USING THE POISSON INVERSE GAUSSIAN
IN BONUS-MALUS SYSTEMS

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Abstract

In this paper, we will cover the bonus-malus system in automobile insurance. Bonus-malus systems are based on the distribution of the number of car accidents Therefore, the modelling and fitting of that distribution are considered. Fitting of data is done using the Poisson inverse Gaussian distribution, which shows a good fit Building the bonus system is done by minimizing the insurer's risk, according to Lemaire's (1985) bonus system.

As Lemaire (1976) put it, bonus-malus systems are based on the random variable number of claim N (frequency), irrespective of their amount To begin with, we must adapt a Poisson process which is not homogeneous The heterogeneity aspect is introduced by mixing the Poisson distribution The parameter λ in the Poisson distribution is considered a random variable. A similar contention is made by Besson and Partrat (1992).

Let us assume that the expected frequency of claims varies within the portfolio. Let us further assume that any particular risk in the portfolio has a Poisson distribution of claim frequencies with mean Λ, where Λ is itself a random variable with distribution representing the expected risks inherent in the given portfolio The distribution function of Λ is given by $U(\Lambda)$ and the unconditional distribution of claim frequencies of an individual drawn from the portfolio is mixed Poisson. N has a Poisson distribution with probability function

$$f_N(n|\Lambda) = e^{-\Lambda} \frac{\Lambda^n}{n!} \quad n = 0, 1, 2, \ldots$$

with

$$E(N) = E(\Lambda) \quad \operatorname{Var}(N) = \operatorname{Var}(\Lambda) + E(\Lambda)$$

It is obvious that mixed Poisson variates have a variance exceeding the mean (unlike the Poisson where mean and variance are equal) This state of fact, which is usually the case in practical situations, is normally desirable from the

insurer's standpoint in that mixed distribution can be thought of as being "safer" than the original Poisson.

One interesting distribution for Λ is the inverse Gaussian. It has thick tails and it also provides the advantage of having closed form expression for the moment generating function It is a reasonable distribution for modeling in many insurance situations. The probability density function of an inverse Gaussian distribution is,

$$f(\lambda) = -\frac{\mu}{\sqrt{2\pi\beta\lambda^3}} e^{-\left\{\frac{(\lambda-\mu)^2}{2\beta\lambda}\right\}}, \qquad \lambda > 0$$

The distribution function is,

$$F(\lambda) = \Phi\left[\frac{(\lambda-\mu)}{\sqrt{\beta\lambda}}\right] + e^{\frac{2\mu}{\beta}} \Phi\left[\frac{-(\lambda+\mu)}{\sqrt{\beta\lambda}}\right], \qquad \lambda > 0$$

where $\Phi(\cdot)$ is the standard Normal (with mean 0 and variance 1) distribution function.
The mean and variance are

$$E(\Lambda) = \mu \qquad \text{Var}(\Lambda) = \mu\beta$$

The Poisson mixed over the inverse Gaussian is thus obtained and called the Poisson inverse Gaussian. One can obtain the probabilities from the probability generating function which is

$$P(z) = e^{\frac{\mu}{\beta}[1-\sqrt{1+2\beta(1-z)}]}$$

The mean and variance are thus obtained,

$$E(N) = \mu$$
$$\text{Var}(N) = \mu(1+\beta)$$

The Poisson inverse Gaussian has two parameters.

Regarding the particular case of liability policies (private cars) which have had k claims, we shall use the results found by PANJER and WILLMOT (1987). BÜHLMANN (1970) published the data that had been gathered in Switzerland in 1961 (see Table 1.1).

We will use the maximum likelihood estimator to estimate a parameter $\theta = (\theta_1, \theta_2, \ldots, \theta_p)$ from our set of independent and identically distributed data $(X_1, X_2, \ldots, X_k)$ with probability function $p_n(\theta)$.

According to PANJER and WILLMOT (1987), the likelihood function is

$$L(\theta) = \prod_{k=0}^{\infty} [p_k(\theta)]^{N_k}$$

TABLE 1 1

CLAIM FREQUENCY DATA

Number of claims per policy (k)	Number of policies (n)	Fitted Values
0	103,704	103,710 03
1	14,075	14,054 65
2	1,766	1,784 65
3	255	254 49
4	45	40 42
5	6	6 94
6	2	1 26
Total	119,853	119,852 44

The log-likelihood is defined to be

$$l(\theta) = \log L(\theta) = \sum_{k=0}^{\infty} N_k \cdot \log p_k(\theta)$$

where $N_k = \{$number of X_i's for which $X_i = k\}$, $k = 0, 1, 2, \ldots$ are the observed frequencies. The maximum likelihood estimator is the random variable $\hat{\theta}$ for which the likelihood (or equivalently the log likelihood) is a maximum

The values of the maximum likelihood estimator of μ and β are $\hat{\mu} = 0,15514$ and $\hat{\beta} = 0.15527$. Then, multiplying the probabilities by 119,853 yields the fitted values of Table 1.1.

The value of $\bar{x}$ is 0.15514 and s^2 is 0 24174.

The goodness of fit statistic D Chi-squared distributed with $(k - r - l)$ degrees of freedom (k intervals and r parameters) defined as

$$D = \sum_{i=0}^{k-1} \frac{(N_i - np_i)^2}{np_i}$$

the value of which is 0.78 on 3 degrees of freedom, yielding a significance level of 85%, which is good This is a better fit than the Poisson-Gamma model, as one can compare with BICHSEL's (1964) results calculated on the same set of data.

We have now a frequency distribution, the Poisson inverse Gaussian, that is fitted to our data. We can now build our bonus-malus system in the same manner than LEMAIRE (1985) did We will minimize the average total risk of the insurer, since the insurer is at risk. Let us consider an insured observed during t years and let us call n_j the number of accidents in fault reported during the jth year For each insured, we have the information vector $(n_1, \quad , n_t)$. Each n_j is a realization of the random variable N_j, for which we assumed

independent and identical distributions. For each observation set $n_1, \ldots, n_t$ we have to relate a number $\lambda_{t+1}(n_1, \ldots, n_t)$, which is the best estimator of λ at time $t+1$. We also choose a quadratic loss function $(\lambda_{t+1} - \lambda)^2$ which yields

$$\lambda_{t+1}(n_1, \ldots, n_t) = E(\Lambda | n_1, \ldots, n_t)$$

Hence, we need to determine the posterior mean of Λ. We already know that $\mu(\lambda)$ is an inverse Gaussian with parameters β and μ. We had N/λ as Poisson distributed, hence the likelihood distribution is

$$P(n_1, \ldots, n_t | \lambda) = \frac{\lambda^n e^{-t\lambda}}{\displaystyle\prod_{j=1}^{t}(n_j!)}, \qquad \text{where} \qquad n = \sum_{i=1}^{t} n_i.$$

Next, the joint distribution of the number of accidents is

$$\overline{P}(n_1, \ldots, n_t) = \int_0^\infty P(n_1, \ldots, n_t | \lambda) \cdot \mu(\lambda)\, d\lambda$$

$$= \int_0^\infty \frac{\lambda^n e^{-t\lambda}}{\displaystyle\prod_{j=1}^{t}(n_j!)} \cdot \frac{\mu}{\sqrt{2\pi\beta\lambda^3}}\, e^{-\left\{\frac{(\lambda-\mu)^2}{2\beta\lambda}\right\}}\, d\lambda$$

$$= \frac{\mu}{\displaystyle\prod_{j=1}^{t}(n_j!)} \int_0^\infty \frac{\lambda^n}{\sqrt{2\pi\beta\lambda^3}}\, e^{-\left\{\frac{(\lambda-\mu)^2}{2\beta\lambda} + t\lambda\right\}}\, d\lambda$$

The Bayes theorem for the posterior distribution of Λ is

$$\mu(\lambda | n_1, \ldots, n_t) = \frac{\dfrac{\lambda^n e^{-t\lambda}}{\displaystyle\prod_{j=1}^{t}(n_j!)} \cdot \dfrac{\mu}{\sqrt{2\pi\beta\lambda^3}}\, e^{-\left\{\frac{(\lambda-\mu)^2}{2\beta\lambda}\right\}}}{\dfrac{\mu}{\displaystyle\prod_{j=1}^{t}(n_j!)} \int_0^\infty \dfrac{\lambda^n}{\sqrt{2\pi\beta\lambda^3}}\, e^{-\left\{\frac{(\lambda-\mu)^2}{2\beta\lambda} + t\lambda\right\}}\, d\lambda}$$

$$= \frac{\lambda^n \, e^{-\left\{\frac{(\lambda-\mu)^2}{2\beta\lambda} + t\lambda\right\}}}{\sqrt{2\pi\beta\lambda^3} \int_0^\infty \dfrac{\lambda^n}{\sqrt{2\pi\beta\lambda^3}}\, e^{-\left\{\frac{(\lambda-\mu)^2}{2\beta\lambda} + t\lambda\right\}}\, d\lambda}$$

One can see that the integral term is not a function of λ once it is solved. We can then find to which distribution $\mu(\lambda | n_1, \ldots, n_t)$ is proportional by omitting

that constant term (the integral);

$$\mu(\lambda|n_1, \ldots, n_t) \propto \frac{\lambda^n e^{\left\{-t\lambda - \frac{\lambda}{2\beta} - \frac{\mu^2}{2\beta\lambda}\right\}} e^{\frac{\mu}{\beta}}}{(2\pi\beta)^{1/2} \lambda^{3/2}}$$

Similarly, we omit all terms that do not depend on λ, since $\mu(\lambda|n_1, \ldots, n_t)$ is a function of λ,

$$\mu(\lambda|n_1, \ldots, n_t) \propto \lambda^{\left(n - \frac{3}{2}\right)} e^{\left\{-\lambda\left(\frac{1}{2\beta} - t\right) - \frac{1}{\lambda}\left(\frac{\mu^2}{2\beta}\right)\right\}}$$

in which

$$a = n - \frac{3}{2} \qquad b = \frac{1}{\frac{1}{2\beta} + t} \qquad c = \frac{\mu^2}{2\beta}$$

to obtain

$$\mu(\lambda|n_1, \ldots, n_t) \propto \lambda^a e^{\left\{-\frac{\lambda}{b} - \frac{c}{\lambda}\right\}}.$$

The above expression is the probability density function of Generalized inverse Gaussian distribution, which can also be presented as

$$P(x) = \frac{\mu'^{(-v)} x^{(v-1)} e^{-\left\{\frac{x^2 + \mu'^2}{2\beta' x}\right\}}}{2 K_v(\mu'/\beta')}, \qquad x > 0$$

$$= \frac{\mu'^{(-v)} x^{(v-1)} e^{\left(-\frac{x}{2\beta'} - \frac{\mu'^2}{2\beta' x}\right)}}{2 K_v(\mu'/\beta')}$$

where $\mu' > 0$, $\beta' > 0$, $-\infty < v < \infty$ and $K_v(x)$ is the modified Bessel function of the third kind with index v. JORGENSEN (1982) shows several results concerning the Generalized inverse Gaussian. One of interest to us is the mean, given by

$$E(\text{G I.G.}) = \mu' \frac{K_{v+1}(\mu'/\beta')}{K_v(\mu'/\beta')}$$

in our case

$$v = a + 1 = n - \frac{1}{2}$$

$$\beta' = \frac{b}{2} = \frac{1}{\dfrac{1}{\beta} + 2t}$$

$$\mu' = \sqrt{2\beta' c} = \mu \sqrt{\frac{1}{1 + 2\beta t}}.$$

When the prior distribution of Λ is inverse Gaussian and the conditional probability function $(N_1, \ldots, N_t)$ given $\Lambda = \lambda$ a t-dimension Poisson distribution, we then see that the posterior distribution of Λ is a Generalized inverse Gaussian.

We already estimated μ and β, then we can write

$$v = n - \frac{1}{2}$$

$$\beta' = \frac{1}{6.44039 + 2t}$$

$$\mu' = 0.15514 \times \sqrt{\frac{1}{1 + 0.31054 t}}$$

and

$$\lambda_{t+1}(n_1, \ldots, n_t) = \mu' \frac{K_{v_{t+1}}(\mu'/\beta')}{K_v(\mu'/\beta')}$$

In our case, the pure premium to be charged is related to the frequency of accidents. That is

$$P_{t+1}(n_1, \ldots, n_t) = \lambda_{t+1}(n_1, \ldots, n_t) = \mu' \frac{K_{v_{t+1}}(\mu'/\beta')}{K_v(\mu'/\beta')}.$$

Hence, we can now build a table of premiums to be charged as a function of accidents (n) and number of years (t). To estimate the modified Bessel functions, we use

$$\frac{K_{v_{t+1}}(\mu'/\beta')}{K_v(\mu'/\beta')} = Q_v(\mu'/\beta')$$

We estimate Q by a Newton Raphson approach. Then we normalize the posterior premium in a way that the premium for a new insured is 100 ($n = 0$ and $t = 0$). We then obtain

$$P_{t+1}(n_1, \ldots, n_t) = 100 \frac{\mu' Q_v(\mu'/\beta')}{\mu_0 Q_{v_0}(\mu_0/\beta_0)}$$

where

$$v_0 = -\frac{1}{2} \qquad \beta_0 = 0\ 15527 \qquad \mu_0 = 0\ 15514$$

But we realize that

$$\mu_0\,Q_{v_0}(\mu_0/\beta_0)$$

is the mean of the prior distribution of Λ The Generalized inverse Gaussian includes the inverse Gaussian when $\lambda = -1/2$. So

$$P_{t+1}(n_1, \quad , n_t) = 100 \cdot \frac{\mu'\,Q_v(\mu'/\beta')}{0\ 15514}$$

Table 1 2 gives the results for P_{t+1} for various t and n. We limited ourselves to $n = 10$ since $n > 10$ accidents is most unlikely to occur An APL program was used (for further calculations also)

This is an equitable system: each insured always pays a premium proportional to the estimation of his frequency of accidents, according to the information accumulated during t years For example, if our insured had 1 accident during his first year, he would have a surcharge of 67 72% ((163.72 − 100) − 100). But if he had no accident during that first year, he is entitled to a reduction of 12.65% ((100 − 87.35) − 100). Later, if he has 1 accident during the second year (and none in his first), he will be penalized 60.60% ((140.28 − 87 35) − 87 35). Similarly, if he has no accident during the second year (and 1 in the first year), he will be granted a bonus of 14 32% ((163.72 − 140 28) − 163.72) This process can be continued indefinitely

TABLE 1 2

BONUS-MALUS TABLE

(based on a Poisson inverse Gaussian frequency distribution)

t \ n	0	1	2	3	4	5	6	7	8	9	10
0	100	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A
1	87 35	163 72	275 71	409 52	553 21	701 11	850 94	1001 76	1153 14	1304 88	1456 85
2	78 54	140 28	229 19	335 61	450 55	569 34	689 96	811 55	933 69	1056 17	1178 88
3	71 95	123 76	197 27	285 31	380 84	479 91	580 73	682 49	784 79	887 42	990 29
4	66 78	111 42	173 94	248 83	330 38	415 23	501 75	589 17	677 13	765 42	853 93
5	62 59	101 80	156 10	221 13	292 16	366 27	441 97	518 56	595 66	673 10	750 75
6	59 10	94 05	142 00	199 37	262 20	327 91	395 15	463 25	531 86	600 80	669 95
7	56 13	87 67	130 54	181 81	238 07	297 05	357 49	418 76	480 54	542 64	604 96
8	53 57	82 30	121 05	167 33	218 22	271 69	326 53	382 20	438 37	494 86	551 56
9	51 33	77 71	113 03	155 18	201 60	250 43	300 63	351 62	403 10	454 89	506 89
10	49 35	73 73	106 17	144 84	187 47	232 40	278 65	325 66	373 16	420 96	468 98
20	37 24	51 12	67 13	86 24	110 43	137 23	160 11	185 51	212 31	239 24	269 44
50	24 60	30 65	36 32	45 23	55 62	65 41	73 76	84 42	94 61	106 22	120 60
100	17 66	20 79	24 21	28 85	33 02	37 60	40 24	44 31	49 51	56 92	64 62

Let us now introduce a utility function of the insurer, as LEMAIRE (1979) did. The insurer assesses the risk according to a utility function $\mu(x)$, and determines the premium by equating the utility function of his present situation and the expected utility of the risk . that is, he is indifferent between his present situation and being at risk

$$\mu(R) = E\{\mu(R+P-x)\} = \int_0^\infty \mu(R+P-x) \cdot dG(x)$$

where R is the reserve of the insurer, P the premium to be charged and $G(x)$ the distribution function of claims in a portfolio. This null utility principle has numerous interesting properties when one uses an exponential utility function,

$$\mu(x) = \frac{1}{c}(1-e^{-cx}), \qquad c > 0.$$

The parameter c characterizes the risk aversion of the insurer We can then evaluate the premium

$$\frac{1}{c}(1-e^{-cR}) = \int_0^\infty \frac{1}{c}(1-e^{-c(R+P-x)}) \, dG(x)$$

which yields

$$P = \frac{1}{c} \log M(c)$$

where $M(c)$ is the moment generating function of the claims distribution. In our situation of a bonus-malus system based on a Poisson inverse Gaussian, we have

$$P = \frac{1}{c} \log \int_0^\infty M(c, \lambda) \, dU(\lambda)$$

where

$$M(c, \lambda) = e^{\lambda(e^c - 1)}$$

is the moment generating function of the Poisson distribution, and $U(\lambda)$ is inverse Gaussian Then

$$P = \frac{1}{c} \log \left[\int_0^\infty e^{\lambda(e^c - 1)} \frac{\mu}{\sqrt{2\pi\beta\lambda^3}} e^{\left\{-\frac{(\lambda-\mu)^2}{2\beta\lambda}\right\}} d\lambda \right].$$

The expression in brackets is the moment generating function of the inverse Gaussian distribution valued at $e^c - 1$. For the Generalized inverse Gaussian,

we have

$$M(z) = (1 - 2\beta z)^{-\frac{\lambda}{2}} \frac{K_\lambda \left(\frac{\mu}{\beta} \sqrt{1 - 2\beta z} \right)}{K_\lambda (\mu/\beta)}, \quad z \le \frac{1}{2\beta}$$

and in particular for the inverse Gaussian when $\lambda = -1/2$. Hence, we replace in P

$$P = \frac{1}{c} \log \left[[1 - 2\beta(e^c - 1)]^{-\frac{\lambda}{2}} \frac{K_\lambda \left(\frac{\mu}{B} \sqrt{1 - 2\beta(e^c - 1)} \right)}{K_\lambda (\mu/\beta)} \right].$$

This formula is valid for all values of parameters λ, β and μ, and in particular for our values of v, β' and μ' of the posterior distribution So,

$$P_{t+1}(n_1, \ldots, n_t) = \frac{1}{c} \log \left[[1 - 2\beta'(e^c - 1)]^{-\frac{v}{2}} \frac{K_v \left(\frac{\mu'}{\beta'} \sqrt{1 - 2\beta'(e^c - 1)} \right)}{K_v (\mu'/\beta')} \right]$$

This premium is a non-decreasing continuous function of c. A choice of $c = 0.25$ (risk aversion) yields a reasonable initial premium $P_1 = 0.18032$, since the pure premium is 0.15514, it corresponds to a safety loading of about 16%. The results are in Table 1.3

TABLE 1 3

BONUS-MALUS TABLE

(based on a Poisson inverse Gaussian frequency distribution)
(with exponential utility function, $c = 0.25$)

t \ n	0	1	2	3	4	5	6	7	8	9	10
0	100	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A
1	86 87	164 15	277 80	413 55	559 18	708 98	860 68	1013 36	1166 59	1320 16	1473 90
2	77 84	139 89	229 47	336 66	452 35	571 85	693 15	815 39	938 18	1061 31	1184 66
3	71 15	122 99	196 67	284 93	380 63	479 84	580 76	682 60	784 98	887 68	990 61
4	65 93	110 44	172 89	247 70	329 13	413 81	500 13	587 34	675 08	763 14	851 42
5	61 72	100 72	154 82	219 60	290 35	364 12	439 46	515 67	592 39	669 43	746 70
6	58 22	92 92	140 58	197 63	260 07	325 36	392 16	459 79	527 92	596 38	665 05
7	55 25	86 51	129 07	179 95	235 78	294 28	354 23	414 99	476 25	537 82	599 61
8	52 70	81 14	119 54	165 42	215 85	268 80	323 14	378 28	433 90	489 84	545 98
9	50 47	76 55	111 52	153 26	199 20	247 53	297 20	347 65	398 57	449 80	501 24
10	48 50	72 59	104 67	142 92	185 08	229 51	275 22	321 70	368 64	415 88	463 33
20	36 51	50 16	69 26	89 63	111 75	134 89	158 72	183 61	209 27	236 41	264 95
50	24 08	30 01	37 81	45 21	54 47	64 08	73 11	83 32	94 47	106 03	118 19
100	17 28	20 33	23 47	27 32	31 76	36 79	39 97	43 31	48 92	55 28	63 24

The Table 1.3 slightly differs from the preceding one (Table 1.2) It can be shown that even for very unreasonable values of c, the differences are small. Finally, these results compare favourably with LEMAIRE's (1985) results where he used a Negative Binomial distribution (Λ was Gamma distributed)

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