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FINITE SUM EVALUATION OF THE NEGATIVE BINOMIAL-EXPONENTIAL MODEL*

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1. INTRODUCTION

The compound negative binomial distribution with exponential claim amounts (severity) distribution is shown to be equivalent to a compound binomial distribution with exponential claim amounts (severity) with a different parameter. As a result of this, the distribution function and net stop-loss premiums for the Negative Binomial-Exponential model can be calculated exactly as finite sums if the negative binomial parameter α is a positive integer. The result is a generalization of LUNDBERG (1940).

2. BINOMIAL-EXPONENTIAL AND NEGATIVE BINOMIAL-EXPONENTIAL MODELS

Consider the distribution of

$$S = X_1 + X_2 + \ldots + X_N \tag{1}$$

where $X_1, X_2, X_3, \ldots$ are independently and identically distributed random variables with common exponential distribution function

$$F_X(x) = 1 - e^{-\lambda x}, \qquad x \geq 0 \tag{2}$$

and N is an integer valued random variable with probability function

$$p_n = Pr\{N = n\}, \qquad n = 0, 1, 2, \ldots . \tag{3}$$

Then the distribution function of S is given by

$$F_S(x) = \sum_{n=0}^{\infty} p_n F_X^{*n}(x) \qquad x > 0. \tag{4}$$

If $M_X(t)$, $M_N(t)$ and $M_S(t)$ are the associated moment generating functions, then

$$\begin{aligned} M_S(t) &= E_N E_X[e^{t(X_1 + \cdots + X_N)} \mid N = n] \\ &= \sum_{n=0}^{\infty} p_n\{M_X(t)\}^n \\ &= M_N(ln\ M_X(t)). \end{aligned} \tag{5}$$

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The moment generating function of the exponential distribution (2) is

$$M_X(t) = \frac{\lambda}{\lambda - t}. \tag{6}$$

First, consider the binomial distribution with probability function

$$p_n = \binom{n}{m} q^n p^{m-n}, \tag{7}$$

and moment generating function

$$M_N(t) = (p + qe^t)^m \tag{8}$$

where $p + q = 1$. Then, for the compound binomial distribution with exponential claim amounts (severity), (5) becomes

$$\begin{aligned} M_S(t) &= \left(p + q\frac{\lambda}{\lambda - t}\right)^m \\ &= \left(\frac{\lambda - pt}{\lambda - t}\right)^m. \end{aligned} \tag{9}$$

Now consider the negative binomial with probability function

$$p_n = \binom{\alpha + n - 1}{n} p^\alpha q^n \tag{10}$$

and the moment generating function

$$M_N(t) = \left(\frac{p}{1 - qe^t}\right)^\alpha \tag{11}$$

where $p + q = 1$. Then, for the compound negative binomial with exponential claim amounts (severity), (5) becomes

$$\begin{aligned} M_S(t) &= \left(\frac{p}{1 - q\dfrac{\lambda}{\lambda - t}}\right)^\alpha \\ &= \left(\frac{p\lambda - pt}{p\lambda - t}\right)^\alpha. \end{aligned} \tag{12}$$

Comparing (9) and (12), one notes that they are of identical form provided that α is integer valued. Hence, the Negative Binomial — Exponential model is equivalent to a Binomial—Exponential model. The negative binomial distribution with integer valued α is sometimes called the Pascal distribution according to JOHNSON and KOTZ (1969).[1]

3. PROBABILITY COMPUTATIONS

When the claim amounts (severity) are exponentially distributed as in (2), the sum of n claim amounts has a gamma distribution with distribution functions.

$$F_X^{*n}(x) = I(n, \lambda x) \tag{13}$$

where

$$I(k, t) = \int_0^t s^{k-1}e^{-s}/\Gamma(k)ds, \qquad k > 0 \tag{14}$$

is an incomplete gamma function. It is well known (formula (6.5.13) of ABRAMOWITZ and STEGUN (1964)) that for positive integer values of k, one can evaluate the incomplete gamma function as

$$I(k, t) = 1 - \sum_{j=0}^{k-1} \frac{t^j e^{-t}}{j!}, \qquad k = 1, 2, 3, \ldots \; . \tag{15}$$

Substituting (15) and (13) into (4) results in

$$F_S(x) = p_0 + \sum_{n=1}^{\infty} p_n\{1 - \sum_{j=0}^{n-1} \frac{(\lambda x)^j e^{-\lambda x}}{j!}\}$$

$$= 1 - \sum_{n=1}^{\infty} p_n \sum_{j=0}^{n-1} \frac{(\lambda x)^j e^{-\lambda x}}{j!}, \qquad x > 0. \tag{16}$$

If N is binomial, (16) becomes

$$F_S(x) = 1 - \sum_{n=1}^{m} \binom{m}{n} q^n p^{m-n} \sum_{j=0}^{n-1} \frac{(\lambda x)^j e^{-\lambda x}}{j!}, \qquad x > 0 \tag{17}$$

which is easily evaluated since it is a finite sum. If N is negative binomial (16) will become the infinite sum

$$F_S(x) = 1 - \sum_{n=1}^{\infty} \binom{\alpha + n - 1}{n} p^\alpha q^n \sum_{j=0}^{n-1} \frac{(\lambda x)^j e^{-p\lambda x}}{j!}, \qquad x > 0 \tag{18}$$

which is computationally inconvenient.

[1] It should be noted that this correspondence is different from the usual correspondence between the negative binomial distribution and the binomial distribution obtained by comparing (8) and (11).

However, since (12) is of the same form as (9), one can use (17) to evaluate the distribution of S for the negative binomial when α is an integer; i.e.

$$(19) \qquad F_S(x) = 1 - \sum_{n=1}^{\alpha} \binom{\alpha}{n} q^n p^{\alpha-n} \sum_{j=0}^{n-1} \frac{(p\lambda x)^j e^{-p\lambda x}}{j!}, \qquad x > 0$$

The result (19) is a generalization of the Geometric-Exponential model studied by LUNDBERG (1940) since the geometric distribution is a special case of the negative binomial distribution (10) with $\alpha = 1$. For the Geometric-Exponential model, (19) reduces to

$$(20) \qquad F_S(x) = 1 - qe^{-p\lambda x}$$

which is the result of LUNDBERG (1940).

When α is an integer, formula (19) makes the exact computation of the distribution function for the Negative Binomial-Exponential model easy to carry out. When α is not an integer, it is suggested that the computation be done for several adjacent integer values so that an interpolation can be carried out to obtain the value at α In order to assess the error involved in the interpolation, one can resort to the standard methods of numerical analysis For example, the error in approximating $F_S(x)$, now denoted $F_S(x \mid \alpha)$, by linearly interpolating between $F_S(x \mid [\alpha])$ and $F_S(x \quad [\alpha + 1])$ is exactly

$$-\tfrac{1}{2}(\alpha - [\alpha])\,([\alpha+1] - \alpha)F_S''(x \mid \xi)$$

where $F''(x \mid \xi)$ is the second derivative with respect to α of $F(x \mid \alpha)$ evaluated at the point $\alpha = \xi$ where $[\alpha] < \xi < [\alpha + 1]$ The unknown derivative can be approximated by a second difference such as $\Delta^2 F_S(x \mid [\alpha - 1])$ or $\Delta^2 F_S(x \mid [\alpha])$ or, better yet, the average of these two values. These methods are found in most standard texts on numerical analysis. By carrying out the calculation for several integral values, interpolation can be carried out and estimates of the error can be calculated.

Rather than provide extensive tables for possible combinations of α, p, λ and x, the authors leave to the reader the evaluation of the error for the specific situations in which the reader may be interested.

4. STOP-LOSS COMPUTATIONS

For a stop-loss level of x, the net stop-loss premium is given by

$$(21) \qquad R(x) = \int_x^\infty (y - x)\,dF_S(y), \qquad x > 0$$

which can be rewritten as

$$(22) \qquad R(x) = E[S] - \int_0^x \{1 - F_S(y)\}\,dy, \qquad x > 0$$

Upon substitution of (19) into (22), the net stop-loss premium for the Negative Binomial-Exponential model becomes

$$R(x) = \frac{\alpha q}{p\lambda} - \int_0^x \sum_{n=1}^{\alpha} \binom{\alpha}{n} q^n p^{\alpha-n} \sum_{j=0}^{n-1} \frac{(p\lambda y)^j e^{-p\lambda y}}{j!} dy$$

$$= \frac{\alpha q}{p\lambda} - \sum_{n=1}^{\alpha} \binom{\alpha}{n} q^n p^{\alpha-n} \sum_{j=0}^{n-1} \frac{1}{j!} \int_0^x (p\lambda y)^j e^{-p\lambda y} dy$$

$$= \frac{\alpha q}{p\lambda} - \sum_{n=1}^{\alpha} \binom{\alpha}{n} q^n p^{\alpha-n} \sum_{j=0}^{n-1} \frac{I(j+1, p\lambda x)}{p\lambda}$$

$$= \frac{1}{p\lambda}[\alpha q - \sum_{n=1}^{\alpha} \binom{\alpha}{n} q^n p^{\alpha-n} \sum_{j=0}^{n-1} \{1 - \sum_{k=0}^{j} \frac{(p\lambda x)^k e^{-p\lambda x}}{k!}\}]$$

$$= \frac{1}{p\lambda}[\alpha q - \sum_{n=1}^{\alpha} n \binom{\alpha}{n} q^n p^{\alpha-n} + \sum_{n=1}^{\alpha} \binom{\alpha}{n} q^n p^{\alpha-n} \sum_{k=0}^{n-1} \sum_{j=k}^{n-1} \frac{(p\lambda x)^k e^{-p\lambda x}}{k!}]$$

$$= \frac{1}{p\lambda}[\alpha q - \alpha q + \sum_{n=1}^{\alpha} \binom{\alpha}{n} q^n p^{\alpha-n} \sum_{k=0}^{n-1} (n-k) \frac{(p\lambda x)^k e^{-p\lambda x}}{k!}]$$

$$= \frac{e^{-p\lambda x}}{p\lambda}[\sum_{n=1}^{\alpha} \binom{\alpha}{n} q^n p^{\alpha-n} \sum_{k=0}^{n-1} (n-k) \frac{(p\lambda x)^k}{k!}] \qquad (23)$$

which is a finite sum consisting of $\alpha(\alpha + 1) / 2$ terms.

When $\alpha = 1$, the net stop-loss premium for the Geometric-Exponential model becomes

$$R(x) = \frac{qe^{-p\lambda x}}{p\lambda} \qquad (24)$$

which can also be obtained directly from (20).

REFERENCES

Abramowitz, M and Stegun, I A (1964) *Handbook of Mathematical Functions*, National Bureau of Standards, Washington.

Johnson, N L and Kotz, S (1969) *Distributions in Statistics: Discrete Distributions*, John Wiley and Sons, New York.

Lundberg, O (1940) On *Random Processes and their Application to Sickness and Accident Statistics*, Almqvist and Wiksells, Uppsala