# LARGE CLAIMS IN INSURANCE MATHEMATICS*

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LADIES, GENTLEMEN,

I take great pleasure in addressing this audience. As you might know I'm a mathematician with a deep interest in insurance mathematics. As such, it is my sincere opinion that the gap between practising actuaries and theoretical researchers can be made substantially smaller. If my contribution can help in bridging the gap, I will feel fully compensated for the effort it took to prepare this lecture and the results contained therein.

The simple fact that we meet on the occasion of the sixteenth ASTIN Colloquium gives me a challenging opportunity to help in creating a platform on which both theoreticians and practitioners can meet.

The subject of my lecture stems from a long interest in large claims: What are they? Are they really dangerous? Is there a way to get them under control? Can one recognize them in practical situations?

I like to express in simple mathematical terms some results that might help in acquiring better insight on the impact of large claims in insurance mathematics. Perhaps, no result will be of immediate applicability as reality is too complicated to be described by the simplicity of the results to follow. Nevertheless the latter can be considered as building blocks of a real world in which one has to tackle large claims in theory as well as in practice.

## 1. WHAT ARE LARGE CLAIMS?

On other occasions I have tried to set up mathematical definitions of what one might call a large claim. None of these approaches seemed to satisfy the practitioners. What could we do better than inquire with people in practice, what they meant by large claims? Here is an anthology of the main answers to the question stated above.

ANSWER 1. *Large claims are the upper* 10% *largest claims.*

It is not quite clear why 10 is used? I see two main reasons why this answer is put forward. The lay out of claim statistics very often has extremely broad intervals for the highest claims; secondly, many reinsurance treaties use proportional reinsurance.

ANSWER 2. *Every claim that consumes at least* 5% *of the sum of claims, or at least* 5% *of the net premiums.*

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This description might be appropriate for small portfolios although it is again not very precise.

ANSWER 3. *Every claim for which the actuary has to go and see one of the chief members of the company.*

Alternatively every claim overshooting a preassigned quantity. Needless to say that stop-loss reinsurance treaties trigger off this reply.

ANSWER 4. *Hidden in the following exchange of thoughts.*

T: Mister actuary, what do you mean by large claims?
A: They don't exist.
T: Sorry, I don't quite understand.
A: Well, large claims don't exist since we reinsure dangerous portfolios.
T: But how do you find out whether or not a portfolio is dangerous?
A: This is clear: we watch for large claims.
T: But, what do you mean by large claims?

From the above answers we can draw some conclusions:

— practitioners believe in large claims;
— they don't precisely know how to define them;
— the reinsurance treaties used in their company give guidelines on how to deal with large claims.

Almost all respondents gave some explicit examples of what they consider to be large claims. There is of course the classical set: earthquakes, tornados, air crashes, floods, etc. At least two other samples were illustrated with actual data.

EXAMPLE 1. Portfolio of fire insurance for wooden houses in Scandinavia. apart from small fires, sometimes a burning house sets the surrounding forest on fire and threatens other houses in the immediate vicinity.

EXAMPLE 2. Portfolio of schoolbus insurance. The typical course of life of a bus looks as follows: the first 8 to 10 years the bus is used on long distance trips; then the bus is employed on one day excursions; the bus ends its career as a schoolbus. It is not hard to forecast that lack of good maintenance makes schoolbuses accident prone.

How does one transform the above vague quotations into hard mathematical terms? Scanning the existing literature dealing with large claims we find that there is often agreement on the claim size distribution of large claims. The non-existence of certain moments or the use of so-called "shadow claims" suggest that fat-tailed distributions like the Pareto-distributions are appropriate models for dealing with large claims.

## 2 HOW DANGEROUS ARE LARGE CLAIMS?

To get some feeling for the danger resulting from large claims we deal with the classical ruinproblem. Assume that consecutive claims occur, according to a Poisson process, $\{N(t);\ t \geq 0\}$ with parameter $\lambda > 0$. Call these successive claim amounts $B_1, B_2, \ldots$ and assume that they are independent with common distribution of $B$, say $B(x) = P\{B \leq x\}$ where $B(0) = 0$. We assume moreover that the claims sizes are independent of the Poisson process. The risk reserve accumulated up to time $t$ is given by

$$Y(t) = u + ct - \sum_{i=1}^{N(t)} B_i$$

where $u$ is the initial reserve and $c$ is the loading corresponding to premium payments.

We are interested in the distribution of the time of ruin, i.e., $T_u = \inf\{t\colon Y(t) < 0\}(= +\infty$ if no such $t$ exists), where $u$ refers to the initial reserve. Our basic assumptions are the following.

- A(i): $c = 1$, this is established by a proper choice of the time scale;
- A(ii): $\rho \equiv \lambda EB < 1$, on the average the income per unit time exceeds the expenses;
- A(iii): $1 - B(x) \sim x^{-\alpha} L(x)$ where $\alpha \geq 1$ and $L$ a slowly varying function ($B$ is of Pareto-type).

We write

$$P\{T_u \leq t\} = P\{T_u < \infty\} - P\{t < T_u < \infty\}. \tag{1}$$

It is well-known (see: P. Embrechts and N. Veraverbeke (1982). Estimates for the probability of ruin with special emphasis on the possibility of large claims. *Insurance Math. Econom.* **1**, 55–72) that under A(i), (ii), (iii) ruin in finite time satisfies the asymptotic equality

$$P\{T_u < \infty\} \sim \frac{\rho}{1-\rho}[1 - \bar{B}(u)] \quad \text{for } x \to \infty \tag{2}$$

where $\bar{B}(u) = (EB)^{-1} \int_0^u [1 - B(y)]\, dy$. (This specific result can also be found in B. von Bahr (1975). Asymptotic ruin probabilities when exponential moments do not exist. *Scand. Actuarial J.* 6–10.) A few trial calculations show that even for very large $u$, this probability may be considerable if $\alpha$ is small or (and) if $\rho$ is close to 1.

Looking back at (1) one might hope that the term $P\{t < T_u < \infty\}$ will lower the above probability considerably. However A(i), (ii), (iii) imply that for all $u \geq 0$ and $t \to \infty$ (a full proof will be published later)

$$P[t < T_u < \infty] \sim \rho(1-\rho)^{1-\alpha}[1 - \bar{B}(t)]\left\{1 + \sum_1^{\infty} \rho^n \bar{B}^{(n)}(u)\right\} \tag{3}$$

which for $u$ large reads

$$P[t < T_u < \infty] \sim \rho(1-\rho)^{-\alpha}[1-\bar{B}(t)]$$

independently of $u$. Hence some of the probability of getting ruined in $[0, t]$ can be shifted to $[t, \infty)$ by increasing $u$; however ruin remains highly probable. It is obvious from the above considerations that something has to be done. Listening to the practitioner reinsurance might be appropriate.

## 3. IS REINSURANCE HELPFUL?

Let us rephrase the question somewhat to get a result with broader applicability. Instead of A(iii) we would like to impose a condition that mainly allows small claims. Although the condition B(iv) is somewhat technical one might vaguely interpret it as meaning that

$$1-B(x) < K e^{-\delta x}$$

for some $\delta > 0$ so that high claims are very improbable. Let $\Lambda(s) = s - \lambda[1-b(s)]$ where $b(s) = E[\exp - sB]$ is the Laplace transform of the claim size $B$. We assume

- B(i): $c = 1$;
- B(ii); $\rho < 1$;
- B(iii): $B(x)$ is a non-lattice distribution;
- B(iv): there exist a value $\kappa > 0$ such that $\Lambda(-\kappa) = 0$.

The results corresponding to (2) and (3) are now: Under B(i), (ii) and (iv), there exists a constant $C_1$ such that (see the above mentioned paper by Embrechts-Veraverbeke)

$$P\{T_u < \infty\} \simeq C_1 e^{-\kappa u} \quad \text{as } u \to \infty. \tag{4}$$

Also (see J. L. Teugels (1982). Estimation of ruin probabilities. *Insurance Math. Econom.* **1**, 163–175) for a constant $C_2$ and $u$ and $t$ large

$$P\{t < T_u < \infty\} \sim C_2 e^{-\upsilon u} u e^{-\theta t} t^{-3/2} \tag{5}$$

for a constant $\upsilon \in (0, \kappa)$ and a constant $\theta > 0$. Actually $\upsilon$ is defined by $\Lambda'(-\upsilon) = 0$ while $\theta = -\Lambda(-\upsilon)$.

The interpretation of (4) and (5) is that if the initial reserve $u$ is large enough, ruin in $[0, T]$ and in $[T, \infty]$ is highly improbable. Alternatively one might say that under the B-conditions no reinsurance is anymore necessary.

Turning back to large claims, stop-loss reinsurance is based on a retention $M$, the corresponding truncated distribution has no tail and hence for constants $C_M$, $\upsilon_M$ and $\theta_M$ by (5)

$$P\{t < T_u < \infty\} \sim C_M u e^{-\upsilon_M u} e^{-\theta_M t} t^{-3/2}. \tag{6}$$

A basic problem is how to determine $M$ in such a way that after one year ruin is only possible with a small probability, starting with initial reserve $u$. Now one can get some rough estimates on $\upsilon_M$ and $\theta_M$.

$$v_M \begin{cases} \geqslant \dfrac{1}{M} \log\left\{1+\dfrac{1-\rho}{\rho}\varphi\right\} \\ \leqslant \dfrac{1}{M}\dfrac{\varphi}{\varphi-1} \log 1+\left\{1+\dfrac{1-\rho}{\rho}\dfrac{1}{A}\right\} \end{cases} \tag{7}$$

where $\varphi = \alpha^{-1}\mu M$, $\mu = EY$, $\alpha = EY^2$ and $A = \mu^{-1}M\{1-B(M)\}$. Also

$$\theta_M \begin{cases} \geqslant v_M(1-\rho)+\dfrac{\alpha\rho v_M}{2M\mu}(1-e^{v_M M}) \\ \leqslant v_M(1-\rho)+v_M\rho\left(1-\exp\dfrac{\alpha v_M}{2\mu}\right). \end{cases} \tag{8}$$

These formulas are interesting as first approximation in that the dependence on $M$ and $\rho$ is emphasized. The larger $M$, the smaller $v_M$ and $\theta_M$; the closer $\rho$ is to 1 the smaller $v_M$ and $\theta_M$.

The more information one has about the claim size distribution the more precision can be obtained for $v_M$ and $\theta_M$. We will return to these results in a forthcoming publication.

To get a quick overview of the differences between the situations described in 2 and 3 I propose to introduce *isoruines*, i.e., curves in the $(t, u)$ plane that give the same probability of ruin. Three typical cases are depicted in fig. 1. Here $\varepsilon = P\{T_u \leqslant t\}$. The full lines correspond to an exponential distribution with $\lambda = \frac{1}{2}$, $\mu = 1$ while the dotted lines come from $\bar{B}(x) = 2/\pi$ arctan $x$ and $\rho = \frac{1}{2}$, $\alpha = 2$.

For fixed time the initial reserves for the Pareto-type case are much larger than for the exponential case. Also the infinite horizon values are very different. For example for the exponential case $u = 7.9$ gives $\varepsilon = 0.01$ while for the Pareto-type case $u = 69.4$.

Let me point out that there are two types of isoruins i.e., $P[T_u \leqslant t]$ and $P[T_u > t]$ with quite different characteristics.

## 4. HOW CAN ONE DETECT LARGE CLAIMS?

We like to formulate an approach which might be useful in practice if suitably adapted. A characteristic of existing reinsurance procedures is that estimated retentions and premiums are based on past year's data. Perhaps one realizes an overall loss in the portfolio; in other cases like largest claims reinsurance and ECOMOR, estimates are based on the largest claims registered during the year. Unfortunately the ordering of claims in increasing order is a time consuming undertaken even for a computer. More important is that the ruin disaster is only discovered at the end of the book-keeping year.

The following procedure tries to do better. We assume that claims are Pareto distributed so that $1-B(x) = x^{-\alpha}$ $(x \geqslant 1)$ with unknown $\alpha$. We look at the claims as they come in: any time a claim is reported bigger than all previous claims we get a *warning*. More precisely we look at the sequence of so-called *record times*

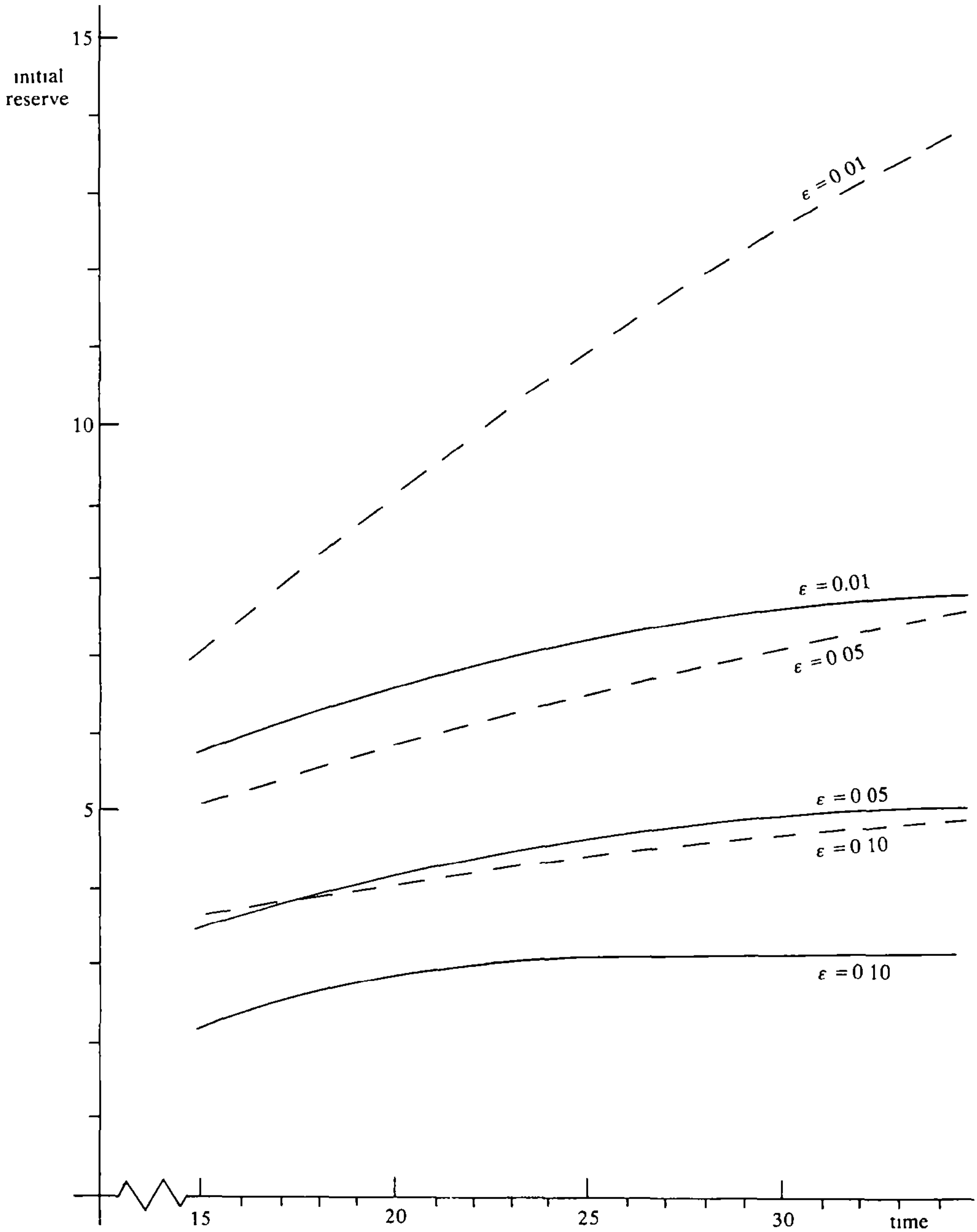


FIGURE 1. Isoruins defined by $\varepsilon = P\{T_u \leq t\}$

and *record values*.

$$N_1 = 1, \qquad X_1 = B_1$$
$$N_2 = \inf\{l : B_l > X_1\}, \qquad X_2 = B_{N_2}$$
$$N_3 = \inf\{l : B_l > X_2\}, \qquad X_3 = B_{N_3}$$
$$\ldots \qquad \ldots .$$

It is easy to show that $1/\alpha$ can be excellently estimated by the sequence of warning values. Indeed, for any $k$, $1/k \log X_k$ is an unbiased, efficient estimator for $1/\alpha$. (Alternatively the sample mean of the $\log B_i$ could be used to estimate $1/\alpha$).

If we are afraid for example that the average claim $EB$ should be infinite we can construct confidence intervals for $1/\alpha$ and check whether or not the value 1 belongs to it. In this sense consecutive warnings could lead to an alarmingly low value of $\alpha$.

To be more precise let us denote by $p_n^{(\alpha)}(\beta)$ the probability that all estimates of $1/\alpha$ based on the first $n$ warning values suggest that $\alpha < \beta^{-1}$, i.e.,

$$p_n^{(\alpha)}(\beta) = P\left\{\bigcap_{k=1}^{n}\left[\frac{\log X_k}{k} > \beta\right]\right\}.$$

For example we give a short table for $\beta = 1$ and $n = 1, 2, 3, 4, 5$.

$n \rightarrow$

| $\beta = 1$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 4 | 0 01832 | 0.00168 | 0 00020 | 0 00001 | $4\ 10^{-6}$ |
| 3 | 0.04979 | 0 00991 | 0.00253 | 0 00072 | 0.00022 |
| 2 | 0 13533 | 0.05495 | 0 02727 | 0.01487 | 0.00858 |
| 1.5 | 0.22313 | 0 12447 | 0.08193 | 0 05825 | 0.04334 |

For example, assume $\alpha = 4$. Any time a warning value suggest that $\alpha < 1$ we sound the alarm. The first warning results in a fake alarm with probability 0.01832. A second fake alarm is so improbable that we better drop the hypothesis that $\alpha = 4$ (or even $\alpha \geqslant 4$).

Similarly, 5 consecutive alarms make $\alpha \geqslant 1.5$ already quite unlikely.

Any time a warning leads to an alarm, the company might ponder to take reinsurance and that while the claims are still coming in.

Although this and allied procedures look promising refinements are necessary since in a sample of size $n$ there are on the average only $\log n$ warning or record values.

## 5. CONCLUSIONS

We have only indicated some major items where recent mathematical developments can help the practitioner to get a better understanding of reality.

As I hope to continue research in the area of primary reinsurance I hope to get inspiring suggestions from you. Let me mention a few topics on which the practitioner has acquired insight, indispensable for the theoretician:

— what are the rules of thumb used in practice to decide about reinsurance?
— how are the retentions chosen?
— what premium principles are used?
— who makes the final decision?
— can you provide examples (or even data) on portfolios where large claims do occur?

Let me finally draw a parallel between insurance mathematics and statistics. In both fields there are researchers and practitioners; in both areas a gap is felt in between theory and practice; fortunately in both domains practitioners and theoreticians meet in fruitful conferences.

There is one more parallel that nicely applies to myself: not too many people in actuarial sciences have to worry about large claims. On the statistical side, few statisticians are involved in the study of the corresponding area of statistics, namely that of so-called outliers. If you feel that my interest in large claims is unsound, then do with me as with outliers in statistics: get them out. But if you feel that large claims are important, help me in getting a better understanding of what they are and what you would really like to do with them.

Thank you for you kind attention.

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