1, \ldots, \theta_\nu$, we call the portfolio a *finite portfolio* and we write

$$\begin{aligned} u_\alpha &= P(\Theta = \theta_\alpha), && (25)\\ p_{i/\alpha} &= P(X_1 = i \mid \Theta = \theta_\alpha). && (26)\end{aligned} \qquad \begin{pmatrix}\alpha = 0, 1, \ldots, \nu\\ i = 0, 1, \ldots, n\end{pmatrix}$$

The numbers (25) and (26) completely describe our portfolio. For example:

$$p_{ijk\cdots} = P(X_1 = i, X_2 = j, X_3 = k, \ldots) = \sum_{\alpha=0}^{\nu} u_\alpha\, p_{i/\alpha}\, p_{j/\alpha}\, p_{k/\alpha} \cdots \tag{27}$$

Note that it is not assumed that the portfolio be finite in the following theorem.

2.2. *Theorem*

The $(n+1) \times (n+1)$ matrix $[p_{ij}]$ is semidefinite positive.

Demonstration.

For every function f of one variable, we have by (2):

$$E(f(X_1)\, f(X_2)) = E f_\Theta^2 \geqslant 0$$

Writing $f(i) = x_j$, this gives

$$Q_p = \sum_{i,j=0}^{n} p_{ij} x_i x_j \geqslant 0$$

for every value of $x_0, x_1, \ldots, x_n$

2.3. *Theorem*

Let $[q_{ij}]$ be an arbitrary $(n+1) \times (n+1)$ symmetric matrix with nonnegative elements adding up to unity. Define $q_i (i = 0, \ldots, n)$ by

$$q_i = \sum_{j=0}^{n} q_{ij}$$

Then, if one of the matrices $[q_{ij}]$ or $[q_{ij} - q_i q_j]$ is semidefinite positive, so is the other.

Demonstration.

Let Q_q and R_q be the quadratic forms

$$Q_q = \sum_{i,j=0}^{n} q_{ij} x_i x_j,$$

$$R_q = \sum_{i,j=0}^{n} (q_{ij} - q_i q_j) x_i x_j = Q_q - \left(\sum_{j=0}^{n} q_i x_i\right)^2$$

Then

$$Q_q = R_q + \left(\sum_{i=0}^{n} q_i x_i\right)^2$$

and if R_q is semidefinite positive, so is Q_q, à fortiori.

Conversely, let Q_q be semidefinite positive. Define the couple of random variables Y_1, Y_2 by

$$P(Y_1 = i, Y_2 = j) = q_{ij} \qquad (i, j = 0, 1, \ldots, n)$$

For every f we have, setting $f(i) = x_i$:

$$E(f(Y_1) f(Y_2)) = \sum_{i,j=0}^{n} f(i) f(j) q_{ij} = \sum_{i,j=0}^{n} q_{ij} x_i x_j \geqslant 0$$

since Q_q is semidefinite positive. In particular, for the function $f - Ef(Y_1) = f - Ef(Y_2)$, we have

$$E((f(Y_1) - Ef(Y_1))(f(Y_2) - Ef(Y_2))) \geqslant 0$$

or

$$R_q = \sum_{i,j=0}^{n} (q_{ij} - q_i q_j) x_i x_j \geqslant 0$$

2.4. *Theorem*

In the finite portfolio the form Q_p equals

$$Q_p = \sum_{\alpha=0}^{\nu} u_\alpha \Big(\sum_{i=0}^{n} p_{i/\alpha} x_i \Big)^2$$

Demonstration.

By (27):

$$Q_p = \sum_{i,j=0}^{n} p_{ij} x_i x_j = \sum_{\alpha=0}^{\nu} u_\alpha \sum_{i=0}^{n} p_{i/\alpha} x_i \sum_{j=0}^{n} p_{j/\alpha} x_j = \sum_{\alpha=0}^{\nu} u_\alpha \Big(\sum_{i=0}^{n} p_{i/\alpha} x_i \Big)^2$$

2.5. *Theorem*

Let $Q_q = \sum_{i,j=0}^{n} q_{ij} x_i x_j$ be a quadratic form with nonnegative symmetric coefficients q_{ij} adding up to unity. Then, to every decomposition

$$Q_q = \sum_{i,j=0}^{n} q_{ij} x_i x_j = \sum_{\alpha=0}^{\nu} \Big(\sum_{i=0}^{n} a_{i\alpha} x_i \Big)^2 \tag{28}$$

of Q_q in a sum of squares of linear forms with nonnegative coefficients $a_{i\alpha}$, there corresponds a finite portfolio for which

$$p_{ij} = q_{ij}, \tag{29}$$

$$u_\alpha = \Big(\sum_{i=0}^{n} a_{i\alpha} \Big)^2, \tag{30}$$

$$p_{i/\alpha} = a_{i\alpha} \Big/ \sum_{i=0}^{n} a_{i\alpha} \tag{31}$$

$$(i = 0, \ldots, n; \quad \alpha = 0, \ldots, \nu)$$

Demonstration.

We suppose of course that no linear form of the decomposition is the zero form.

Define u_α and $p_{i/\alpha}$ by (30) and (31). From (31) we have

$$\sum_{i=0}^{n} p_{i/\alpha} = 1 \qquad (\alpha = 0, \ldots, \nu).$$

By setting $x_0 = x_1 = \ldots = x_n = 1$ in (28), we have $\sum_{\alpha=0}^{\nu} u_\alpha = 1$

Also

$$q_{ij} = \sum_{\alpha=0}^{\nu} a_{i\alpha} a_{j\alpha} = \sum_{\alpha=0}^{\nu} u_\alpha p_{i/\alpha} p_{j/\alpha} = p_{ij}$$

by taking the coefficient of $x_i x_j$ in (28) and using (30) and (31).

2.6. *Remarks*

(I) Given the matrix $[p_{ij}]$, every possible finite portfolio for which (19) is valid thus results from a decomposition of Q_p in a sum of squares of linear forms with nonnegative coefficients. For all such possible portfolios, the credible premium (optimal or linear) will be the same.

(II) By 2.2., a necessary condition on a given matrix $[q_{ij}]$ to be the $[p_{ij}]$ matrix of some portfolio, finite or not, is that $[q_{ij}]$ be semidefinite positive.

(III) In the classical theory of decomposition of a quadratic form in a sum of squares of linear forms, the latter are generally independent and in number not larger than the dimension of the matrix of the quadratic form. For a decomposition giving rise to a portfolio, this is no longer needed. On the other side, we need linear forms with nonnegative coefficients, which is not the case in the classical theory.

(IV) As a simple illustration, we consider the form Q in two variables

$$Q = \frac{1}{29}\,(3x^2 + 12xy + 14y^2)$$

Among a lot of others, three possible decompositions are

$$Q = \frac{4}{29}\left(\frac{x}{2} + \frac{y}{2}\right)^2 + \frac{9}{29}\left(\frac{x}{3} + \frac{2y}{3}\right)^2 + \frac{16}{29}\left(\frac{x}{4} + \frac{3y}{4}\right)^2,$$

$$Q = \frac{27}{29}\left(\frac{x}{3} + \frac{2y}{3}\right)^2 + \frac{2}{29}\left(0\,x + 1\,y\right)^2,$$

$$Q = \frac{200}{203}\left(\frac{3x}{10} + \frac{7y}{10}\right)^2 + \frac{3}{203}\left(1\,x + 0y\right)^2$$

To these three decompositions correspond three different finite portfolios with same $[p_{ij}]$ matrix equal to

$$\begin{bmatrix} 3/29 & 6/29 \\ 6/29 & 14/29 \end{bmatrix}$$

For each of the three portfolios we would find the same optimal premium and the same linear credibility premium.

If we had a decomposition with only one square of a linear form, the two variables X_1 and X_2 should be independent. So the third decomposition shows that, in the present case, these variables are "nearly" independent.

3. ADJUSTMENT OF A $[p_{ij}]$ MATRIX

3.1. *The Problem*

In the next section, we apply the theory to a concrete portfolio in automobile insurance. We limit ourselves to the consideration of the number of claims. Then p_{ij} is the probability of i claims in one year, say the first, and j claims in another year, say the second, for a contract chosen at random in the portfolio.

Practically, the probability p_{ij} is estimated by an observed frequency q_{ij}. Except perhaps for estimates from very large samples, the matrix $[q_{ij}]$, of course symmetrized in the obvious way, does not fit in the theory because generally it is not semidefinite positive. So it must be transformed, as slightly as possible, in a usable matrix $[p_{ij}]$.

3.2. *Smoothing on a Fixed Ascending Diagonal*

Suppose, for a moment, that the parameter θ of each fixed contract is interpreted as the mean number of claims in one year, and that the arrivals are poissonnian. Then we should have

$$P(X_1 = i \mid \Theta = \theta) = e^{-\theta} \frac{\theta^i}{i!} \quad (i = 0, 1, 2, \ldots) \tag{32}$$

But since, for practical reasons, we do not consider a number of claims in one year greater than a fixed integer n, we replace (32) by

$$P(X_1 = i \mid \Theta = \theta) = c_{n,\theta}\, e^{-\theta} \frac{\theta^i}{i!} \quad (i = 0, 1, \ldots, n) \tag{33}$$

where $c_{n,\theta}$ is the suitable norming factor.

Denoting by $U(\theta)$ the structure function of the portfolio, we have, for a contract chosen at random

$$p_{ij} = \int_0^\infty c_{n,\theta}^2\, e^{-2\theta} \frac{\theta^{i+j}}{i!\,j!}\, dU(\theta) \qquad (i, j = 0, 1, \ldots, n)$$

For the probability of k ($k = 0, 1, \ldots, 2n$) claims in two years, we have then

$$_2p_k = \sum_{\substack{i,j=0\\ i+j=k}}^{n} p_{ij} = \left(\sum_{\substack{i,j=0\\ i+j=k}}^{n} \frac{1}{i!\,j!} \right) \int_0^\infty c_{n,\theta}^2\, e^{-2\theta}\, \theta^{i+j}\, dU(\theta) \tag{34}$$

So, for $i + j = k$ ($i, j = 0, 1, \ldots, n$), p_{ij} and $_2p_k$ are related by

$$p_{ij} = a_{ij}\, {}_2p_k \tag{35}$$

where

$$(36) \qquad a_{ij} = \frac{\dfrac{1}{i!\,j!}}{\sum\limits_{\substack{i,j=0\\ i+j=k}}^{n} \dfrac{1}{i!\,j!}}, \quad (i, j = 0, \ldots, n;\ i+j=k)$$

If we take

$$(37) \qquad {}_2q_k = \sum_{\substack{i,j=0\\ i+j=k}}^{n} q_{ij}$$

and then use (35) with ${}_2p_k = {}_2q_k$, we have a first adjustment of the matrix $[q_{ij}]$. Since, for fixed k, the elements a_{ij} of (36) add up to unity, it is immediate that the sum of the elements of each ascending diagonal is the same in the initial and the adjusted matrix.

We reached (35), starting from a poissonnian hypothesis. Now we keep only (35) and abandon the poissonnian hypothesis, because this relation is in fact true in a more general situation. For example, if the factor $c_{n,0}^2\, e^{-2\theta}$ is replaced by another one not depending on i or j, then (35) remains true with a_{ij} given by (36).

3.3. *Extrapolation for the Last Ascending Diagonals*

For statistics deriving from small samples, the above method does not yet furnish a semidefinite positive $[p_{ij}]$ matrix. So a preliminary smoothing of the ${}_2q_k$'s is necessary.

If, again for one moment, we make the poissonnian hypothesis and do not neglect claims in number greater than n in one year, then we have

$$(38) \qquad {}_2p_k = \int_0^\infty e^{-2\theta}\, \frac{(2\theta)^k}{k!}\, dU(\theta), \qquad (k = 0, 1, 2, \ldots)$$

Writing

$$(39) \qquad r_k = k!\ {}_2p_k \qquad (k = 0, 1, 2, \ldots)$$

we have

$$r_k = \int e^{-2\theta}\, (2\theta)^k\, dU(\theta) \qquad (k = 0, 1, 2, \ldots)$$

From this relation it can be proved that

$$(40) \qquad r_k^2 \leqslant r_{k-1}\, r_{k+1}, \qquad (k = 1, 2, \ldots)$$

and that equality for some k can only hold in a portfolio of homogeneous composition (that means: Θ = constant a.e.), in which case it holds for every

k. In the case of a binomial negative distribution for the total number of claims in a fixed period (here 2 years), which amounts to a gamma density for Θ, it can be verified that, for $k \to \infty$, we have

$$\frac{r_{k-1}\, r_{k+1}}{r_k} \to 1$$

These considerations suggest the following method of adjustment. We take

$$r_0 = 0!\ {}_2q_0,\ r_1 = 1!\ {}_2q_1,\ \ldots,\ r_{k_0} = k_0!\ {}_2q_{k_0}$$

and, from k_0 on, taken as large as possible, we set

$$r_k = (1 + \varepsilon_{k,\alpha,\beta,\ldots}) \frac{r_{k-1}^2}{r_{k-2}} \qquad (k \geqslant k_0 + 1) \tag{41}$$

where $\varepsilon_{k,\alpha,\beta,\ldots}$ is a positive quantity, decreasing with increasing k and containing parameters α, β, ... to be determined in function of some requirements for the adjusted matrix. There is of course some arbitrariness in the choice of $\varepsilon_{k,\alpha,\beta,\ldots}$, but as we shall see in our numerical illustration of next section, this quantity, when properly chosen, introcuces only very small probabilities.

From the preceding discussion we only retain (41) and (39), because it is not difficult to see that (40) is valid in a more general situation than the poissonnian from which we started.

4. NUMERICAL ILLUSTRATION

4.1. *Basic Statistics*

The statistics used are those of Table 1.

TABLE 1: BASIC STATISTICS

$i \backslash j$	0	1	2	3	4	5
0	784	103	13	2	2	0
1	119	33	5	1	0	0
2	18	5	3	2	0	0
3	1	1	0	0	1	0
4	0	0	0	0	0	0
5	1	0	0	0	0	0

The number at the intersection of row i and column j in this table is the number of automobiles with i claims one year and j claims the following year among 1094 automobiles.

These statistics were established by *P. Thyrion* and used in THYRION (1972) and afterwards in DE VYLDER (1975).

On dividing by 1094 and symmetrizing, we obtain the matrix $[q_{ij}]$ of Table 2.

Most of our following numerical results were computed with a precision of 15 à 16 significant digits. Often, however, we reproduce the intermediate results with 3 significant digits only.

TABLE 2: NON ADJUSTED SYMMETRIZED MATRIX $[q_{ij}]$

		.717	.203	.0585	.0119	.00640	
							.00274
$i=0$	.717	.101	.0142	.00137	.000914	.000457	o
$i=1$	.101	.0302	.00457	.000914	o	o	.000914
$i=2$	.0142	.00457	.00274	.000914	o	o	o
$i=3$	.00137	.000914	.000914	o	.000457	o	o
$i=4$	.000914	o	o	.000457	o	o	o
$i=5$	.000457	o	o	o	o	o	
	.835	.137	.0224	.00366	.00137	.000457	
	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	

TABLE 3: ADJUSTED MATRIX $[p_{ij}]$

		.717	.203	.0585	.0119	.00493	
							.00261
$i=0$	.717	.101	.0146	.00149	.000308	.0000815	.00139
$i=1$	.101	.0293	.00446	.00123	.000408	.000134	.000676
$i=2$	.0146	.00446	.00185	.000815	.000335	.000127	000296
$i=3$	.00149	.00123	.000815	.000447	.000211	.0000909	.000116
$i=4$	.000308	.000408	.000335	.000211	.000114	.0000579	.0000410
$i=5$	.0000815	.000134	.000127	.0000909	.0000579	.0000410	
	.835	.137	.0222	.00428	.00143	.000532	
	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	

4.2. *Adjustment*

Our aim is to find a semidefinite positive matrix $(p_{ij}]$ as close as possible to the matrix $[q_{ij}]$.

Following the method explained in the preceding section, we take

$$\begin{aligned} {}_2p_0 &= q_{00} &&= .717 \\ {}_2p_1 &= q_{01} + q_{10} &&= .203 \\ {}_2p_2 &= q_{02} + q_{11} + q_{20} &&= .0585 \\ {}_2p_3 &= q_{03} + q_{12} + q_{21} + q_{30} &&= .0119 \end{aligned}$$

We tried of course to keep also for ${}_2p_4$ the observed corresponding frequency .00640, but this was unsuccessfull. From the above values, we have the value of r_0, r_1, r_2, r_3 by (39). We set

$$r_k = \left(1 + \frac{\alpha}{\beta^{k-4}}\right) \frac{r_{k-1}^2}{r_{k-2}} \qquad (k = 4, 5, \ldots, 10)$$

because we observed that a quantity $\varepsilon_{k,\alpha,\beta,\ldots}$ in (41) rapidly converging to zero gives a ${}_2p_4$ closer to .00640 than one converging more slowly to zero. From the values of the r_k ($k = 4, 5, \ldots, 10$) we deduce those of the p_k by (39) and choose α and β to satisfy

$$\sum_{k=0}^{10} {}_2p_k = 1 \tag{42}$$

From the values of the ${}_2p_k$ we then deduce those of the p_{ij} by (35).

For fixed β it is not difficult to determine α, with the required precision, from (42). So we still dispose of β. For a previously indicated reason, we try to take β as large as possible. Now, by calculating the characteristic values, we observed that for $\beta = 2$, we obtained a semidefinite positive matrix $[p_{ij}]$, while for $\beta = 4$, there appeared one negative characteristic value. We then tried the values $\beta = 2.1$, $\beta = 2.2$, $\ldots$, $\beta = 3.8$, $\beta = 3.9$ and found that for $\beta = 3$ all characteristic were still positive, while for $\beta = 3.1$ there appeared a negative one. In fact, for $\beta = 3$ there was a characteristic value so small that we preferred to take $\beta = 2.9$, although this was not essential. The corresponding value of α is $\alpha = 1.723\,569\,981\,730\,550$. The characteristic values of the adjusted $[p_{ij}]$ matrix are .732 .0151 .00154 .0000835 .0000096 .0000000081. For the adjusted matrix, the mean value of the number of claims in one year is .202607, while for the original matrix it is .200640. Instead of (42), we could have used the relation making these mean values equal, but then, unless we introduced a new parameter, we would have had to change proportionally the now kept fixed quantities ${}_2p_0$, ${}_2p_1$, ${}_2p_2$, ${}_2p_3$. Since the difference between the two means is small in our actual adjustment, we keep it as it is.

A glance at Tables 2 and 3 is enough to be convinced of the quality of our adjustment, especially when one looks at the partial sums indicated in the margins.

A characteristic of our adjustment is that it used only the numbers ${}_2p_k$ and not the decomposition of such a number on the corresponding ascending diagonal. In other words, instead of Table 1, we used only the frequencies of k claims in two years. It seems that our method can be adapted for the case were the frequency of k claims in one year is the only statistical material.

4.3. *A Theorically Possible Portfolio Compatible with the $[p_{ij}]$ Matrix*

If we decompose the quadratic form Q_p by *Lagrange*'s method (successive completion of squares), taking the variables in the order $x_0, x_1, \ldots, x_5$, we find after some normalisations:

$$Q_p = \sum_{i,j=0}^{5} p_{ij}\, x_i\, x_j =$$

$$
\begin{array}{llllllll}
.972 & (.859x_0 + & .122x_1 + & .0175x_2 + & .00178x_3 + & .000369x_4 + & .0000983x_5)^2 \\
+\ .0237 & & (.793x_1 + & .127\ x_2 + & .0545\ x_3 + & 0194\ \ x_4 + & .00652\ \ x_5)^2 \\
+\ .00401 & & & (.540\ x_2 + & .287\ \ x_3 + & .125\ \ x_4 + & .0488\ \ x_5)^2 \\
+\ .000285 & & & & (.392\ \ x_3 + & .373\ \ x_4 + & .235\ \ x_5)^2 \\
+\ .000025 & & & & & (.32\ \ x_4 + & .68\ \ x_5)^2 \\
+\ 0000032 & & & & & & (\ \ x_5)^2
\end{array}
$$

As explained in section 2, this decomposition defines a portfolio for which the $[p_{ij}]$ matrix is our adjusted $[p_{ij}]$.

This portfolio does not serve in the sequel, but we calculated it to make sure that our adjusted $[p_{ij}]$ matrix is not a theorically impossible one.

4.4. *The Optimal Premium and the Linear Premium*

To make comparisons sensefull, these premiums are of course calculated both for the adjusted $[p_{ij}]$ matrix.

4.4.1. The optimal premium

From (18), we obtain, in table 4, the values of the f_i^* for the indicated values of $t + 1$.

TABLE 4: COMPONENTS OF THE OPTIMAL PREMIUM
$E^*(X_{t+1} \mid X_1, \ldots, X_t) = f^*_{x_1} + f^*_{x_2} + \ldots + f^*_{x_t}$

$t+1$	f_0^*	f_1^*	f_2^*	f_3^*	f_4^*	f_5^*
2	.163922	.322485	.566282	1.285385	1.712988	2.060772
3	.070165	.201312	.385665	.938154	1.252583	1.495804
4	.041312	.154117	.301413	.748922	.993612	1.174104
5	.027911	.127399	.249519	.624949	822816	962363
6	.020394	.109677	.213655	.536605	.701129	.812356
7	015681	.096841	.187171	.470247	.609979	.700767
8	.012500	.087009	166728	.418507	539185	.614733
9	.010237	.079179	.150432	.377009	482654	.546539
10	.008562	.072763	.137116	.342977	.436504	.491274
20	002613	.041181	.073446	.179860	.219454	.238560
30	001290	.029042	.050507	.121616	144603	.155734
50	.000526	.018364	.031328	.073604	.084674	.091804
99	.000159	.009688	.016461	.037222	.040897	.046423
100	.000156	.009596	.016305	.036848	.040458	.045969

TABLE 5: PROBABILITY p_i OF i CLAIMS IN ONE YEAR

p_0	p_1	p_2	p_3	p_4	p_5
.834599	.136944	.022208	004283	.001434	.000532

From this table it follows, for example, that the optimal semilinear forecast of the number of claims in the 4th year, for a driver with 2, 2, 0 claims in the preceding years is

$$E^*(X_3|X_1=2, X_2=2, X_3=0) = f_2^*+f_2^*+f_0^* =$$
$$.301413+.301413+.041312 = .644138$$

To make a verification possible of relation (16) which amounts to

$$t\,E(f^*_{X_1}) = E(X_1)$$

or

$$t \sum_{i=0}^{5} p_i f_i^* = E(X_1)$$

where

$$E(X_1) = .202607$$

we give, in table 5, the values of p_i, the probability of i claims in one year, with a precision greater than in Table 3.

4.4.2. The linear premium

The credibility factor Z in (21), given in (22), is expressed in Table 6 for various values of $t+1$. Intermediate values computed from the not printed 15 digits precise $[p_{ij}]$ matrix are also indicated.

TABLE 6: CREDIBILITY FACTOR Z IN LINEAR FORECAST

$$E(X_{t+1} \mid X_1, \ldots, X_t) = (1-Z)\,E(X_1) + Z/t\,(X_1+\ldots+X_t)$$

$t+1$	Z
2	.231545
3	.376024
4	.474773
5	.546537
6	.601048
7	.643859
8	.678373
9	.706788
10	.730590
20	.851300
30	.897310
50	.936566
99	.967244
100	.967564
$E(X_1)$	= .202607
$E(X_1^2)$	= .300577
$E(X_1X_2)$	= .101142
$\mathrm{var}(X_1)$	= .259527
$\mathrm{cov}(X_1, X_2)$	= .060092

The linear forecast for the above considered driver is

$$\bar{E}(X_3 \mid X_1 = 2, X_2 = 2, X_3 = 0) = (1 - Z)\, E(X_1) + Z\,(2 + 2 + 0)\,/\,3 = .739445$$

4.4.3. The mean quadratic errors

Table 7 gives, for different values o. $t+1$, the mean square error in the approximation of the risk premium m_Θ by the optimal premium and the linear premium. The formulae used are (7) and (23).

As expected, the optimal premium is always closer to m_Θ, and thus to X_{t+1}, than the linear premium.

TABLE 7: MEAN SQUARE ERROR FOR THE OPTIMAL AND THE LINEAR PREMIUM

$t+1$	Optimal	Linear
2	.0438	.0462
3	.0347	.0375
4	.0288	.0316
5	.0247	.0272
6	.0217	.0240
7	.0193	.0214
8	.0175	.0193
9	.0164	.0176
10	.0147	.0162
20	.00822	.00894
30	.00574	.00617
50	.00359	.00381
99	.00188	.00197
100	.00186	.00195

4.4.4. Comparative Tables

The values of the optimal premium and the linear one are given in Tables 8 and 9 for $t+1=2$ and $t+1=3$ respectively. As is seen, these values may differ very much, even for relatively small values of X_1, X_2. Consider, for example the case $X_1 = 0$, $X_2 = 3$ in Table 9.

TABLE 8: OPTIMAL AND LINEAR FORECAST FOR SECOND YEAR ($t+1=2$)

X_1	Optimal	Linear
0	.163922	.155694
1	.322485	.387239
2	.566282	.618784
3	1.285385	.850329
4	1.712988	1.081873
5	2.060772	1.313419

TABLE 9: OPTIMAL AND LINEAR FORECAST FOR THE THIRD YEAR ($t+1=3$) [a]

$X_1 \backslash X_2$	0	1	2	3	4	5
0	140330	.271477	.455830	1.008319	1.322748	1.565969
	.126422	.314434	502446	690458	.878470	1.066482
1	.271477	.402624	.586977	1 139466	1.453895	1 697116
	.314434	.502446	.690458	.878470	1.066482	1 254494
2	.455830	.586977	771330	1.323819	1.638248	1.881469
	.502446	.690458	.878470	1 066482	1.254494	1.442506
3	1.008319	1.139466	1.323819	1.876308	2.190737	2.433958
	690458	.878470	1.066482	1 254494	1.442506	1.630518
4	1 322748	1 453895	1.638248	2 190737	2.505166	2.748387
	.878470	1.066482	1.254494	1 442506	1 630518	1.818530
5	1.565969	1.697116	1.881469	2.433958	2.748387	2 991608
	1.066482	1.254494	1.442506	1.630518	1.818530	2.006542

[a] The first number indicated is the optimal premium, the number beneath it, the linear one

In Table 9, the linear premium does of course not very on an ascending diagonal. This is not the case for the optimal premium. For example, 3 and 0 claims respectively in the first and the second year is much worse than 2 and 1 claim.

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