TWO STOCHASTIC APPROACHES FOR DISCOUNTING ACTUARIAL FUNCTIONS

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Abstract

Two approaches used to model interest randomness are presented. They are the modeling of the force of interest accumulation function and the modeling of the force of interest. The expected value, standard deviation and coefficient of skewness of the present value of annuities-immediate are presented as illustrations. The implicit behavior of the force of interest under the two approaches is investigated by looking at a particular conditional expectation of the force of interest accumulation function.

Keywords

Force of interest; Force of interest accumulation function; White Noise process; Wiener process; Ornstein-Uhlenbeck process; Present value function; Annuity-immediate.

1. Introduction

A wide variety of stochastic processes have been used to model interest randomness in the present value function and other actuarial functions. Not only are different processes used but they are also used in different ways. Two approaches that are used in existing literature are, firstly, the modeling of the force of interest accumulation function (see, for example, Devolder (1986), Beekman and Fuelling (1990, 1991, 1993), De Schepper et al. (1992a, 1992b), De Schepper and Goovaerts (1992)), and secondly, the modeling of the force of interest (see, for example, Panjer and Bellhouse (1980), Dhaene (1989), Frees (1990), Parker (1992, 1993a, 1993b, 1994), Norberg (1993)). The particular assumption that the forces of interest are independent and identically distributed (i.e. a White Noise process) will be seen to have an equivalent process for the force of interest accumulation function. IID interest notes have been used by Waters (1978, 1990), Dufresne (1990) and Papachristou and Waters (1991) among others.

Although in the deterministic situation the two approaches are equivalent, they are truly different in the stochastic situation.

In this paper, we compare these two approaches for some simple Gaussian processes (see Parker (1993c) for an earlier version presented at the XXIV ASTIN Colloquium). In Section 2, we define the random present value function and give an expression for its moments about the origin.

In Section 3, we present two stochastic processes, namely, the Wiener process and the Ornstein-Uhlenbeck process, for the force of interest accumulation function. The following section presents three stochastic processes, the White Noise, Wiener and Ornstein-Uhlenbeck processes, for modeling the force of interest.

In Section 5, we find the first three moments about the origin of the random present value of a n-year annuity-immediate of equal payments of 1. Some illustrations are presented in Section 6. Section 7 takes a closer look at an implicit difference between the two approaches. Finally, Section 8 summarizes the findings.

2. PRESENT VALUE FUNCTION

Let δ_s denote the force of interest at time s and let $y(t)$ denote the force of interest accumulation function at time t. We then have

$$y(t) = \int_0^t \delta_s \, ds . \tag{1}$$

The random present value at time 0 of a payment of 1 at time t is given by $e^{-y(t)}$.

Assuming that $y(t)$ is Gaussian, then the present value function is log-normally distributed with parameters $E[-y(t)]$ and $V[y(t)]$, and its mth moment about the origin is:

$$E[(e^{-y(t)})^m] = E[e^{-m \cdot y(t)}] = \exp\{-m \cdot E[y(t)] + .5 m^2 \cdot V[y(t)]\} \tag{2}$$

(see, for example, AITCHISON and BROWN (1963, p. 8)).

In the next section we will use two Gaussian stochastic processes to model the force of interest accumulation function. And, in the following section, Section 4, we will look at three Gaussian stochastic processes to model the force of interest.

3. MODELING THE FORCE OF INTEREST ACCUMULATION FUNCTION

A first approach to consider interest randomness is to model $y(t)$, the force of interest accumulation function. Here we present a Wiener process with deterministic drift δ and an Ornstein-Uhlenbeck process also with deterministic drift δ.

3.1. Wiener process

Let $y(t)$ be the sum of a deterministic drift of slope δ and a perturbation modeled by a Wiener process. That is

$$y(t) = \delta \cdot t + \sigma \cdot W_t , \tag{3}$$

where $\sigma \geq 0$ and W_t is the standardized Wiener process.

It can be shown that the expected value and autocovariance function of $y(t)$ are given by

$$E[y(t)] = \delta \cdot t , \tag{4}$$

and

$$\text{cov}\,[y(s), y(t)] = \sigma^2 \cdot \min(s, t). \tag{5}$$

(see ARNOLD (1974, Section 3.2)).

3.2. Ornstein-Uhlenbeck process

Let $y(t)$ be the sum of a deterministic drift of slope δ and a perturbation modeled by an Ornstein-Uhlenbeck process. That is

$$y(t) = \delta \cdot t + X(t), \tag{6}$$

where $X(t)$ is an Ornstein-Uhlenbeck process with parameters $\alpha \geq 0$ and $\sigma \geq 0$ and with an initial condition $X(0) = 0$. Therefore,

$$dX(t) = -\alpha \cdot X(t)\,dt + \sigma\,dW_t. \tag{7}$$

Using the results of ARNOLD (1974, p. 134), one can obtain the expected value and autocovariance function of $y(t)$ as defined in (6) and they are given by

$$E[y(t)] = \delta \cdot t, \tag{8}$$

and

$$\text{cov}\,[y(s), y(t)] = \frac{\sigma^2}{2\alpha} \cdot (e^{-\alpha(t-s)} - e^{-\alpha(t+s)}), \qquad s \leq t \tag{9}$$

or

$$\text{cov}\,[y(s), y(t)] = \rho^2 \cdot (e^{-\alpha(t-s)} - e^{-\alpha(t+s)}), \qquad s \leq t \tag{10}$$

where

$$\rho^2 = \frac{\sigma^2}{2\alpha}. \tag{11}$$

4. MODELING THE FORCE OF INTEREST

A second approach to model interest randomness is to model δ_s, the force of interest. Here we present a White Noise process, a Wiener process and an Ornstein-Uhlenbeck process. Note that the three processes will be defined so that they start at δ, not at the origin.

4.1. White Noise process

Let the force of interest be a White Noise process with mean δ and variance σ^2. That is, for $t > 0$,

$$\delta_t \sim N(\delta, \sigma^2). \tag{12}$$

The forces of interest are therefore modeled by Gaussian, independent and identically distributed random variables. Note that, in continuous time, White Noise is not a physical process but a mathematical abstraction (see KARLIN and TAYLOR (1981, p. 343)).

One may consider, in some sense, that the White Noise process is the derivative of the Wiener process (see, for example, ARNOLD (1974, p. 53) of KARLIN and TAYLOR (1981, p. 342)). (This indicates that assuming a stochastic process for $y(t)$ does not necessarily imply that a meaningful physical process for δ_t exist).

Then, $y(t)$, as defined in (1), is a Wiener process with expected value

$$E[y(t)] = \delta \cdot t, \tag{13}$$

and autocovariance function

$$\operatorname{cov}[y(s), y(t)] = \sigma^2 \cdot \min(s, t). \tag{14}$$

(see, for example, ARNOLD (1974, Section 3.2)).

Therefore, the model presented above is merely an alternative description of the Wiener process for the force of interest accumulation function presented in 3.1.

4.2. Wiener process

A second model for the force of interest is the Wiener process. Let the force of interest be defined as

$$\delta_t = \delta + \sigma \cdot W_t, \qquad \sigma \geq 0. \tag{15}$$

Adapting the results in Section 3.1 we find that the expected value and autocovariance function of this process are

$$E[\delta_t] = \delta, \tag{16}$$

and

$$\operatorname{cov}[\delta_s, \delta_t] = \sigma^2 \cdot \min(s, t). \tag{17}$$

Then, from the definition of $y(t)$ (see (1)), it follows that $y(t)$ is normally distributed with expected value

$$E[y(t)] = \delta \cdot t, \tag{18}$$

and autocovariance function

$$\operatorname{cov}[y(s), y(t)] = \int_0^s \int_0^t \operatorname{cov}[\delta_u, \delta_v]\, du\, dv, \tag{19}$$

which gives

$$\operatorname{cov}[y(s), y(t)] = \sigma^2 \cdot (s^2 t/2 - s^3/6), \qquad s \leq t. \tag{20}$$

4.3. Ornstein-Uhlenbeck process

As a third model for the force of interest we consider an Ornstein-Uhlenbeck process. Let the force of interest be defined by the following stochastic differential equation

$$(21) \qquad d\delta_t = -\alpha(\delta_t - \delta)\cdot dt + \sigma\cdot dW_t \qquad \alpha \geq 0, \quad \sigma \geq 0,$$

with initial value $\delta_0 = \delta$ (see, for example, ARNOLD (1974, p. 134)).

Then, it can be shown that the expected value of δ_t is

$$(22) \qquad E[\delta_t] = \delta,$$

and that its autocovariance function is

$$(23) \qquad \operatorname{cov}[\delta_s, \delta_t] = \frac{\sigma^2}{2\alpha}\cdot(e^{-\alpha(t-s)} - e^{-\alpha(t+s)}), \qquad s \leq t.$$

Again, we will denote $\sigma^2/2\alpha$ by ρ^2.

The force of interest accumulation function, $y(t)$, is therefore a Gaussian process with expected value

$$(24) \qquad E[y(t)] = \delta\cdot t,$$

and autocovariance function

$$(25) \quad \operatorname{cov}[y(s), y(t)] = \frac{\sigma^2}{\alpha^2}\min(s,t) + $$

$$+ \frac{\sigma^2}{2\alpha^3}[-2 + 2e^{-\alpha s} + 2e^{-\alpha t} - e^{-\alpha|t-s|} - e^{-\alpha(t+s)}].$$

(see, for example, PARKER (1994, Section 6)).

Note that the two models considered in Section 3 and the three models considered in this section have all been defined such that their expected values of the force of interest accumulation function are the same (i.e. $E[y(t)] = \delta\cdot t$). What varies over the models is the variance of $y(t)$ and the expected response in a given situation. This will be discussed further in Section 7.

5. ANNUITY-IMMEDIATE

We now consider a n-year annuity-immediate contract. Let $a_{\overline{n|}}$ be the present value of n equal payments of 1 made at the end of each of the next n years. Then, we have

$$(26) \qquad a_{\overline{n|}} = \sum_{t=1}^{n} e^{-y(t)}.$$

We now consider the first three moments of $a_{\overline{n|}}$ using its assumed true probability distribution so that all moments have their usual interpretations. Note however that the expected value will be different than the market price of the annuity which requires that such price be in equilibrium for any purchasing strategy (see BÜHLMANN (1992)).

The expected value of $a_{\overline{n}|}$ may be obtained in the following way:

$$(27)\qquad E[a_{\overline{n}|}] = E\left[\sum_{t=1}^{n} e^{-y(t)}\right] = \sum_{t=1}^{n} E[e^{-y(t)}],$$

where from equation (2),

$$(28)\qquad E[e^{-y(t)}] = \exp\{-E[y(t)] + .5 \cdot V[y(t)]\}.$$

The particular values for $E[y(t)]$ and $V[y(t)]$ were given in Sections 3 and 4 for different modeling approaches and different stochastic processes.

The second moment about the origin of $a_{\overline{n}|}$ may be shown to be equal to

$$(29)\qquad E[(a_{\overline{n}|})^2] = \sum_{t=1}^{n}\sum_{s=1}^{n} E[e^{-y(t)-y(s)}].$$

Similarly, the third moment about the origin of $a_{\overline{n}|}$ is given by

$$(30)\qquad E[(a_{\overline{n}|})^3] = \sum_{t=1}^{n}\sum_{s=1}^{n}\sum_{r=1}^{n} E[e^{-y(t)-y(s)-y(r)}].$$

In order to evaluate the expected values to be summed in (29) and (30), one simply notes that the exponential random variables involved are log-normally distributed. For example,

$$(31)\qquad e^{-y(t)-y(s)-y(r)} \sim \Lambda(\mu, \beta),$$

where

$$(32)\qquad \mu = -E[y(t)] - E[y(s)] - E[y(r)],$$

and

$$(33)\qquad \beta = V[y(t)] + V[y(s)] + V[y(r)] + 2\operatorname{cov}[y(t), y(s)] + \\ + 2\operatorname{cov}[y(t), y(r)] + 2\operatorname{cov}[y(s), y(r)].$$

Therefore, from (2), we have:

$$(34)\qquad E[e^{-y(t)-y(s)-y(r)}] = \exp\{\mu + .5 \cdot \beta\}.$$

6. ILLUSTRATIONS

As a way to illustrate the different approaches and the different stochastic processes considered in this paper, we will evaluate their expected values, standard deviations and coefficients of skewness (see, for example, MOOD, GRAYBILL and BOES (1974, pp. 68, 76)) of $a_{\overline{n}|}$, for certain values of the parameters.

Some expected values are found in Table 1. Results are presented for values of the parameters δ set at .06 and .1 in each process. For the White Noise and Wiener processes, we let the parameter σ take the values .01 and .02. For the Ornstein-Uhlenbeck process, the parameter α is chosen to be .17 (this is the value obtained by BEEKMAN and FUELLING (1990, p. 186) from certain U.S. Treasury bill returns). We let the parameter ρ take the values .01 and .02 which correspond to σ equal

$.01 \cdot (.34)^{.5}$ and $.02 \cdot (.34)^{.5}$ respectively. This is consistent with some of the values used by BEEKMAN and FUELLING (1990, Tables 1 and 2).

It should be pointed out that an estimation procedure for finding the values of the different parameters from a data set of past interest rates would generally produce different values of the estimates of the parameters σ, α or ρ depending on the modeling approach used and on the stochastic process chosen. The estimators of the parameter δ, however, are likely to be roughly the same in all cases considered here. Using the same parameters under both approaches is believed to be appropriate to illustrate certain differences between these two approaches.

TABLE 1

EXPECTED VALUE OF $a_{\overline{n}|}$

Modeling the force of interest accumulation function								
				n				
				5	10	20	30	40
Wiener:		δ	σ					
		.06	.01	4.1920	7.2983	11.3057	13.5061	14.7143
		.06	.02	4.1938	7.3038	11.3202	13.5289	14.7435
		.10	.01	3.7418	6.0118	8.2246	9.0390	9.3387
		.10	.02	3.7433	6.0161	8.2337	9.0511	9.3524
O-U:	δ	α	ρ					
	.06	.17	.01	4.1915	7.2967	11.3013	13.4991	14.7052
	.06	.17	.02	4.1919	7.2975	11.3027	13.5008	14.7071
	.10	.17	.01	3.7413	6.0106	8.2218	9.0353	9.3346
	.10	.17	.02	3.7417	6.0113	8.2228	9.0364	9.3357
Modeling the force of interest								
				n				
				5	10	20	30	40
Wiener:		δ	σ					
		.06	.01	4.1943	7.3273	11.5925	14.4863	17.0285
		.06	.02	4.2030	7.4217	12.6140	19.5880	48.6888
		.10	.01	3.7437	6.0327	8.3788	9.4388	10.0567
		.10	.02	3.7510	6.1008	8.9232	11.3948	18.0414
O-U:	δ	α	ρ					
	.06	.17	.01	4.1920	7.3007	11.3221	13.5410	14.7658
	.06	.17	.02	4.1938	7.3135	11.3862	13.6702	14.9531
	.10	.17	.01	3.7417	6.0135	8.2336	9.0548	9.3586
	.10	.17	.02	3.7432	6.0229	8.2703	9.1151	9.4331

O-U: Ornstein-Uhlenbeck

From Table 1, one can see that the expected value of $a_{\overline{n}|}$ does not depend very much on the modeling approach used nor does it depend on the parameters of the process, except for the parameter δ, of course. The Wiener process, for n larger than say 20, when used to model the force of interest, is another exception.

Table 2 presents some standard deviations of $a_{\overline{n}|}$. It indicates that for a given stochastic process and a given modeling approach, the standard deviation is more or less proportional to the parameter σ (or ρ). It would appear that adjusting the parameters of a model cannot produce similar standard deviations to those of a different model for all n since the standard deviation exhibits significantly different patterns depending on the modeling approach and/or stochastic process selected.

TABLE 2

STANDARD DEVIATION OF $a_{\overline{n}|}$

Modeling the force of interest accumulation function

				n = 5	10	20	30	40
Wiener:		δ	σ					
		.06	.01	.0605	.1342	.2623	.3503	.4053
		.06	.02	.1211	.2687	.5258	.7028	.8137
		.10	.01	.0530	.1058	.1734	.2037	.2160
		.10	.02	.1061	.2118	.3476	.4085	.4332
O-U:	δ	α	ρ					
	.06	.17	.01	.0258	.0457	.0645	.0705	.0724
	.06	.17	.02	.0517	.0913	.1291	.1411	.1448
	.10	.17	.01	.0228	.0368	.0463	.0479	.0482
	.10	.17	.02	.0456	.0736	.0926	.0959	.0964

Modeling the force of interest

				n = 5	10	20	30	40
Wiener:		δ	σ					
		.06	.01	.1251	.5171	1.9640	4.2762	8.6273
		.06	.02	.2515	1.0710	5.1457	27.4239	1111.8356
		.10	.01	.1073	.3880	1.1483	1.9504	2.9114
		.10	.02	.2157	.8019	2.8968	10.1266	240.2379
O-U:	δ	α	ρ					
	.06	.17	.01	.0576	.1968	.5294	.7975	.9767
	.06	.17	.02	.1152	.3952	1.0736	1.6334	2.0169
	.10	.17	.01	.0495	.1495	.3263	.4202	.4610
	.10	.17	.02	.0991	.3001	.6604	.8563	.9433

O-U: Ornstein-Uhlenbeck

For example, we can compare the standard deviations of $a_{\overline{n}|}$ produced by the Ornstein-Uhlenbeck model with parameters $\delta = .06$, $\alpha = .17$ and $\rho = .02$ for the force of interest accumulation function, with those produced by the Ornstein-Uhlenbeck model with parameters $\delta = .06$, $\alpha = .17$ and $\rho = .01$ for the force of interest. Then the standard deviations presented for $n = 5$ are roughly the same (.0517 compared to .0576) while for $n = 40$, the latter (.9767) is almost 7 times larger than the former (.1448). Multiplying the value of ρ in the former by 7 would produce similar standard deviations for $n = 40$ but then the standard deviation in the former model would be about 7 times larger than in the latter model for $n = 5$.

Similar comparisons can be made between different processes under the same approach or different approaches.

This suggests that it is not possible to select different models that would be equivalent in the sense of producing similar standard deviations for all n.

The coefficient of skewness of $a_{\overline{n}|}$ for the same four models are contained in Table 3.

TABLE 3

COEFFICIENT OF SKEWNESS OF $a_{\overline{n}|}$

Modeling the force of interest accumulation function								
				n				
				5	10	20	30	40
Wiener:		δ	σ					
		.06	.01	.0481	.0640	.0841	.0963	.1040
		.06	.02	.0963	.1282	.1686	.1932	.2087
		.10	.01	.0530	.0616	.0772	.0844	.0876
		.10	.02	.0946	.1233	.1547	.1693	.1757
O-U:	δ	α	ρ					
	.06	.17	.01	.0197	.0202	.0185	.0171	.0165
	.06	.17	.02	.0394	.0404	.0370	.0343	.0330
	.10	.17	.01	.0194	.0198	.0183	.0176	.0175
	.10	.17	.02	.0389	.0395	.0366	.0353	.0349
Modeling the force of interest								
				n				
				5	10	20	30	40
Wiener:		δ	σ					
		.06	.01	.1338	.3488	.9732	2.1347	6.5145
		.06	.02	.2690	.7266	2.8689	56.9320	1.3×10^5
		.10	.01	.1311	.3336	.8718	1.7175	4.0382
		.10	.02	.2636	.6940	2.5013	41.5591	1.2×10^5
O-U:	δ	α	ρ					
	.06	.17	.01	.0585	.1205	.2157	.2773	.3166
	.06	.17	.02	.1172	.2421	.4379	.5693	.6564
	.10	.17	.01	.0573	.1154	.1961	.2383	.2580
	.10	.17	.02	.1148	.2318	.3977	.4874	.5311

O-U: Ornstein-Uhlenbeck

The coefficient of skewness also exhibits significantly different patterns depending on the model considered. This supports the observation made earlier that no two models can be seen as equivalent.

7. IMPLICIT BEHAVIOR OF THE FORCE OF INTEREST

Clearly, modeling the force of interest accumulation function has quite different implications on the random present value function and other actuarial functions than modeling the force of interest. Basically, when modeling the force of interest, it is δ_s that varies according to the chosen stochastic process. When modeling $y(t)$, then δ_s varies so that $y(t)$ follows the chosen stochastic process. Those differences have already been illustrated by the standard deviation and coefficient of skewness of $a_{\overline{n}|}$. Another useful way of illustrating the differences between the two approaches is to look at the conditional expected value of $y(t)$ given $y(s)$ and δ_s for $s < t$. This conditional expectation will provide some insight into the implicit behavior of each process.

7.1. Modeling the force of interest accumulation function

The conditional expected value of $y(t)$ given $y(s)$ and δ_s for $s < t$ when $y(t)$ follows an Ornstein-Uhlenbeck process may be obtained in the following way.

Using (6), we have

$$E[y(t) \mid y(s) = x, \delta_s = \varepsilon] = E[\delta \cdot t + X(t) \mid \delta \cdot s + X(s) = x, \delta_s = \varepsilon] \tag{35}$$

$$= \delta \cdot t + E[X(t) \mid X(s) = x - \delta \cdot s, \delta_s = \varepsilon], \tag{36}$$

since $X(t) | X(s)$ is independent of δ_s for $s < t$ from the Markovian property of $X(t)$, then

$$E[y(t) | y(s) = x, \delta_s = \varepsilon] = \delta \cdot t + E[X(t) | X(s) = x - \delta \cdot s], \tag{37}$$

which is [see, for example, BEEKMAN and FUELLING (1990, Section 2)]

$$E[y(t) | y(s) = x, \delta_s = \varepsilon] = \delta \cdot t + (x - \delta \cdot s) \cdot e^{-\alpha(t-s)}, \quad s < t. \tag{38}$$

One can proceed in a similar way to find the corresponding result when the force of interest accumulation function is modeled by a Wiener process.

7.2. Modeling the force of interest

The conditional expected value of $y(t)$ given $y(s)$ and δ_s for $s < t$ when δ_s follows an Ornstein-Uhlenbeck process may be obtained in the following way.

Using (1), we have

$$E[y(t) \mid y(s) = x, \delta_s = \varepsilon] = E\left[\int_0^t \delta_r \, dr \;\middle|\; \int_0^s \delta_r \, dr = x, \delta_s = \varepsilon\right] \tag{39}$$

$$= E\left[\int_0^s \delta_r \, dr + \int_s^t \delta_r \, dr \;\middle|\; \int_0^s \delta_r \, dr = x, \delta_s = \varepsilon\right] \tag{40}$$

and conditioning on $y(s) = x$, (40) becomes

$$(41)\quad E[y(t) \mid y(s) = x, \delta_s = \varepsilon] = x + E\left[\int_s^t \delta_r \, dr \;\Big|\; \int_0^s \delta_u \, du = x, \delta_s = \varepsilon\right]$$

$$(42)\quad = x + \int_s^t E\left[\delta_r \;\Big|\; \int_0^s \delta_u \, du = x, \delta_s = \varepsilon\right] dr.$$

From the Markovian property of the process, $\delta_r \mid \delta_s$ with $r > s$ is independent of all values of δ_u for $u < s$, we then have

$$(43)\quad E[y(t) \mid y(s) = x, \delta_s = \varepsilon] = x + \int_s^t E[\delta_r \mid \delta_s = \varepsilon] \, dr.$$

Finally, adapting the result for the conditional expectation of an Ornstein-Uhlenbeck process found in ARNOLD (1974, p. 134), we may write (43) as

$$(44)\quad E[y(t) \mid y(s) = x, \delta_s = \varepsilon] = x + \int_s^t \delta + (\varepsilon - \delta) \cdot e^{-\alpha(r-s)} \, dr.$$

$$(45)\quad = x + \delta(t - s) + (\varepsilon - \delta) \cdot \left(\frac{1 - e^{-\alpha(t-s)}}{\alpha}\right).$$

We can proceed similarly to find the corresponding conditional expectations when the force of interest is modeled by a White Noise or a Wiener process.
Table 4 summarizes these results and those obtained earlier in this paper.

TABLE 4

SUMMARY OF RESULTS ABOUT $y(t)$

Process	$E[y(t)]$	$V[y(t)]$	$E[y(t) \mid y(s) = x, \delta_s = \varepsilon]$
		Modeling the force of interest accumulation function	
Wiener	$\delta \cdot t$	$\sigma^2 \cdot t$	$x + \delta(t - s)$
O-U	$\delta \cdot t$	$\rho^2 \cdot (1 - e^{-2\alpha t})$	$\delta \cdot t + (x - \delta \cdot s) \cdot e^{-\alpha(t-s)}$
		Modeling the force of interest	
Wiener	$\delta \cdot t$	$\sigma^2 \cdot t^3/3$	$x + \varepsilon(t - s)$
O-U	$\delta \cdot t$	$\dfrac{2\rho^2 t}{\alpha} + \dfrac{\rho^2}{2\alpha}(-3 + 4e^{-\alpha t} - e^{-2\alpha t})$	$x + \delta(t - s) + (\varepsilon - \delta)\left(\dfrac{1 - e^{-\alpha(t-s)}}{\alpha}\right)$

We note from Table 4, as mentioned earlier, that the expected value of $y(t)$ is the same for all four models presented. Also, as noted earlier, the variances are quite different from one model to another. The salient feature of Table 4, however, is the fact that when modeling the force of interest accumulation function, the conditional expectation of $y(t)$ given $y(s)$ and δ_s does not depend on the values of δ_s. But when modeling the force of interest, this conditional expectation does depend on the value of δ_s.

In order to illustrate the possible implications of the conditional expected values of $y(t)$ presented in Table 4, we now consider the Consumer Price Index (CPI) for Canada for the 1960-1992 period (see Canadian Institute of Actuaries (1993, Table 1A)). Here, the CPI plays the role of the force of interest.

The results presented in Sections 2.2 and 6.4 of PANDIT and WU (1983) were used to estimate the parameters of the different models. The estimator for δ is .05335. The estimator of the parameter α when modeling the force of interest accumulation function is .01955, and when modeling the force of interest, it is .05389.

Using these values, the expected values of $y(t)$, $t > 10$, given $y(10) = .2771$ and $\delta_{10} = .0131$ were computed. The results are presented in Figure 1 where $t = 0$ corresponds to 1960. It is difficult to determine from this figure whether the fact that some models do not use the value of δ_{10} makes a significant difference.

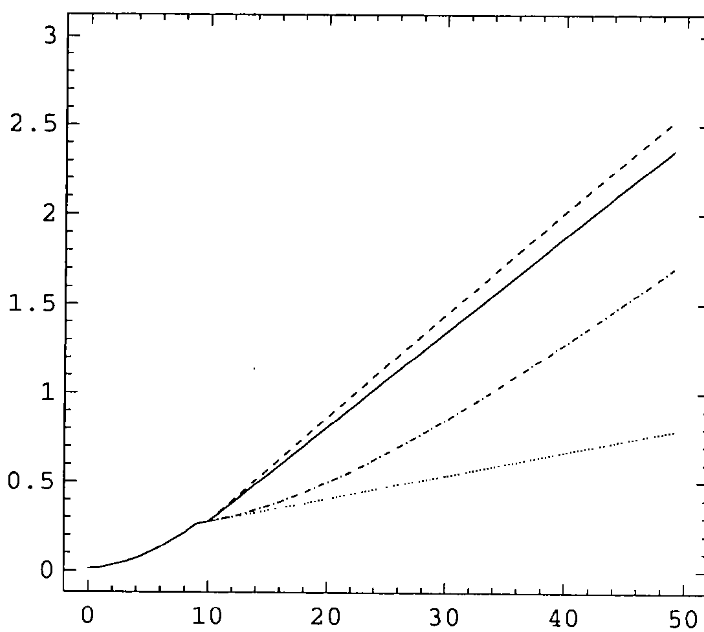

FIGURE 1. Conditional Expected Value of $y(t)$ given $y(10)$ and δ_{10}.

——	$\{y(s)\}_{s=0}^{10}$		with $y(10) = .2771$		$\delta_{10} = .0131$	
——	δ_t:	W-N	$\delta = .05335$	or	$y(t)$: Wiener	$\delta = .05335$
- -	$y(t)$:	O-U	$\delta = .05335$	$\alpha = .01955$		
· · ·	δ_t:	Wiener	$\delta = .05335$			
- · -	δ_t:	O-U	$\delta = .05335$	$\alpha = .05389$		

Figure 2 presents the expected values of δ_t, $t > 10$, given $y(10) = .2771$ and $\delta_{10} = .0131$. This last figure clearly indicates a possible implication resulting from modeling the force of interest accumulation function instead of the force of interest. That is, an expected value of the force of interest, in the immediate future, which can be significantly different from its current value.

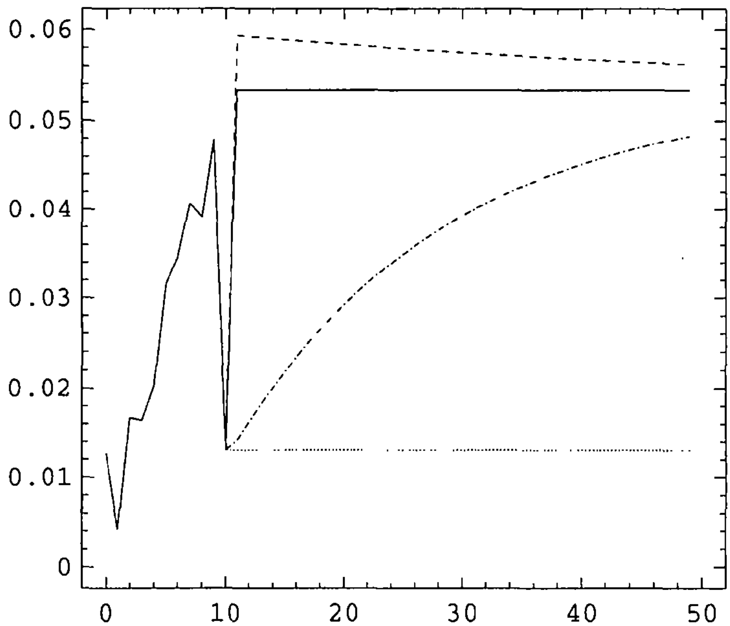

FIGURE 2. Conditional Expected Value of δ_t given $y(10)$ and δ_{10}.

——	$\{\delta_s\}_{s=0}^{10}$		with $y(10) = .2771$	$\delta_{10} = .0131$		
——	δ_t:	W-N	$\delta = .05335$	or	$y(t)$: Wiener	$\delta = .05335$
– –	$y(t)$:	O-U	$\delta = .05335$	$\alpha = .01955$		
· · ·	δ_t:	Wiener	$\delta = .05335$			
– · –	δ_t:	O-U	$\delta = .05335$	$\alpha = .05389$		

8. REMARKS AND SUMMARY

It should be noted that the numerical values presented in Tables 1 and 2 of this paper are not entirely comparable with those in BEEKMAN and FUELLING (1990, 1991). BEEKMAN and FUELLING (1990, 1991) study the continuous annuity, $\bar{a}_{\overline{n}|}$, and we chose to study the annuity-immediate, $a_{\overline{n}|}$. The choice of a discrete annuity was made in order to avoid errors involved in doing numerical integrations that would have been needed for the continuous annuity for some of the models considered.

In this paper, we have studied different models under two approaches to model the interest randomness. An annuity-immediate was used to present some illustrations.

As measured by the agreement of the expected values, standard deviations and coefficients of skewness, no two models can be seen as equivalent, even if one would try to select particular values of the parameters. The one exception to this is that a White Noise process for the force of interest is equivalent to a Wiener process for the force of interest accumulation function.

Further, when modeling the force of interest accumulation function, defined as $y(t)$, the conditional expected value of $y(t)$ given $y(s)$ and δ_s, $s < t$, does not depend on the value of the force of interest at time s. However, when modeling the force of interest, the expected value of $y(t)$ given $y(s)$ and δ_s, $s < t$, does depend on the value of the force of interest at time s.

Finally, another advantage to using one of the models presented for the force of interest is that they are special cases of one-factor interest rate term structure models. This means that the work that has already been done in finance could be used by actuaries interested in arbitrage-free pricing.

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