ON A CLASS OF SEMI-MARKOV RISK MODELS OBTAINED AS CLASSICAL RISK MODELS IN A MARKOVIAN ENVIRONMENT

JEAN-MARIE REINHARD
Groupe AG, Brussels, Belgium

ABSTRACT

We consider a risk model in which the claim inter-arrivals and amounts depend on a markovian environment process. Semi-Markov risk models are so introduced in a quite natural way. We derive some quantities of interest for the risk process and obtain a necessary and sufficient condition for the fairness of the risk (positive asymptotic non-ruin probabilities). These probabilities are explicitly calculated in a particular case (two possible states for the environment, exponential claim amounts distributions).

KEYWORDS
Semi-Markov processes, ruin theory.

1. INTRODUCTION

Several authors have used the semi-Markov processes in Queuing Theory and in Risk Theory [e.g., CINLAR (1967), NEUTS (1966), NEUTS and SHUN-ZER CHEN (1972), PURDUE (1974), JANSSEN (1980), REINHARD (1981)]. Besides, some duality results lead to nice connections between the two theories [FELLER (1971), JANSSEN and REINHARD (1982)].

Semi-Markov risk models may be defined as follows. Consider a risk model in continuous time; let B_n $(n \in N_0)$* and U_n $(n \in N_0)$ denote respectively the amount and the arrival time of the nth claim. Put $A_0 = B_0 = U_0 = 0$ and define $A_n = U_n - U_{n-1}$ $(n \in N_0)$. We suppose that the A_n and B_n are random variables defined on a complete probability space $(\Omega, \mathscr{A}, P)$; the variables A_n $(n \in N_0)$ are a.s. positive. Let now J_n $(n \in N)$ be random variables defined on $(\Omega, \mathscr{A}, P)$ and taking their values in $J = \{1, \ldots, m\}$ $(m \in N_0)$. Suppose finally that $\{(J_n, A_n, B_n); n \in N\}$ is a Markov chain with transition probabilities defined by a bivariate semi-Markov kernel:

$$P[J_{n+1} = j, A_{n+1} \leq t, B_{n+1} \leq x \mid J_k, A_k, B_k; k = 0, \ldots, n] = Q_{J_n j}(x, t) \quad \text{a.s.} \tag{1.1}$$
$$(j \in J, \quad t \geq 0, \quad x \in R, \quad n \in N)$$

where $Q_{ij}(x, \cdot)$ and $Q_{ij}(\cdot, t)$ are right continuous nondecreasing functions satisfying:

$$Q_{ij}(x, t) \geq 0, \quad Q_{ij}(\infty, 0) = 0 \qquad (i, j \in J; t \geq 0)$$
$$\sum_{j=1}^{m} Q_{ij}(\infty, \infty) = 1 \qquad (i \in J)$$
$$Q_{ij}(-\infty, \infty) = 0 \qquad (i, j \in J).$$

* $N_0 = \{1, 2, 3, \ldots\}$, $N = \{0, 1, 2, 3, \ldots\}$

Such processes, called $(J-Y-X)$ processes, were studied by JANSSEN and REINHARD (1982) and REINHARD (1982). In the particular case where

$$Q_{ij}(x,t)=(1-e^{-\lambda t})Q_{ij}(x), \qquad \lambda>0, \tag{1.2}$$

the processes $\{A_n\}$ and $\{(J_n, B_n)\}$ being independent, JANSSEN (1980) interpreted the variables J_n as the types of the successive claims. The next section will show that another subclass of semi-Markov kernels appears if we assume that the risk depends on an environment process.

2. RISK PROCESSES IN A MARKOVIAN ENVIRONMENT

Suppose that the claim frequency and amounts depend on the external environment (economic situation . . .) and that the external environment may be characterized at any time by one of the m states $1, \dots, m$ ($m \in N_0$). Let I_0 denote the state of the environment at time $t=0$ and let I_n, $n=1, \dots$, be the state of the environment after its nth transition. Put $T_0=0$ and let T_n ($n \in N_0$) be the time at which occurs the nth transition of the environment process. We suppose that I_n and T_n ($n \in N$) are random variables defined on $(\Omega, \mathcal{A}, P)$ and taking their values in J and R^+ respectively. Define now $Y_n=T_n-T_{n-1}$ ($n \in N_0$), $Y_0=0$ and assume that

$$P[I_{n+1}=j, Y_{n+1}\leq t \mid (I_k, Y_k), k=0,\dots,n, I_n=i]=h_{ij}(1-e^{-\lambda_i t}) \tag{2.1}$$

$$(i,j\in J; \quad t\geq 0; \quad n\in N)$$

where the λ_i are strictly positive real numbers and $H=(h_{ij})$ is a transition matrix:

$$h_{ij}\geq 0, \qquad \sum_{k=1}^{m} h_{ik}=1 \qquad (i,j\in J).$$

$\{I_n, n\in N\}$ is then a Markov chain with a matrix of transition probabilities $H=(h_{ij})$:

$$h_{ij}=P[I_{n+1}=j \mid I_n=i]. \tag{2.2}$$

Define $N_e(t)=\sup\{n: T_n\leq t\}$ and $I(t)=I_{N_e(t)}$ ($t\geq 0$). The process $\{I(t), t\geq 0\}$ is a finite-state Markov process; it is known that the number of transitions of the environment process $\{I(t)\}$ in any finite interval $(s,t]$, i.e., $N_e(t)-N_e(s)$, is a.s. finite.

Denote now by J_n the state of the environment process at the arrival of the nth claim:

$$J_n=I(U_n) \qquad (n\in N). \tag{2.3}$$

We will suppose that the following assumptions are satisfied:

(H1) The sequences of random variables (A_n) and (B_n) are conditionally independent given the variables J_n.

(H2) The distribution of a claim depends uniquely on the state of the environment at the time of arrival of that claim. Let

$$F_i(x)=P[B_n\leq x \mid J_n=i] \qquad (i\in J, \quad n\in N, \quad x\in R) \tag{2.4}$$

(H3) Let $N(t)$ be the number of claims occurring in $(0, t]$. If $I(u)=i$ for all u in some interval $(t, t+h]$, then the number of claims occurring in that interval, i.e., $N(t+h)-N(t)$, has a Poisson distribution with parameter α_i ($\alpha_i>0$); we assume further that given the process $\{I(t)\}$ the process $\{N(t)\}$ has independent increments. So

$$P[N(t+h)=n+1 \mid N(t)=n, I(u)=i \text{ for } t<u\leqslant t+h]=\alpha_i h+o(h). \tag{2.5}$$

The process $\{N(t); t\geqslant 0\}$ appears thus as a Poisson process with parameter modified by the transitions of the environment process.

Under the above assumptions it may be shown that $\{(J_n, A_n, B_n), n\in N\}$ is a $(J\text{–}Y\text{–}X)$ process with semi-Markov kernel $\mathcal{Q}$ defined by (1.1). $\{(J_n, A_n), n\in N\}$ is a Markov renewal process [see PYKE (1961)]; we denote its kernel by $\mathcal{V}=(V_{ij}(\cdot))$:

$$V_{ij}(t)=P[J_{n+1}=j, A_n\leqslant t \mid (J_k, A_k), k=0,\dots,n; J_n=i] \tag{2.6}$$

$$(i, j\in J, \quad n\in N, \quad t\geqslant 0).$$

Moreover it follows from the assumptions that

$$Q_{ij}(x, t)=V_{ij}(t)F_j(x) \qquad (i, j\in J, \quad t\geqslant 0, \quad x\in R). \tag{2.7}$$

$\{J_n, n\in N\}$ is a Markov chain with matrix P of transition probabilities defined by

$$P_{ij}=P[J_{n+1}=j \mid J_n=i]=Q_{ij}(\infty,\infty)=V_{ij}(\infty) \qquad (i, j\in J). \tag{2.8}$$

In the next section it will be shown how the semi-Markov kernel $\mathcal{Q}$ (or equivalently $\mathcal{V}$) can be deduced from the instantaneous rates α_i, the transition matrix H, the constants λ_i and the distributions $F_i(\cdot)$.

3. COMPUTATION OF THE KERNEL

Let us first introduce some notations: for any mass function (i.e., right continuous and non-decreasing) $G(t)$ defined on R^+ let

$$\tilde{G}(s)=\int_0^\infty e^{-st}G(t)\,dt, \qquad g(s)=\int_{0-}^\infty e^{-st}\,dG(t)$$

provided the above integrals converge.

The following system of integral equations may be easily deduced from the hypothesis

$$V_{ij}(t)=\delta_{ij}\frac{\alpha_i}{\alpha_i+\lambda_i}(1-e^{-(\alpha_i+\lambda_i)t})+\lambda_i\sum_{k=1}^{m}h_{ik}\int_0^t e^{-(\alpha_i+\lambda_i)u}V_{kj}(t-u)\,du \tag{3.1}$$

$$(i, j\in J, \quad t\geqslant 0).$$

The first term in the right side of (3.1) corresponds to the case where a claim occurs before the environment changes, the second term to the case where the environment changes before a claim occurs.

For $s\geqslant 0$, define now the following matrices:

$$L(s)=(h_{ij}\lambda_i/(\alpha_i+s+\lambda_i)), \qquad E(s)=(\delta_{ij}\alpha_i/(\alpha_i+s+\lambda_i)).$$

By taking the Laplace transforms of both sides in (3.1) we obtain

$$\tilde{V}_{ij}(s)=\delta_{ij}\frac{\alpha_i}{s(\alpha_i+\lambda_i+s)}+\frac{\lambda_i}{\alpha_i+\lambda_i+s}\sum_{k=1}^{m}h_{ik}\tilde{V}_{kj}(s) \tag{3.2}$$

$$(i,j\in J;\quad s>0),$$

or, in matrix notation,

$$[I-L(s)]\tilde{V}(s)=(1/s)E(s)\qquad (s>0) \tag{3.3}$$

(we will always use the same symbol for a matrix and its elements whenever this causes no ambiguity). As for any $s\geqslant 0$

$$L_i(s)=\sum_{j=1}^{m}L_{ij}(s)=\frac{\lambda_i}{\alpha_i+\lambda_i+s}<1,$$

$I-L(s)$ is regular for $s\geqslant 0$ and consequently (3.3) has as unique solution

$$\tilde{V}(s)=(1/s)[I-L(s)]^{-1}E(s)\qquad (s>0), \tag{3.4}$$

or equivalently

$$v(s)=[I-L(s)]^{-1}E(s)\qquad (s>0). \tag{3.5}$$

As $p_{ij}=V_{ij}(\infty)=\lim_{s\searrow 0}v_{ij}(s)$, the matrix P of the transition probabilities of the chain $\{J_n\}$ can be directly deduced from (3.5):

$$P=[I-L(0)]^{-1}E(0). \tag{3.6}$$

Notice that the semi-Markov kernel $\mathscr{V}$ is solution of a first order linear differential system: by deriving (3.1) with respect to t we obtain

$$V'_{ij}(t)=\alpha_i\delta_{ij}+\sum_{k=1}^{m}[\lambda_i h_{ik}-(\alpha_i+\lambda_i)\delta_{ik}]V_{kj}(t)\qquad (i,j\in J;\quad t\geqslant 0). \tag{3.7}$$

4. SOME RESULTS ABOUT QUANTITIES RELATED TO THE RISK PROCESS

In this section we derive some explicit expressions or equations related to the semi-Markov risk-process defined in the preceding sections.

4.1. *Stationary Probabilities of the Chain* $\{J_n\}$

From now on we suppose that the chain $\{J_n\}$ is irreducible. As m is finite there exists a unique probability distribution $\bar{\eta}=(\eta_1,\ldots,\eta_m)$ such that

$$\eta_i>0\qquad (i\in J), \tag{4.1}$$

$$\sum_{i=1}^{m}\eta_i h_{ij}=\eta_j\qquad (j\in J).$$

We have then:

THEOREM 1

The Markov chain $\{J_n; n \in N\}$ is irreducible and aperiodic (thus ergodic as $m < \infty$). Its stationary probabilities are given by

$$\pi_i = \frac{\alpha_i \eta_i}{\lambda_i} \left\{ \sum_{j=1}^{m} \frac{\alpha_j \eta_j}{\lambda_j} \right\}^{-1} \qquad (i \in J). \tag{4.2}$$

Proof

Let $i, j \in J$. As the chain $\{I_n\}$ is irreducible, there exists $n \in N$ such that $h_{ij}^{(n)} > 0$. It may be easily seen that this implies $(L^n(0))_{ij} > 0$. Now we obtain from (3.6):

$$p_{ij} = \sum_{n=0}^{\infty} (L^n(0))_{ij} \frac{\alpha_j}{\alpha_j + \lambda_j}. \tag{4.3}$$

The probabilities p_{ij} are thus strictly positive for all $i, j \in J$.

It remains to show that $\bar{\pi}P = \bar{\pi}$. Define the diagonal matrices

$$D = \left\langle \delta_{ij} \frac{\lambda_i}{\alpha_i + \lambda_i} \right\rangle, \qquad A = \left(\delta_{ij} \frac{\alpha_i}{\lambda_i} \right). \tag{4.4}$$

We have then $L(0) = DH$, $E(0) = I - D$, $\bar{\pi} = K\bar{\eta}A$ (where K is the norming factor in the right side of (4.2)), $AD = I - D$; (3.6) may be written as follows:

$$P = I - D + DHP. \tag{4.5}$$

Now

$$\bar{\pi}P = \bar{\pi} - \bar{\pi}D + \bar{\pi}DHP = \bar{\pi} - K[\bar{\eta}(I - D) - \bar{\eta}(I - D)HP].$$

As $\bar{\eta}H = \bar{\eta}$, we obtain

$$\bar{\pi}P = \bar{\pi} - K\bar{\eta}[(I - D) - (I - DH)P] = \bar{\pi}, \tag{4.6}$$

the last equality resulting from (4.5).

Note that (4.2) has an immediate intuitive interpretation: η_i is the asymptotic probability of finding the chain $\{I_n; n \in N\}$ in state i; $(\lambda_i)^{-1}$ is the mean time spent by the process $\{I(t); t \geq 0\}$ in state i before its next transition; α_i is the mean number of claims occurring per time unit when the process $\{I(t); t \geq 0\}$ sojourns in state i; π_i appears thus well as the asymptotic average number of claims occurring in environment i.

4.2. *Number of Claims Occurring in* $(0, t)$

The equations obtained here could be derived from the general theory of semi-Markov processes. It is, however, interesting to restate them directly as the semi-Markov kernel $\mathscr{V}$ is itself expressed as the solution of the differential system (3.7)

Define

$$N_j(t) = \begin{cases} \sum_{k=1}^{N(t)} 1_{[J_k = j]} & \text{if } N(t) > 0, \\ 0 & \text{if } N(t) = 0, \end{cases} \tag{4.7}$$

where as previously $N(t)$ is the number of claims occurring in $(0, t)$. $N_j(t)$ is clearly the number of claims occurring in environment j before t. Let

$$M_{ij}(t) = E[N_j(t) \mid J_0 = i]$$

and

$$M_i(t) = E[N(t) \mid J_0 = i] = \sum_{j=1}^{m} M_{ij}(t) \qquad (t \geq 0).$$

The following system of integral equations is easily obtained:

$$M_{ij}(t) = \delta_{ij} e^{-\lambda_i t} \alpha_i t + \int_0^t \lambda_i e^{-\lambda_i u} \left[\delta_{ij} \alpha_i u + \sum_k h_{ik} M_{kj}(t-u) \right] du$$

or

$$M_{ij}(t) = \delta_{ij} \alpha_i \frac{1 - e^{-\lambda_i t}}{\lambda_i} + \sum_{k=1}^{m} \lambda_i h_{ik} \int_0^t e^{-\lambda_i u} M_{kj}(t-u)\, du \qquad (t \geq 0). \tag{4.8}$$

Taking the derivatives of both sides with respect to t we obtain

$$M'_{ij}(t) = \alpha_i \delta_{ij} - \lambda_i M_{ij}(t) + \lambda_i \sum_{k=1}^{m} h_{ik} M_{kj}(t) \qquad (t \geq 0), \tag{4.9}$$

and after summation over j

$$M'_i(t) = \alpha_i - \lambda_i M_i(t) + \lambda_i \sum_{k=1}^{m} h_{ik} M_k(t) \qquad (t \geq 0). \tag{4.10}$$

(4.9) with the boundary condition $M_{ij}(0) = 0$ $(i, j \in J)$ has a unique solution.

4.3. *Further Properties of the Claim Arrival Process*

We extend first to the $(J\text{–}Y\text{–}X)$ processes a well known property of Markov chains and $(J\text{–}X)$ processes.

THEOREM 2

Let $\{(J_n, A_n, B_n);\ n \in N\}$ be a $(J\text{–}Y\text{–}X)$ process with state space $J \times R^+ \times R$ and kernel $\mathscr{Q}$ defined by (1.1). Suppose that the Markov chain $\{J_n\}$ is irreducible (and thus positive recurrent as m is finite). Let $Z_{ij}(x, t)$, $i, j \in J$, be real measurable functions defined on $R \times R^+$ such that the integrals

$$\int_{-\infty}^{\infty} \int_0^{\infty} |Z_{ij}(x, t)| Q_{ij}(dx, dt) \qquad (i, j \in J)$$

are finite. Let

$$z_i = \sum_{j=1}^{m} \int_{-\infty}^{\infty} \int_0^{\infty} Z_{ij}(x, t) Q_{ij}(dx, dt) = E(Z_{J_{n-1}J_n}(B_n, A_n) | J_{n-1} = i).$$

Define then $n_{i,0} = 0$, $n_{i,k} = \inf\{n > n_{i,k-1}: J_n = i\}$ for $k \in N_0$ (recurrence indices of state i) and let

$$\zeta_{i,r} = E\left(\sum_{k=n_{i,r}+1}^{n_{i,r+1}} Z_{J_{k-1}J_k}(B_k, A_k) \right) \qquad (i \in J, \quad r \in N).$$

The random variables $\zeta_{i,r}$, $r = 1, 2, \ldots$, are i.i.d. and we have

$$E(\zeta_{i,r}) = \frac{1}{\pi_i} \sum_{j=1}^{m} \pi_j z_j \qquad (i \in J, \quad r \in N_0) \tag{4.11}$$

where the π_i are the stationary probabilities of the chain $\{J_n\}$.

Proof
Define

$${}_i p_{ij}^{(n)} = P[J_n = j, J_k \neq i \text{ for } k = 1, \ldots, n-1 | J_0 = i] \qquad (i, j \in J; \quad n \in N_0).$$

We have then

$$E(\zeta_{i,r}) = \sum_{k \neq i} \sum_{n=1}^{\infty} {}_i p_{ik}^{(n)} z_k + z_i \qquad (i \in J, \quad r \in N_0).$$

(4.11) follows since we know from Markov chain theory that $\sum_{n=1}^{\infty} {}_i p_{ik}^{(n)} = \pi_k / \pi_i$.

Mean Recurrence Time of Claims Occurring in a Given Environment

We return now to the risk model. Define

$$G_{ij}(t) = P[N_j(t) > 0 | J_0 = i] \qquad (i, j \in J; \quad t \geq 0). \tag{4.12}$$

$G_{ij}(\cdot)$ is the distribution function of the first time at which a claim occurs in environment j given that the initial environment is i. Let

$$\gamma_{ij} = \int_{0-}^{\infty} t \, dG_{ij}(t) \qquad (i, j \in J). \tag{4.13}$$

We could obtain a system of integral equations for the distributions $G_{ij}(\cdot)$ and derive from it after passage to the Laplace–Stieltjes transforms a linear system for the γ_{ij}. We may, however, proceed more directly as follows:

$$(4.14)\qquad \gamma_{ij} = \sigma_{ij}\int_0^\infty e^{-(\alpha_i+\lambda_i)t}\left[\alpha_i t + \lambda_i \sum_{k=1}^m h_{ik}(t+\gamma_{kj})\right] dt$$

$$+(1-\delta_{ij})\int_0^\infty e^{-(\alpha_i+\lambda_i)t}\left[\alpha_i(t+\gamma_{ij}) + \lambda_i \sum_{k=1}^m h_{ik}(t+\gamma_{kj})\right] dt;$$

we thus get a linear system:

$$(4.15)\qquad \frac{\lambda_i + \delta_{ij}\alpha_i}{\alpha_i+\lambda_i}\gamma_{ij} = \frac{1}{\alpha_i+\lambda_i} + \frac{\lambda_i}{\alpha_i+\lambda_i}\sum_{k=1}^m h_{ij}\gamma_{kj} \qquad (i, j \in J).$$

The diagonal elements γ_{ii} (mean recurrence time of claims occurring in state i) may be explicitly expressed by using Theorem 2. Define $Z_{ij}(x, t) = t$; then $z_i = E(A_1 | J_0 = i)$. We have

$$z_i = \int_0^\infty e^{-(\alpha_i+\lambda_i)t}\left[\alpha_i t + \lambda_i \sum_{j=1}^m h_{ij}(t+z_j)\right] dt \qquad (i \in J).$$

Hence

$$z_i = \frac{1}{\alpha_i+\lambda_i} + \frac{\lambda_i}{\alpha_i+\lambda_i}\sum_{j=1}^m h_{ij} z_j \qquad (i \in J),$$

or, if $\bar{z} = (z_1, \ldots, z_m)^t$ and $\bar{y} = (\alpha_1^{-1}, \ldots, \alpha_m^{-1})^t$,

$$\bar{z} = (I - L(0))^{-1}E(0)\bar{y} = P\bar{y};$$

we have thus

$$(4.16)\qquad z_i = E(A_1 | J_0 = i) = \sum_{j=1}^m p_{ij}\frac{1}{\alpha_j} \qquad (i \in J)$$

and consequently

$$(4.17)\qquad \sum_{i=1}^m \pi_i z_i = E_\pi(A_1) = \sum_{j=1}^m \pi_j \frac{1}{\alpha_j}.$$

Using finally theorem 2 we have:

THEOREM 3

For any $i \in J$:

$$(4.18)\qquad \gamma_{ii} = \frac{1}{\pi_i}\sum_{j=1}^m \pi_j \frac{1}{\alpha_j}.$$

Renewal Theorem—Stationary Probabilities

Given that $J_0 = i$, the times at which claims occur in environment j form a pure renewal process if $i = j$ and a delayed renewal process if $i \neq j$. We have the classical renewal equations:

$$M_{ij}(t)=\int_0^t [1+M_{jj}(t-u)]\,d\dot{G}_{ij}(u) \qquad (i,j\in J;\quad t\geq 0). \tag{4.19}$$

As the distribution functions $G_{jj}(\cdot)$ are clearly not arithmetic, the expected number of claims occurring in environment j within $(t,t+h)$ tends to $h(\gamma_{jj})^{-1}$ when $t\to\infty$ whatever the initial environment i, i.e.,

$$\lim_{t\to\infty}[M_{ij}(t+h)-M_{ij}(t)]=\frac{h}{\gamma_{jj}} \qquad (i,j\in J;\quad h\geq 0). \tag{4.20}$$

[see FELLER (1971), Chapt. XI]. From (4.20) it follows that

$$\lim_{t\to\infty}\frac{M_{ij}(t)}{t}=\frac{1}{\gamma_{jj}} \qquad (i,j\in J). \tag{4.21}$$

Define now

$$F_{ij}(t)=(p_{ij})^{-1}V_{ij}(t) \tag{4.22}$$

$$R_{jk}^{(i)}(u,t)=P[J_{N(t)}=j,J_{N(t)+1}=k,\,U_{N(t)+1}\leq t+u\,|\,J_0=i];$$

the last quantity is thus the probability, given that $J_0=i$, that the last claim before t occurred in environment j and that the next claim will occur in environment k before time $t+u$. We deduce immediately from Theorem 7.1 of PYKE (1961b) that

$$\lim_{t\to\infty}R_{jk}^{(i)}(u,t)=p_{jk}\frac{1}{\gamma_{jj}}\int_0^u [1-F_{jk}(y)]\,dy, \tag{4.23}$$

which limit is independent of i; we denote it by $R_{jk}^0(u)$. Let now

$$V_{ij}^*(u)=\gamma_{ii}z_i^{-1}R_{ij}^0(u)$$

and define a chain $\{(\bar{J}_n,\bar{A}_n,\bar{B}_n);\,n\in N\}$ as follows:

$$\begin{cases}\bar{A}_0=\bar{B}_0=0 \quad \text{a.s.}\\ P[\bar{J}_1=j,\bar{A}_1\leq u,\bar{B}_1\leq x\,|\,\bar{A}_0,\bar{B}_0;\bar{J}_0=i]=V_{ij}^*(u)F_j(x)\\ P[\bar{J}_n=j,\bar{A}_n\leq u,\bar{B}_n\leq x\,|\,\bar{A}_k,\bar{B}_k,\bar{J}_k(k=0,\dots,n-1);\bar{J}_{n-1}\\ \qquad =i]=V_{ij}(u)F_j(x)\end{cases} \tag{4.24}$$

$$(i,j\in J;\quad u\in R^+,\quad x\in R,\quad n\geq 2).$$

where z_i is defined by (4.16).

We define for that chain the same quantities and adopt the same notations as for the chain $\{(J_n,A_n,B_n);\,n\in N\}$. The risk processes associated with the two chains are identical except that for the second one the time of occurrence of the first claim is distributed according to the semi-Markov kernel $(V_{ij}^*(\cdot))$ instead of $(V_{ij}(\cdot))$. Suppose now that

$$a_i=P[\bar{J}_0=i]=\frac{z_i}{\gamma_{ii}} \qquad (i\in J). \tag{4.25}$$

Then [see PYKE (1961b)]:

$$P[\bar{J}_{\bar{N}(t)}=j, \bar{J}_{\bar{N}(t)+1}=k, \bar{U}_{\bar{N}(t)+1}\leqslant t+u]=R^0_{jk}(u). \tag{4.26}$$

5. PREMIUM INCOME—RUIN PROBABILITIES

We assume that the company managing the risk receives premiums at a constant rate $c_i>0$ during any time interval the environment process remains in state i. The premium income process is thus characterized by a vector $(c_1, \ldots, c_m)$ with positive entries. Denote by $A^c(t)$ the aggregate premium received during $(0, t)$:

$$A^c(t)=\sum_{k=1}^{N_e(t)} c_{I_{k-1}}(T_k-T_{k-1})+c_{I_{N_e(t)}}(t-T_{N_e(t)}) \tag{5.1}$$

and by $B(t)$ the aggregate amount of the claims occurring in $(0, t)$:

$$B(t)=\sum_{k=0}^{N(t)} B_k \qquad (t\geqslant 0). \tag{5.2}$$

Assume now that the initial amount of free assets of the company is $u\geqslant 0$. The amount of free assets at time t is then

$$Z_u(t)=u+S(t) \tag{5.3}$$

where

$$S(t)=A^c(t)-B(t). \tag{5.4}$$

Define then

$$R_i(u, t)=P[Z_u(v)\geqslant 0 \text{ for } 0\leqslant v\leqslant t\,|\,J_0=i] \qquad (i\in J;\quad u, t\geqslant 0), \tag{5.5}$$

$$R_i(u)=R_i(u,\infty)=P[Z_u(v)\geqslant 0 \text{ for all } v\geqslant 0\,|\,J_0=i] \qquad (i\in J,\quad u\geqslant 0). \tag{5.6}$$

We will refer to the probabilities (5.5) as to the finite time non-ruin probabilities and to the probabilities (5.6) as to the asymptotic non-ruin probabilities.

5.1. *Random Walk of the Free Assets*

Denote by A^c_n the premium received between the occurrences of the $(n-1)$th and nth claims $(n\geqslant 1)$. Define then

$$X_k=A^c_k-B_k \qquad (k=1,2,\ldots);\quad X_0=0 \quad \text{a.s.}, \tag{5.7}$$

$$S_n=\sum_{k=0}^{n} X_k \qquad (n\in N). \tag{5.8}$$

Clearly the chain $\{(J_k, X_k); k\in N\}$ is a $(J\text{–}X)$ process, $\{S_n\}$ is a random walk defined on the finite Markov chain $\{J_n\}$ [see JANSSEN (1970); MILLER (1962); NEWBOULD (1973)]. The amount of free assets just after the occurrence of the

nth claim is given by

$$Z_u(A_0+\cdots+A_n)=u+S_n$$

and clearly

$$R_i(u)=P\left[\inf_k S_k \geq -u \mid J_0=i\right]. \tag{5.9}$$

From now on we assume that the d.f. $F_i(\cdot)$ has a finite expectation μ_i $(i\in J)$. We get then

$$b_i=E[B_k\mid J_{k-1}=i]=\sum_{j=1}^m p_{ij}\mu_j \tag{5.10}$$

and

$$z_i^c=E[A_k^c\mid J_{k-1}=i]=\int_0^\infty e^{-(\alpha_i+\lambda_i)t}\left[\alpha_i c_i t+\lambda_i \sum_{j=1}^m h_{ij}(c_i t+z_j^c)\right]dt$$

so that, concluding as to obtain (4.16),

$$z_i^c=\sum_{j=1}^m p_{ij}\frac{c_j}{\alpha_j} \qquad (i\in J). \tag{5.11}$$

If the premium rates are constant whatever the state of the environment, i.e., if $\bar{c}=(c,\ldots,c)$, we obtain naturally $z_i^c=cz_i$. We conclude from (5.10) and (5.11) that

$$\zeta_i=E[X_k\mid J_{k-1}=i]=\sum_{j=1}^m p_{ij}\left(\frac{c_j}{\alpha_j}-\mu_j\right). \tag{5.12}$$

Notice that we would obtain the same result for a semi-Markov risk model with kernel $\mathcal{Q}^*$ defined by

$$Q_{ij}^*(x,t)=p_{ij}(1-e^{-\alpha_j t})F_j(x). \tag{5.13}$$

Define now

$$D_{i,r}=\sum_{k=n_{i,r}+1}^{n_{i,k+1}} X_k \qquad (i\in J,\quad r\in N_0)$$

where the $n_{i,r}$ are the recurrence indices of claims occurring in environment i as defined in section 4.3; for i fixed the variables $D_{i,r}$ $(r=1,2,\ldots)$ are i.i.d.; $D_{i,r}$ is clearly the variation of the free assets between the rth and $(r+1)$th claims occurring in environment i. We obtain from theorem 2

$$E(D_{i,r})=\frac{1}{\pi_i}\sum_{j=1}^m \pi_j\left(\frac{c_j}{\alpha_j}-\mu_j\right) \qquad (i\in J,\quad r\in N_0). \tag{5.14}$$

As the variables A_k^c are absolutely continuous and conditionally (given the J_k) independent of the variables B_k, the process $\{(J_n,S_n);\, n\in N\}$ is not degenerate [see NEWBOULD (1973)], i.e., there exist no constants $w_1, \ldots, w_m$ such that $P[X_n = w_j - w_i | J_{n-1} = i, J_n = j] = 1$, or equivalently there exists no i such that $D_{i,r} = 0$ a.s. (NEWBOULD (1973), lemma 2). Using Proposition 3A of JANSSEN (1970) we obtain then

THEOREM 4

Let

$$d = \sum_{j=1}^{m} \pi_j \left(\frac{c_j}{\alpha_j} - \mu_j \right). \tag{5.15}$$

Then (i) If $d > 0$, the random walk $\{S_n\}$ drifts to $+\infty$, i.e. $\lim_{n\to\infty} S_n = \infty$ a.s.; $R_i(u) > 0$, $\forall u \geq 0$, $i \in J$. (ii) If $d < 0$, the random walk $\{S_n\}$ drifts to $-\infty$, i.e. $\lim_{n\to\infty} S_n = -\infty$ a.s.: $R_i(u) = 0$, $\forall u \geq 0$, $i \in J$. (iii) If $d = 0$, the random walk $\{S_n\}$ is oscillating, i.e. $\limsup S_n = +\infty$ a.s. and $\liminf S_n = -\infty$ a.s.; $R_i(u) = 0$, $\forall u \geq 0$, $i \in J$.

Notice that when $m = 1$ theorem 4 reduces evidently to the classical result for the Poisson model.

5.2. *Distribution of the Aggregate Net Pay-out in* $(0, t)$

From now on we suppose that the claim amounts are a.s. positive:

$$F_i(0-) = 0, \qquad F_i(0) < 1 \qquad \forall i \in J. \tag{5.16}$$

Recall that $A^c(t)$ and $B(t)$ denote respectively the aggregate premium received and the aggregate amount of claims occurred during $(0, t)$. Then denote by $C(t)$ the net pay-out of the company in $(0, t)$:

$$C(t) = B(t) - A^c(t) = -S(t) \qquad (t \geq 0)$$

Let then

$$W_{ij}(x, t) = P[C(t) \leq x, I(t) = j | I(0) = i] \qquad (i, j \in J; \quad t \geq 0). \tag{5.17}$$

Define now

$$c_0 = \max\{c_i;\, i \in J\}, \qquad J_0 = \{i \in J : c_i = c_0\}.$$

It is easy to prove the following

LEMMA

(i) $W_{ij}(x, t) = 0$ for $i, j \in J$ and $x < -c_0 t$;

(ii) $W_{ij}(x, t) > 0$ for $i, j \in J$ and $x > -c_0 t$;

(iii) $W_{ij}(-c_0 t, t) > 0$ if $i, j \in J_0$ and either $i = j$ or there exist $r \in N_0$ and $i_1, \ldots, i_r \in J_0$ such that $h_{ii_1} h_{i_1 i_2} \ldots h_{i_r j} > 0$; $W_{ij}(-c_0 t, t) = 0$ otherwise.

Let now

$$\tilde{W}_{ij}(s,t)=\int_{-c_0t}^{\infty} e^{-sx}W_{ij}(x,t)\,dx; \qquad \tilde{W}(s,t)=(\tilde{W}_{ij}(s,t)) \qquad (s>0),$$

$$w_{ij}(s,t)=\int_{-c_0t-}^{\infty} e^{-sx}\,d_xW_{ij}(x,t)=s\tilde{W}_{ij}(s,t); \qquad w(s,t)=(w_{ij}(s,t)) \qquad (s>0),$$

$$\varphi_i(s)=\int_{0-}^{\infty} e^{-sx}\,dF_i(x) \qquad (s\geq 0).$$

The following theorem gives an explicit expression for the transform matrix $\tilde{W}(s,t)$.

THEOREM 5

For $s>0$ and $t\geq 0$,

$$\tilde{W}(s,t)=1/s\exp\{-T(s)t\} \tag{5.18}$$

where

$$T_{ij}(s)=\delta_{ij}(\alpha_i+\lambda_i-\alpha_i\varphi_i(s)-c_is)-\lambda_ih_{ij}. \tag{5.19}$$

Proof

For $x\geq -c_0t$, $t\geq 0$ and $h>0$ we obtain easily

$$\begin{aligned} W_{ij}(x,t+h)=&(1-(\alpha_i+\lambda_i)h)W_{ij}(x+c_ih,t)\\ &+\alpha_ih\int_{0-}^{x+c_ih+c_0t}W_{ij}(x+c_ih-y,t)\,dF_i(y)\\ &+\lambda_ih\sum_{k=1}^{m}h_{ik}W_{kj}(x+c_ih,t)+o(h). \end{aligned} \tag{5.20}$$

Dividing (5.20) by h and letting h tend to 0, we get

$$\begin{aligned} \frac{\partial}{\partial t}W_{ij}(x,t)-c_i\frac{\partial}{\partial x}W_{ij}(x,t)=&-(\alpha_i+\lambda_i)W_{ij}(x,t)\\ &+\alpha_i\int_{0-}^{x+c_0t}W_{ij}(x-y,t)\,dF_i(y)\\ &+\lambda_i\sum_{k=1}^{m}h_{ik}W_{kj}(x,t)\\ &(x\geq -c_0t,\ t\geq 0). \end{aligned} \tag{5.21}$$

We multiply now each term in (5.21) by e^{-sx} and integrate from $-c_0t$ to ∞. We obtain so

$$\frac{\partial}{\partial t}\tilde{W}_{ij}(s,t)+\sum_{k=1}^{m}[\delta_{ik}(\alpha_i+\lambda_i-\alpha_i\varphi_i(s)-c_is)-\lambda_ih_{ik}]\tilde{W}_{kj}(s,t) \tag{5.22}$$
$$=(c_0-c_i)\,e^{sc_0t}W_{ij}(-c_0t,t) \qquad (s>0,\,t\geq 0).$$

According to the above lemma the right side of (5.22) is always zero. In matrix notation, the solution of (5.22) is then easily seen to be

$$\tilde{W}(s,t)=\exp\{-T(s)t\}K \tag{5.23}$$

where

$$K=\tilde{W}(s,0)=(1/s)w(s,0)=(1/s)I \qquad (s>0).$$

The proof is complete.

Notice that when $m=1$ (5.18) reduces to the known result for the classical Poisson model.

5.3. *Seal's Integral Equation for the Finite Time non-ruin Probabilities*

We show in this subsection that the SEAL's integral equation (1974) may be extended to the here considered semi-Markov model. We still assume that the claim amounts are a.s. positive.

Define for $u,t\geq 0$ and $i,j\in J$

$$R_{ij}(u,t)=P[Z_u(v)\geq 0 \text{ for } 0\leq v\leq t, I(t)=j\,|\,I(0)=i]; \tag{5.24}$$

we have clearly

$$R_i(u,t)=\sum_{j=1}^{m}R_{ij}(u,t) \qquad (i\in J;\quad u,t\geq 0).$$

Define further for $s>0$ and $t\geq 0$

$$\tilde{R}_{ij}(s,t)=\int_0^\infty e^{-su}R_{ij}(u,t)\,du; \qquad \tilde{R}(s,t)=(\tilde{R}_{ij}(s,t)),$$

$$r_{ij}(s,u)=\int_{0-}^\infty e^{-su}\,d_uR_{ij}(u,t)=s\tilde{R}_{ij}(s,t); \qquad r(s,t)=(r_{ij}(s,t)).$$

We obtain easily for $u,t\geq 0$ and $h>0$

$$R_{ij}(u,t+h)=[1-(\alpha_i+\lambda_i)h]R_{ij}(u+c_ih,t) \tag{5.25}$$
$$+\alpha_ih\int_{0-}^{u+c_ih}R_{ij}(u+c_ih-y,t)\,dF_i(y)$$
$$+\lambda_ih\sum_{k=1}^{m}h_{ik}R_{kj}(u+c_ih,t)+o(h).$$

Dividing (5.25) by h and letting h tend to 0, we find

$$(5.26)\quad \frac{\partial}{\partial t}R_{ij}(u,t)-c_i\frac{\partial}{\partial u}R_{ij}(u,t)=-(\alpha_i+\lambda_i)R_{ij}(u,t)+\alpha_i\int_0^u R_{ij}(u-y,t)\,dF_i(y)+\lambda_i\sum_{k=1}^m h_{ik}R_{kj}(u,t)\qquad (u,t\geqslant 0).$$

Taking the Laplace transform of each term ir (5.26), we obtain

$$(5.27)\quad \frac{\partial}{\partial t}\tilde{R}_{ij}(s,t)+\sum_{k=1}^m[\delta_{ik}(\alpha_i+\lambda_i-c_is-\alpha_i\varphi_i(s))-\lambda_ih_{ik}]\tilde{R}_{kj}(s,t)+c_iR_{ij}(0,t)=0\qquad (s>0,\quad t\geqslant 0).$$

The solution of the differential system (5.27) is easily seen to be

$$(5.28)\quad \tilde{R}(s,t)=\exp\{-T(s)t\}K-\int_0^t\exp\{-T(s)(t-u)\}CR(0,u)\,du\qquad (s>0,\quad t\geqslant 0)$$

where $C=(\delta_{ij}c_i)$; the constant matrix K is determined by the boundary condition $r(s,0)=s\tilde{R}(s,0)=sI$. Thus $K=s^{-1}I$. Using finally (5.18), (5.28) may be written as follows

$$(5.29)\quad \tilde{R}_{ij}(s,t)=\tilde{W}_{ij}(s,t)-s\sum_{k=1}^m\int_0^t\tilde{W}_{ik}(s,t-u)c_kR_{kj}(0,u)\,du\qquad (s>0,\quad t\geqslant 0).$$

Suppose now that the distributions $F_i(\cdot)$ are absolutely continuous and denote their densities by $f_i(\cdot)$. The mass functions $W_{ij}(\cdot,t)$ are then absolutely continuous too; we denote their densities by $W'_{ij}(\cdot,t)$ $(t\geqslant 0)$. Taking the inverse Laplace transforms in (5.29) we obtain then

$$(5.30)\quad R_{ij}(x,t)=W_{ij}(x,t)-\sum_{k=1}^m c_k\int_0^t W'_{ik}(x,u)R_{kj}(0,t-u)\,du\qquad (x,t\geqslant 0).$$

The unknown constants (with respect to x) $R_{kj}(0,u)$ are solutions of the Volterra type integral system obtained by putting $x=0$ in (5.30):

$$(5.31)\quad R_{ij}(0,t)=W_{ij}(0,t)-\sum_{k=1}^m c_k\int_0^t W'_{ik}(0,u)R_{kj}(0,t-u)\,du\qquad (t\geqslant 0).$$

Define now

$$S_{ij}(x,t)=P[B(t)\leqslant x, I(t)=j\,|\,I(0)=i]\qquad (x,t\geqslant 0)$$

and denote the correspondıng densities by $S'_{ij}(x,t)$. In the particular case where

$c_i = c$ ($i \in J$) we have clearly $W_{ij}(x, t) = S_{ij}(x + ct, t)$; (5.30) and (5.31) become then

$$(5.32)\quad R_{ij}(x, t) = S_{ij}(x + ct, t) - c \sum_{k=1}^{m} \int_0^t S'_{ik}(x + cu, u) R_{kj}(0, t - u)\, du \qquad (x, t \geq 0),$$

$$(5.33)\quad R_{ij}(0, t) = S_{ij}(ct, t) - c \sum_{k=1}^{m} \int_0^t S'_{ik}(cu, u) R_{kj}(0, t - u)\, du \qquad (t \geq 0).$$

When $m = 1$ (5.32) and (5.33) reduce exactly to Seal's system.

5.4. *Asymptotic Non-ruin Probabilities*

We suppose here that the number d defined by (5.15) is strictly positive; then for all $i \in J$ and $u \geq 0$, $R_i(u) > 0$ and $R_i(\cdot)$ is a probability distribution. After summation over j (5.26) gives for $t = \infty$:

$$(5.34)\quad c_i R'_i(u) = (\alpha_i + \lambda_i) R_i(u) - \alpha_i \int_{0-}^{u} R_i(u - y)\, dF_i(y) - \lambda_i \sum_{k=1}^{m} h_{ik} R_k(u)$$
$$(i \in J; \quad u \geq 0).$$

It can be shown that (5.34) has a unique solution such that $R_i(\infty) = 1$, $\forall i \in J$. Integrating (5.34) from 0 to t we get

$$(5.35)\qquad c_i R_i(t) = c_i R_i(0) + \alpha_i \int_0^t R_i(t - y)[1 - F_i(y)]\, dy$$
$$+ \lambda_i \int_0^t \left[R_i(u) - \sum_{k=1}^{m} h_{ik} R_k(u) \right] du \qquad (i \in J, \quad t \geq 0).$$

For $m = 1$ (5.35) is the well known defective renewal equation from which the famous Cramer estimate may be derived (see FELLER, Chapter XI). For $m > 1$, (5.35) is unfortunately not more a renewal type equation. Letting t tend to ∞ in (5.35) does not give an explicit value for the probabilities $R_i(0)$ as is the case when $m = 1$:

$$(5.36)\qquad R_i(0) = 1 - \frac{\alpha_i \mu_i}{c_i} - \frac{\lambda_i}{c_i} \int_0^{\infty} \left[R_i(u) - \sum_{k=1}^{m} h_{ik} R_k(u) \right] du.$$

However, when the claim amounts distributions are exponential,

$$F_i(x) = 1 - e^{-x/\mu_i} \qquad (x \geq 0),$$

a further differentiation of both sides of (5.34) shows that the asymptotic non-ruin probabilities are solution of the differential system

$$(5.37)\qquad R''_i(u) = \left(\frac{\alpha_i + \lambda_i}{c_i} - \frac{1}{\mu_i} \right) R'_i(u) - \frac{\lambda_i}{c_i} \sum_{j=1}^{m} h_{ij} R'_j(u) + \frac{\lambda_i}{c_i \mu_i} R_i(u)$$
$$- \frac{\lambda_i}{c_i \mu_i} \sum_{j=1}^{m} h_{ij} R_j(u) \qquad (i \in J, \quad u \geq 0)$$

with the boundary conditions

$$R_i(\infty)=1; \qquad R_i'(0)=\frac{\alpha_i+\lambda_i}{c_i}R_i(0)-\frac{\lambda_i}{c_i}\sum_{j=1}^{m} h_{ij}R_j(0) \qquad (i\in J). \tag{5.38}$$

6. EXAMPLE

Assume that

$$m=2, \qquad h_{12}=h_{21}=1, \qquad h_{11}=h_{22}=0; \tag{6.1}$$

there are thus two possible states for the environment, the sojourn times in each state being exponentially distributed.

The solution of system (3.7) is then

$$\begin{cases} V_{11}(t)=-\dfrac{\alpha_1(\alpha_1+\lambda_2+r_1)}{r_1(r_1-r_2)}(1-e^{r_1t})+\dfrac{\alpha_1(\alpha_2+\lambda_2+r_2)}{r_2(r_1-r_2)}(1-e^{r_2t}), \\ V_{12}(t)=-\dfrac{\lambda_1\alpha_2}{r_1(r_1-r_2)}(1-e^{r_1t})+\dfrac{\lambda_1\alpha_2}{r_2(r_1-r_2)}(1-e^{r_2t}), \\ V_{22}(t)=-\dfrac{\alpha_2(\alpha_1+\lambda_1+r_1)}{r_1(r_1-r_2)}(1-e^{r_1t})+\dfrac{\alpha_2(\alpha_1+\lambda_1+r_2)}{r_2(r_1-r_2)}(1-e^{r_2t}), \\ V_{21}(t)=-\dfrac{\lambda_2\alpha_1}{r_1(r_1-r_2)}(1-e^{r_1t})+\dfrac{\lambda_2\alpha_1}{r_2(r_1-r_2)}(1-e^{r_2t}) \qquad (t\geqslant 0), \end{cases} \tag{6.2}$$

where r_1 and r_2 are the solutions (always distinct and negative as α_i, $\lambda_i>0$) of

$$(\alpha_1+\lambda_1+r)(\alpha_2+\lambda_2+r)=\lambda_1\lambda_2. \tag{6.3}$$

The stationary probabilities for the chain $\{J_n\}$ are given by (4.2) which becomes here

$$\pi_1=\frac{\alpha_1\lambda_2}{\alpha_1\lambda_2+\alpha_2\lambda_1}, \qquad \pi_2=\frac{\alpha_2\lambda_1}{\alpha_1\lambda_2+\alpha_2\lambda_1}. \tag{6.4}$$

Expectations of the number of claims occurring in environment i ($i=1,2$) before t are obtained by solving system (4.9) with the boundary conditions $M_{ij}(0)=0$:

$$\begin{aligned} M_{11}(t)&=\frac{\alpha_1\lambda_2}{\lambda_1+\lambda_2}t+\frac{\alpha_1\lambda_1}{(\lambda_1+\lambda_2)^2}(1-e^{-(\lambda_1+\lambda_2)t}), \\ M_{12}(t)&=\frac{\alpha_2\lambda_1}{\lambda_1+\lambda_2}t-\frac{\alpha_2\lambda_1}{(\lambda_1+\lambda_2)^2}(1-e^{-(\lambda_1+\lambda_2)t}). \end{aligned} \tag{6.5}$$

$M_{22}(t)$ and $M_{21}(t)$ are obtained by replacing in the expressions of $M_{11}(t)$ and $M_{12}(t)$ respectively $\alpha_{1(2)}$ by $\alpha_{2(1)}$ and $\lambda_{1(2)}$ by $\lambda_{2(1)}$.

The mean recurrence time of claims occurring in environment i ($i=1,2$) is given by (4.18).

$$\gamma_{11}=\frac{\lambda_1+\lambda_2}{\alpha_1\lambda_2},\qquad \gamma_{22}=\frac{\lambda_1+\lambda_2}{\alpha_2\lambda_1}; \tag{6.6}$$

We obtain then from (4.15)

$$\gamma_{12}=\frac{\alpha_2+\lambda_1+\lambda_2}{\alpha_2\lambda_1},\qquad \gamma_{21}=\frac{\alpha_1+\lambda_1+\lambda_2}{\alpha_1\lambda_2}. \tag{6.7}$$

The characteristic number d defined by (5.15) takes the following form:

$$d=\frac{\lambda_2(c_1-\alpha_1\mu_1)+\lambda_1(c_2-\alpha_2\mu_2)}{\alpha_1\lambda_2+\alpha_2\lambda_1}. \tag{6.8}$$

From now on we assume that $d>0$ and that the claim amount distributions $F_i(\cdot)$ are exponential, i.e.,

$$F_i(x)=1-e^{-x/\mu_i}\qquad (x\geqslant 0;\quad i=1,2). \tag{6.9}$$

From (5.37) and (5.38) we obtain that the asymptotic non-ruin probabilities are solution of the following differential system

$$\begin{cases} c_1R_1''(u)=\left(\alpha_1+\lambda_1-\dfrac{c_1}{\mu_1}\right)R_1'(u)+\dfrac{\lambda_1}{\mu_1}R_1(u)-\dfrac{\lambda_1}{\mu_1}R_2(u)-\lambda_1R_2'(u)\\ c_2R_2''(u)=\left(\alpha_2+\lambda_2-\dfrac{c_2}{\mu_2}\right)R_2'(u)+\dfrac{\lambda_2}{\mu_2}R_2(u)-\dfrac{\lambda_2}{\mu_2}R_1(u)-\lambda_2R_1'(u) \end{cases} \tag{6.10}$$

$$(u\geqslant 0)$$

with the boundary conditions

$$\begin{cases} R_1(\infty)=R_2(\infty)=1\\ c_1R_1'(0)-(\alpha_1+\lambda_1)R_1(0)+\lambda_1R_2(0)=c_2R_2'(0)\\ \qquad -(\alpha_2+\lambda_2)R_2(0)+\lambda_2R_1(0)=0. \end{cases} \tag{6.11}$$

Define

$$\rho_i=\frac{1}{\mu_i}-\frac{\alpha_i}{c_i}\qquad (i=1,2) \tag{6.12}$$

and assume without restriction that $\rho_1\geqslant\rho_2$.
The condition $d>0$ is then equivalent to the following

$$\frac{\lambda_2}{c_2\mu_2}\rho_1+\frac{\lambda_1}{c_1\mu_1}\rho_2>0. \tag{6.13}$$

As $\rho_1 \geqslant \rho_2$, then ρ_1 is clearly strictly positive. We obtain then that the general solution of (6.10) takes the form

$$(6.14)\qquad \begin{cases} R_1(u) = A_0 + A_1\, e^{k_1 u} + A_2\, e^{k_2 u} + A_3\, e^{k_3 u}, \\ R_2(u) = A_0 - D(k_1) A_1\, e^{k_1 u} - D(k_2) A_2\, e^{k_2 u} \\ \qquad - D(k_3) A_3\, e^{k_3 u}, \end{cases}$$

where

$$(6.15)\qquad D(k_i) = \frac{c_1\mu_1 k_i^2 + (c_1 - \alpha_1\mu_1 - \lambda_1\mu_1)k_i - \lambda_1}{\lambda_1\mu_1 k_i + \lambda_1} = \frac{\lambda_2\mu_2 k_i + \lambda_2}{c_2\mu_2 k_i^2 + (c_2 - \alpha_2\mu_2 - \lambda_2\mu_2)k_i - \lambda_2},$$

and where k_1, k_2, k_3 are the roots of the characteristic equation

$$(6.16)\qquad P(k) = k^3 + \left(\rho_1 + \rho_2 - \frac{\lambda_1}{c_1} - \frac{\lambda_2}{c_2}\right)k^2 + \left[\left(\rho_1 - \frac{\lambda_1}{c_1}\right)\left(\rho_2 - \frac{\lambda_2}{c_2}\right) - \frac{\lambda_2}{c_2\mu_2} - \frac{\lambda_1}{c_1\mu_1} - \frac{\lambda_1\lambda_2}{c_1c_2}\right]k - \left(\frac{\lambda_2}{c_2\mu_2}\rho_1 + \frac{\lambda_1}{c_1\mu_1}\rho_2\right) = 0.$$

From (6.13) we see that $k_1k_2k_3 > 0$. It is easily verified that

$$P(-\rho_1) = \frac{\alpha_1\lambda_1}{c_1^2}(\rho_1 - \rho_2) \geqslant 0; \qquad P(-\rho_2) = \frac{\alpha_2\lambda_2}{c_2^2}(\rho_2 - \rho_1) \leqslant 0;$$
$$P(0) < 0.$$

From this we may deduce that $P(k)$ has a negative root, say k_2, between $-\rho_1$ and $-\rho_2$. As the product of the three roots is positive we deduce further that the two other roots, k_1 and k_3, are real (if k_1 and k_3 were complex conjugate roots, their product would be positive; we would then have $k_1k_2k_3 < 0$). As $P(+\infty) = +\infty$ and $P(-\infty) = -\infty$, we conclude finally that when $\rho_1 > \rho_2$ one of the roots, say k_1, is strictly less than $-\rho_1$ and that the other, k_3, is positive. When $\rho_1 = \rho_2 = \rho$ (we have then $k_2 = -\rho$), we obtain the same conclusions by verifying that $P'(-\rho) < 0$. We summarize this as follows:

$$(6.17)\qquad \begin{array}{lll} k_1 < -\rho_1 < k_2 < \min\{0, -\rho_2\}, & k_3 > 0 & \text{if } \rho_1 > \rho_2, \\ k_1 < k_2 = -\rho < 0 < k_3 & & \text{if } \rho_1 = \rho_2 = \rho. \end{array}$$

From the boundary conditions (6.11) we obtain that

$$(6.18)\qquad A_0 = 1, \qquad A_3 = 0$$

and that A_1 and A_2 are the solutions of

$$[c_1k_1-\alpha_1-\lambda_1-\lambda_1D(k_1)]A_1+[c_1k_2-\alpha_1-\lambda_1-\lambda_1D(k_2)]A_2=\alpha_1$$

$$[(-c_2k_1+\alpha_2+\lambda_2)D(k_1)+\lambda_2]A_1+[(-c_2k_2+\alpha_2+\lambda_2)D(k_2)+\lambda_2]A_2=\alpha_2$$

or, which is equivalent in view of (6.15),

$$\text{(6.19)}\qquad \begin{cases} \dfrac{A_1}{\mu_1k_1+1}+\dfrac{A_2}{\mu_1k_2+1}=-1 \\ \dfrac{D(k_1)}{\mu_2k_1+1}A_1+\dfrac{D(k_2)}{\mu_2k_2+1}A_2=1. \end{cases}$$

We can obtain a lower bound for k_1. Verify first that $P(\mu_1^{-1})<0$ if $\mu_1\leq\mu_2$ and that $P(\mu_2^{-1})<0$ if $\mu_2\leq\mu_1$. We can then easily conclude that

$$\text{(6.20)}\qquad -\min\{\mu_1,\mu_2\}^{-1}<k_1.$$

We summarize the above results in

THEOREM 6

If $m=2$, $h_{12}=h_{21}=1$, $d>0$ and if the claim amount distributions are exponential, the asymptotic non-ruin probabilities are given by

$$R_1(u)=1+A_1\,e^{k_1u}+A_2\,e^{k_2u},$$

$$R_2(u)=1-D(k_1)A_1\,e^{k_1u}-D(k_2)A_2\,e^{k_2u}\qquad (u\geq 0),$$

where k_1 and k_2 are the two negative roots of (6.16), where the constants $D(k_i)$ are given by (6.15) and where A_1 and A_2 are solutions of (6.19).

When $\alpha_1=\alpha_2=\alpha$, $\mu_1=\mu_2=\mu$, $c_1=c_2=c$ and if λ_1 and λ_2 are arbitrary positive numbers, then $k_2=-\rho$ and k_1 is the negative root of

$$\text{(6.21)}\qquad k^2+\left(\rho-\frac{\lambda_1+\lambda_2}{c}\right)k-\frac{\lambda_1+\lambda_2}{c\mu}=0.$$

When obtain then $D(k_2)=-1$, $D(k_1)=\lambda_2/\lambda_1$ and the solution of (6.19) is $A_1=0$, $A_2=-\alpha\mu/c$. As expected the ruin probabilities $R_1(u)$ and $R_2(u)$ are in this case identical and equal to the ruin probabilities obtained for the classical Poisson model with exponentially distributed claim amounts:

$$\text{(6.22)}\qquad R_1(u)=R_2(u)=1-\frac{\alpha\mu}{c}e^{-\rho u}.$$

REFERENCES

CINLAR, E (1967) Time dependance of queues with semi-markovian services *J Appl Prob.*, **4**, 356–364. Queues with semi-markovian arrivals *Ibid*, 365–379

FELLER, W. (1971) An introduction to probability theory and its applications, Vol II Wiley New York

JANSSEN, J. (1970) Sur une généralisation du concept de promenade aléatoire sur la droite réelle. *Ann Inst H. Poincaré*, B, **VI**, 249–269)

JANSSEN, J (1980) Some transient results on the M/SM/1 special semi-Markov model in risk and queuing theories *Astin Bull*, **11**, 41–51

JANSSEN, J and REINHARD, J M (1982). Some duality results for a class of multivariate semi-Markov processes *J. Appl Prob.*, March 1982

MILLER, H D (1962). A matrix factorization problem in the theory of random variables defined on a finite Markov chain *Proc Camb Phil Soc*, **58**, 268–285 Absorption probabilities for sums of random variables defined on a finite Markov chain *Ibid*, 286–298

NEUTS, M F (1966). The single server queue with Poisson input and semi-Markov service times *J Appl Prob*, **3**, 202–230

NEUTS, M. F and SHUN-ZER CHEN (1972) The infinite-server queue with Poisson arrivals and semi-Markov service times *J Op Res*, **20**, 425–433.

NEWBOULD, M (1973) A classification of a random walk defined on a finite Markov chain. *Z Wahrsch verw Geb*, **26**, 95–104

PYKE, R (1961a) Markov renewal processes· definitions and preliminary properties *Ann. Math Stat.*, **32**, 1231–1242.

PYKE, R (1961b) Markov renewal processes with finitely many states *Ibid*, 1243–1259

PURDUE, P (1974) The M/M/1 queue in a markovian environment *J Op Res*, **22**, 562–569

REINHARD, J M (1981) A semi-markovian game of economic survival *Scand Act. J*, 1981, 23–38

REINHARD, J M (1982) Identités du type Baxter–Spitzer pour une classe de promenades aléatoires semi-markoviennes To appear in *Ann. Inst H Poincaré.*

SEAL, H L (1974) The numerical calculation of $U(w, t)$, the probability of non-ruin in interval $(0, t)$ *Scand Act J*, 1974, 121–139