SHORT CONTRIBUTIONS

A NOTE ON THE NORMAL POWER APPROXIMATION

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ABSTRACT

The normal power (NP) approximation essentially approximates the random variable X as the quadratic polynomial $\tilde{X} \approx Y+\gamma(Y^2-1)/6$ where $\tilde{X} = (X-\mu)/\sigma$ is the standardized variable, $Y \sim N(0, 1)$, and μ, σ γ are the mean, variance skewness of X respectively. The coefficients of this polynomial are not determined by equating the lower moments. It is shown that matching these moments does not improve the overall accuracy of the approximation.

1. INTRODUCTION

Let X be the aggregate claims in one year, Z_k be the size of the k^{th} claim and N be the total number of claims, i.e.,

$$X = \sum_{k=1}^{N} Z_k,$$

with $X = 0$ if $N = 0$. Let $F(x)$ be the cumulative distribution function (cdf) of X. It is well known that $F(x)$ is given by

$$F(x) = \sum_{k=0}^{\infty} p_k G^{*k}(x), \qquad x \geq 0$$

where $G(x)$ is the cdf of Z_k, $G^{*k}(x)$ is the k^{th} convolution of G with itself, $G^{*0}(x) = 1$ for $x > 0$, and $p_k = P[N = k]$

Direct evaluation of $F(x)$ is possible only in very special cases, so approximations are needed. A simple and easy approximation to $F(x)$, is the normal power (NP) approximation. The essential idea of the NP approximation is to transform the standarized original variable $\tilde{X} = (X-\mu)/\sigma$, where $\mu = E[X]$ and $\sigma^2 = \mathrm{Var}[X]$, into a symmetric variable $Y = v(\tilde{X})$. In particular v is chosen so that Y is a standard normal variable or is nearly so. By inverting the Edgeworth expansion of the unknown cdf of $\tilde{X}$ and using Newton's method (see BEARD et al. (1984), pp. 108–111), it can be proved that

(1) $$\tilde{X} \approx Y + \frac{\gamma}{6}(Y^2-1)$$

where $Y \sim N(0, 1)$ and γ is the skewness of X. This results in the NP approximation

$$F(x) \approx N\left(\frac{-3}{\gamma} + \sqrt{\frac{9}{\gamma^2} + 1 + \frac{6\tilde{x}}{\gamma}}\right), \tag{2}$$

where $\tilde{x} = (x-\mu)/\sigma$. This approximation is valid for $\tilde{x} > 1$, and is fairly accurate if $0 \leq \gamma \leq 1$, with the accuracy decreasing as γ increases.

2. THE MAIN RESULT

Since the inverse transform $v^{-1}(Y)$ approximates $\tilde{X}$, one would expect the left hand side (LHS) and the right hand side (RHS) of equation (1) to have approximately equal moments However this is not the case because

$$E\left[\left(Y + \frac{\gamma}{6}(Y^2-1)\right)^k\right] = \begin{cases} 0 & \text{if } k = 1 \\ 1+\gamma^2/18 & \text{if } k = 2 \\ \gamma+\gamma^3/27 & \text{if } k = 3 \end{cases}$$

while

$$E[\tilde{X}^k] = \begin{cases} 0 & \text{if } k = 1 \\ 1 & \text{if } k = 2 \\ \gamma & \text{if } k = 3. \end{cases}$$

If γ is small, the terms $\gamma^2/18$ and $\gamma^3/27$ can be neglected, giving an approximate equality between the first 3 moments of the LHS and RHS. On the other hand if γ is large, the variance and skewness of the RHS of equation (1) will be inflated, possibly leading to poorer approximations

The important question at this point is this: can the accuracy of the NP approximation be improved by equating the first three moments of the LHS and RHS of equation (1)? To this end, consider the quadratic

$$\tilde{X} = aY+b(Y^2-1) \tag{3}$$

where a and b are real constants and, once again, $Y \sim N(0, 1)$ Matching the first three moments yield the following equations

$$1 = a^2+2b^2$$
$$\gamma = 6a^2b+8b^3$$

These equations reduce to

$$a = \sqrt{1-2b^2} \quad \text{for} \quad [-1/\sqrt{2} \leq b \leq 1/\sqrt{2}] \tag{4}$$
$$\gamma = 6b-4b^3.$$

TABLE 1

$\gamma = 0\ 08$			
$\tilde{x}$	$F(\tilde{x})$	NP	Adj-NP
1 0	0 1586	0 1587	0 1586
2 0	0 0249	0 0249	0 0249
3 0	0 0019	0 0019	0 0019
4 0	0 0001	0 0001	0 0001
$\gamma = 0\ 238$			
$\tilde{x}$	$F(\tilde{x})$	NP	Adj-NP
1 0	0 1579	0 1587	0 1583
2 0	0 0288	0 0289	0 0286
3 0	0 0031	0 0031	0 0030
4 0	0 0002	0 0002	0 0002
$\gamma = 0\ 593$			
$\tilde{x}$	$F(\tilde{x})$	NP	Adj-NP
1 0	0 1529	0 1587	0 1566
2 0	0 0362	0 0372	0 0362
3 0	0 0068	0 0064	0 0062
4 0	0 0011	0 0009	0 0008
$\gamma = 0\ 779$			
$\tilde{x}$	$F(\tilde{x})$	NP	Adj-NP
1 0	0 1526	0 1587	0 1553
2 0	0 0394	0 0411	0 0395
3 0	0 0084	0 0084	0 0080
4 0	0 0011	0 0014	0 0014
$\gamma = 1\ 082$			
$\tilde{x}$	$F(\tilde{x})$	NP	Adj-NP
1 0	0 1376	0 1587	0 1525
2 0	0 0376	0 0470	0 0440
3 0	0 0125	0 0119	0 0109
4 0	0 0042	0 0027	0 0024
$\gamma = 1\ 628$			
$\tilde{x}$	$F(\tilde{x})$	NP	Adj-NP
1 0	0 1280	0 1587	0 1452
2 0	0 0370	0 0562	0 0496
3 0	0 0140	0 0184	0 0159
4 0	0 0069	0 0057	0 0048

It is clear that for $-2\sqrt{2} \leq \gamma \leq 2\sqrt{2}$, equation (4) has exactly one root in the region $-1/\sqrt{2} \leq b \leq 1/\sqrt{2}$. Since the distribution of insurance claims are usually positively skewed, only the case where $0 \leq \gamma \leq 2\sqrt{2}$ is considered.

For $0 \leq \gamma 2\sqrt{2}$, let b_0 be the unique root of equation (4) which lies in the region $0 \leq b_0 \leq 1/\sqrt{2}$, and let

$$a_0 = \sqrt{1 - 2b_0^2}\,. \tag{5}$$

Substituting the values into equation (4), the following approximation results

$$F(x) \approx N\left(\frac{-a_0}{2b_0} + \sqrt{\frac{a_0^2}{4b_0^2} + 1 + \frac{\tilde{x}}{b_0}}\right). \tag{6}$$

This approximation will be called the "adjusted" NP approximation

Table 1 shows the values produced by the traditional NP approximation (equation (2)) and by the adjusted NP approximation (equation (6)). The values of F and NP are taken from PENTIKAINEN (1987, pp. 32–34, cases 1, 3, 5, 6, 7, 8). Following PENTIKAINEN, F is actually $1-F$ (the right tail probability) for $\tilde{x} > 0$. From this table, it is clear that both NP approximations yield similar values. As a result, equation (2) must be viewed as being superior because it is easier to use, i.e., it requires fewer steps to derive this approximation

Finally, it should be noted that these approximations have not been properly calculated; $F(x)$ should be approximated as follows:

$$\begin{aligned} F(x) &= P[\tilde{X} \leq \tilde{x}] \\ &\approx P[aY+b(Y^2-1) \leq \tilde{x}] \\ &= P[r_1 \leq Y \leq r_2] \quad \text{because} \quad b > 0 \\ &= N(r_2)-N(r_1) \end{aligned} \tag{7}$$

where $r_1 \leq r_2$ are the roots of the equation

$$\tilde{x} = ay+b(y^2-1), \tag{8}$$

with $a = 1$, $b = \gamma/6$ for the traditional NP approximation, and $a = a_0$, $b = b_0$ for the adjusted NP approximation. The approximation (7) will serve to increase the estimates of the right tail probabilities $1-F(x)$. However, over the range of applicability of the NP approximations, i.e , for $\tilde{x} > 1$ and $0 \leq \gamma \leq 1$, the extra term $N(r_1)$ is insignificant.

REFERENCES

BEARD, R E , PENTIKAINEN, T and PESONEN, E (1984) *Risk Theory* Chapman and Hall, London

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