RECURSIONS FOR CERTAIN BIVARIATE COUNTING DISTRIBUTIONS AND THEIR COMPOUND DISTRIBUTIONS

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ABSTRACT

We consider three classes of bivariate counting distributions and the corresponding compound distributions For each class we derive a recursive algorithm for calculating the bivariate compound distribution.

KEYWORDS

Bivariate counting distributions; Compound distributions; Recursions; Bivariate mixed Poisson

1. INTRODUCTION

In this paper we develop recursive algorithms for bivariate compound distributions of the type

$$g(x, y) = \sum_{n\ m=0}^{\infty} p(n, m) f_1^{*n}(x) f_2^{*m}(y), \quad x, y = 0, 1, \quad , \tag{1.1}$$

which is the joint distribution of

$$(X, Y) = \left(\sum_{i=0}^{N} U_i, \sum_{i=0}^{M} V_i \right),$$

where (N, M) has a probability function (pf)

$$p(n, m) = \mathsf{P}(N = n, M = m), \tag{1.2}$$

and all the severities U_i, V_i are mutually independent and independent of (N, M) with pf's

$$f_1(u) = \mathsf{P}(U_i = u), \ f_2(v) = \mathsf{P}(V_i = v)$$

on the non-negative integers

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For a counting variable K we write $K \sim R_1(a, b)$ when its pf q satisfies the recursion

$$q(k) = \left(a + \frac{b}{k}\right) q(k-1), \quad k \geq 1, \tag{1.3}$$

and $K \sim R_1$ means that (1.3) holds for some constants a and b

For a compound variable X with counting distribution $R_1(a, b)$ and severity distribution f we remind of the fact that the pf of X can be calculated from PANJER's (1981) recursive formula

$$g(x) = \frac{1}{1 - af(0)} \sum_{u=1}^{x} \left(a + \frac{bu}{x}\right) f(u)\, g(x-u), \tag{1.4}$$

when f is concentrated on the non-negative integers. Also the identity

$$\frac{1}{n+1} f^{*(n+1)}(x) = \sum_{u=1}^{x} \frac{u}{1x} f^{*1}(u) f^{*n}(x-n), \tag{1.5}$$

from SCHROTER (1990, Lemma 1) will be used in the following

In sections 2, 3 and 4 we consider three different models

— **Model A.** With $K = N + M$ it holds that $(N \mid K = k) \sim \text{Binomial}(k, \rho_1)$, and $K \sim R_1(a, b)$

— **Model B.** $N = R_0 + R_1$ and $M = R_0 + R_2$, where R_0, R_1 and R_2 are mutually independent and $R_j \sim R_1(a_j, b_j)$.

— **Model C.** N and M are conditionally independent given $\theta = \vartheta$ and Poisson distributed with parameters $\vartheta\lambda_1$ and $\vartheta\lambda_2$, respectively. The parameter θ has a density u which satisfies

$$\frac{d}{d\vartheta} \log u(\vartheta) = \frac{\sum_{i=0}^{k} b_i \vartheta^i}{\sum_{i=0}^{k} a_i \vartheta^i}.$$

We also consider the marginal pf's

$$g(x) = \sum_{y=0}^{\infty} g(x, y), \quad p(n) = \sum_{m=0}^{\infty} p(n, m),$$

for X and N, and the conditional pf's

$$g(y \mid x) = g(x, y)/g(x), \quad p(m \mid n) = p(n, m)/p(n),$$

for $(Y \mid X = x)$ and $(M \mid N = n)$ In particular, we derive recursions for the conditional moments,

$$\mu_{l,x} = \mathsf{E}(Y^l \mid X = x) = \sum_{y=0}^{\infty} y^l g(y \mid x),$$

based on the auxiliary functions

$$\tilde{\mu}_{l,x} = g(x)\mu_{l,x}.$$

The bivariate recursions are of interest in prediction problems involving the conditional pf $g(y \mid x)$ of Y, given that $X = x$ has been observed. Using the recursions derived in this paper this will involve the calculation of $g(u, y)$ for all $u = 0, \ldots, x$ and $y = 0, 1, \ldots$. It should be noted that if one is only interested in the distribution of $X + Y$, then there are simpler ways of calculating this distribution than via $g(x, y)$, and the bivariate recursions should not be used in this case.

Model A has a natural application in claims reserving where K denotes the total number of claims incurred in a fixed exposure period. If W_i, $i = 1, \ldots, K$, denote the waiting times until notification, which can be assumed to be iid and independent of $K \sim R_1(a, b)$, then the numbers of reported and outstanding claims at time τ,

$$N = \sum_{i=1}^{K} I(W_i \leq \tau), \quad M = \sum_{i=1}^{K} I(W_i > \tau),$$

satisfy assumption A with $p_1 = P(W_i \leq \tau)$.

The recursion obtained in Section 2 can be viewed as a bivariate version of the Panjer recursion).

The situation with a binomial distribution of N given $N + M$ as assumed in model A arises in a variety of insurance problems. Consider for instance the case where a stop-loss contract with retention limit d has been written for a one-year period $[0, 1]$. At time $t \in [0, 1]$ the aggregate claim amount has reached the level $X = x$, and the problem is to determine the probability $\sum_{y>d-x} g(y \mid x)$ that the final claim amount $X + Y$ will exceed the limit d, given the information $X = x$ (or to determine the expected reinsurance recoveries $\sum_{x+y>d} (x + y - d) g(y \mid x)$ in this case). If the claim occurrences are generated by a mixed Poisson process with (random) intensity $\theta\eta(s)$, $0 \leq s \leq 1$, then the claim numbers (N, M) occurring in $[0, t]$ and $(t, 1]$ satisfy the binomial assumption of model A with $p_1 = \int_0^t \eta(s)\,ds / \int_0^1 \eta(s)\,ds$. The recursion derived in Section 2 is then applicable if θ has a gamma distribution, in which case the total number of claims $K = N + M$ has a negative binomial distribution $(\in R_1)$, and of course in the Poisson case where θ is degenerate. More generally, if the mixing distribution satisfies the condition given in model C, we may use the recursion derived in Section 4 with $\lambda_1 = \int_0^t \eta(s)\,ds$ and $\lambda_2 = \int_t^1 \eta(s)\,ds$. Another application of model C arises in connection with customer based rating where a customer with unknown risk characteristics represented by θ has reported a total claim amount $X = x$ on the existing policies. For a new policy, this customer will report a claim amount of Y during the next year, and this policy can then be rated on the basis of the experience $X = x$ using the conditional pf $g(y \mid x)$.

The class of mixed Poisson distributions considered in Section 4 were investigated by WILLMOT (1993) and HESSELAGER (1993) in the univariate case, and the recursions derived in Section 4 give a bivariate extension of their results.

Model B uses a standard way of constructing bivariate distributions (see e.g. KOCHERLAKOTA et al. 1992), which is useful in risk theory when two risk classes are affected by the same events. Let R_1 and R_2 denote the numbers of events causing a claim in class 1 and class 2, respectively, and let R_0 denote the number of events causing a claim to both classes.

Model B was studied in the Poisson case where $a_j = 0$ for $j = 0, 1, 2$ by TEICHER (1954), who obtained a recursion for the bivariate pf $p(n, m)$. In Section 3 we extend the result of TEICHER (1954) to the more general case where $R_j \sim R_1$, and derive a recursion for the corresponding compound pf $g(x, y)$

2. A BIVARIATE VERSION OF THE PANJER RECURSION

With $K = N + M$ we consider the following model:

A. The conditional distribution of N given K is binomial,

$$\mathsf{P}(N = n \mid K = k) = \binom{k}{n} \rho_1^n \rho_2^{k-n}, \quad \rho_1 + \rho_2 = 1, \tag{2.1}$$

and $K \sim R_1(a, b)$.

2.1. Bivariate distributions

Let

$$\varphi(s, t) = \mathsf{E}[s^N t^M] = \sum_{n, m \geq 0} p(n, m) s^n t^m$$

denote the pgf for (N, M), and let $\psi(s) = \mathsf{E}s^K$ be the pgf for $K = N + M$. From the assumption (2.1) we find that

$$\begin{aligned} \varphi(s, t) &= \mathsf{EE}[s^N t^M \mid K] = \mathsf{E}[t^K \mathsf{E}[(s/t)^N \mid K]] \\ &= \mathsf{E}[t^K (\rho_1 (s/t) + \rho_2)^K] = \psi(\rho_1 s + \rho_2 t), \end{aligned} \tag{2.2}$$

where we have made use of the fact that the pgf for the binomial distribution with parameters (k, ρ) is $(\rho z + (1 - \rho))^k$. When $K \sim R_1(a, b)$ we also remind of the fact that the pgf ψ satisfies the differential equation

$$(1 - as)\psi'(s) = (a + b)\psi(s). \tag{2.3}$$

From this we readily obtain a recursion for the bivariate pf $p(n, m)$:

Theorem 2.1: *Under condition A it holds that*

$$p(n, m) = \rho_1 \left(a + \frac{b}{n}\right) p(n - 1, m) + a\rho_2 p(n, m - 1), \quad n \geq 1$$

$$p(n, m) = \rho_2 \left(a + \frac{b}{m}\right) p(n, m - 1) + a\rho_1 p(n - 1, m), \quad m \geq 1$$

with $p(n, -1) = p(-1, m) = 0$.

Proof Differentiating (2.2) with respect to s yields

$$(1 - a\rho_1 s - a\rho_2 t) \frac{d}{ds} \varphi(s, t) = (a + b)\rho_1 \varphi(s, t),$$

and with $\varphi(s, t) = \sum_{n, m>0} p(n, m) s^n t^m$ we compare the coefficients of $s^{n-1} t^m$ in this equation for $n \geq 1$ to obtain

$$np(n, m) - a\rho_1(n-1)p(n-1, m) - a\rho_2 np(n, m-1) = (a+b)\rho_1 p(n-1, m)$$

This proves the first relation, and the second follows by symmetry. QED

Theorem 2.1 gives the following recursion for $g(x, y)$.

Theorem 2.2: *Under condition A it holds that*

$$g(0, 0) = \psi(\rho_1 f_1(0) + \rho_2 f_2(0)), \tag{2.4}$$

where $\psi(z)$ is the pgf for K. For $x \geq 1$ it holds that

$$g(x, y) = \rho_1 \sum_{u=0}^{x} \left(a + \frac{bu}{x}\right) f_1(u)\, g(x-u, y) + a\rho_2 \sum_{v=0}^{y} f_2(v)\, g(x, y-v), \tag{2.5}$$

and for $y \geq 1$,

$$g(x, y) = \rho_2 \sum_{v=0}^{y} \left(a + \frac{bv}{y}\right) f_2(v)\, g(x, y-v) + a\rho_1 \sum_{u=0}^{x} f_1(u)\, g(x-u, y). \tag{2.6}$$

Proof. The initial value is

$$g(0, 0) = \sum_{n, m=0}^{\infty} p(n, m) f_1(0)^n f_2(0)^m = \varphi(f_1(0), f_2(0)),$$

and (2.4) follows from (2.2). Using (1.5) for $i = 1$ yields

$$\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{1}{n} p(n-1, m) f_1^{*n}(x) f_2^{*m}(y)$$

$$= \sum_{n, m=0}^{\infty} \frac{1}{n+1} p(n, m) f_1^{*(n+1)}(x) f_2^{*m}(y) = \sum_{u=1}^{x} \frac{u}{x} f_1(u)\, g(x-u, y)$$

For $x \geq 1$ we then obtain from Theorem 2.1 that

$$g(x, y) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} p(n, m) f_1^{*n}(x) f_2^{*m}(y)$$

$$= \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left[\rho_1 \left(a + \frac{b}{n}\right) p(n-1, m) + a\rho_2 p(n, m-1)\right] f_1^{*n}(x) f_2^{*m}(y)$$

$$= \rho_1 \sum_{u=0}^{x} \left(a + \frac{bu}{x}\right) f_1(u)\, g(x-u, y) + \rho_2 a \sum_{v=0}^{y} f_2(v)\, g(x, y-v).$$

The second identity (2.6) follows by symmetry. QED

The bivariate compound distribution is calculated recursively from (2.5) or (2.6) by collecting the terms involving $g(x, y)$ on the left-hand side. It is seen that the

number of multiplications involved with the calculation of $g(u, v)$ for $(u, v) \leq (x, y)$ is of order $O(xy(x+y))$ If the severity distributions f_1 and f_2 have bounded supports, as will typically be the case in practical applications, the number of multiplications will be of order $O(xv)$.

By summing (2 5) over $y \geq 0$ we obtain the Panjer recursion (1 4) for the marginal distribution of X, and the recursion obtained from Theorem 2.2 may therefore be viewed as a bivariate extension of the Panjer recursion

2.2. Marginal and conditional distributions

Theorem 2 1 can also be used to identify the marginal and conditional distributions of (N, M). We have the following.

Theorem 2.3 *Under condition A it holds that*

$$N \sim R_1\left(a\frac{\rho_1}{1-a\rho_2}, b\frac{\rho_1}{1-a\rho_2}\right),$$

$$(M \mid N=n) \sim R_1(a\rho_2, (b+an)\rho_2)$$

Proof By summing the first relation in Theorem 2.1 over $m \geq 0$ we find that

$$p(n) = \frac{\rho_1}{1-a\rho_2}\left(a + \frac{b}{n}\right)p(n-1), \quad n \geq 1,$$

which proves the first assertion For the conditional pf $p(m \mid n)$ we observe from the second relation in Theorem 2.1 that

$$\frac{p(m \mid n)}{p(m-1 \mid n)} = \frac{p(n, m)}{p(n, m-1)} = \rho_2\left(a + \frac{b}{m}\right) + a\rho_1\frac{p(n-1, m)}{p(n, m-1)}, \quad m \geq 1$$

Since $p(n, m) = q(n+m)\binom{n+m}{n}\rho_1^n\rho_2^m$, where q is the pf for K, we have that

$$\frac{p(n-1, m)}{p(n, m-1)} = \frac{n}{m}\frac{\rho_2}{\rho_1},$$

which proves the result. QED

The class R_1 contains the binomial distributions $(a<0)$, the Poisson distributions $(a=0)$, and the negative binomial distributions $(0<a<1)$ It is seen from Theorem 2 3 that the marginal and conditional distributions are binomial, Poisson or negative binomial when the distribution of K is binomial, Poisson or negative binomial, respectively. In particular, it is seen that N and M are independent in the Poisson case $(a=0)$

Note also from Theorem 2 3 that the marginal distribution of X and the conditional distribution of $(Y \mid N=n)$ can be calculated by use of the Panjer recursion (1 4)

In order to calculate the conditional pf $g(u \mid x)$ for $y = 0, \ldots, y_{max}$, one needs to calculate pf$g(u, v)$ for all $(u, v) \leq (x, y_{max})$. In some cases one may settle for an approximation to this distribution, based on the conditional moments $\mu_{k,1}$.

Theorem 2.4 *With* $e_j = \sum_{v=0}^{\infty} v^j f_2(v)$, *it holds that*

$$(2.7) \qquad (1 - a\rho_2)\tilde{\mu}_{l,1} = a\rho_1 \sum_{u=0}^{1} f_1(u)\tilde{\mu}_{l,1-u} + a\rho_2 \sum_{i=0}^{l-1} c(l, i)\tilde{\mu}_{i,1} e_{l-i},$$

where $c(l, i) = a\binom{l}{i} + b\binom{l-1}{i}$.

Theorem 2.4 follows from Theorem 2.2 by straightforward calculations, and the details can be found in Appendix A. With $\tilde{\mu}_{l,1}$ given by Theorem 2.4, and the marginal pf $g(x)$ of X being calculated from the Panjer recursion, we then have a recursive procedure for obtaining the conditional moments $\mu_{l,1} = \tilde{\mu}_{l,1}/g(x)$.

Note that $\mathrm{E}Y^l = \sum_x \tilde{\mu}_{l,1}$. By summing (2.7) over x we obtain that

$$\mathrm{E}Y^l = \frac{a\rho_2}{1-a} \sum_{i=0}^{l-1} c(l, i)\mathrm{E}Y^i e_{l-i},$$

which is De Pril's (1986) recursion for the moments of the compound distribution when $M \sim R_1\left(a\dfrac{\rho_2}{1-a\rho_1}, b\dfrac{\rho_2}{1-a\rho_1}\right)$.

3 BIVARIATE R_2 DISTRIBUTIONS

In this section we consider the following situations:

B. $N = R_0 + R_1$ and $M = R_0 + R_2$, where R_0, R_1 and R_2 are mutually independent and $R_j \sim R_1(a_j, b_j)$.

3.1. Bivariate distributions

The case where R_0, R_1 and R_2 are independently Poisson distributed has previously been considered in the literature. Teicher (1954) (see also Johnson and Kotz, 1969, p. 298) showed that this bivariate Poisson distribution satisfies the recurrence relations

$$np(n, m) = \lambda_1 p(n-1, m) + \lambda_0 p(n-1, m-1),$$
$$mp(n, m) = \lambda_2 p(n, m-1) + \lambda_0 p(n-1, m-1),$$

where λ_0, λ_1 and λ_2 denote the Poisson parameters for R_0, R_1 and R_2, respectively. Holgate (1954) treated the estimation problems for this distribution.

Since the bivariate Poisson distribution appears as a special case of condition B with $a_j = 0$ and $b_j = \lambda_j$, the following result is seen to generalize that of Teicher (1954).

Theorem 3.1 *Under condition B it holds that*

$$p(n, m) = \left(a_0 + \frac{b_0}{n}\right) p(n-1, m-1) + \left(a_1 + \frac{b_1}{n}\right) p(n-1, m) - \left(a_0 a_1 + \frac{a_1 b_0 + b_1 a_0}{n}\right) p(n-2, m-1), \quad n \geq 1, \tag{3.1}$$

$$p(n, m) = \left(a_0 + \frac{b_0}{m}\right) p(n-1, m-1) + \left(a_2 + \frac{b_2}{m}\right) p(n, m-1) - \left(a_0 a_2 + \frac{a_2 b_0 + b_2 a_0}{m}\right) p(n-1, m-2), \quad m \geq 1, \tag{3.2}$$

where $p(0, 0) = \pi_0 \pi_1 \pi_2$, $\pi_j = \mathsf{P}(R_j = 0)$, *and* $p(n, m) = 0$ *when* $n < 0$ *or* $m < 0$.

Proof The pgf for (N, M) is in this case

$$\varphi(s, t) = \mathsf{E}[s^N t^M] = \mathsf{E}[(st)^{R_0} s^{R_1} t^{R_2}] = \psi_0(st)\psi_1(s)\psi_2(t), \tag{3.3}$$

where ψ_j denotes the pgf for $R_j \sim R_1(a_j, b_j)$ Differentiating (3.3) with respect to s and making use of (2.3) yields

$$(1 - a_1 s - a_0 st + a_0 a_1 t s^2) \frac{d}{ds} \varphi(s, t) = [t(1 - a_1 s)(a_0 + b_0) + (a_1 + b_1)(1 - a_0 st)] \varphi(s, t),$$

and with $\varphi(s, t) = \sum_{n, m \geq 0} p(n, m) s^n t^m$ we compare the coefficients of $s^{n-1} t^m$ in this equation for $n \geq 1$ to obtain

$$\begin{aligned} &np(n, m) - a_1(n-1)p(n-1, m) - a_0(n-1)p(n-1, m-1) \\ &\quad + a_0 a_1 (n-2) p(n-2, m-1) \\ &= (a_1 + b_1) p(n-1, m) + (a_0 + b_0) p(n-1, m-1) \\ &\qquad - [a_1(a_0 + b_0) + a_0(a_1 + b_1)] p(n-2, m-1). \end{aligned}$$

This verifies (3.1), and (3.2) follows analogously. QED

Remark 1. Note from the proof of Theorem 3.1 that (3.1) only requires that R_0 and R_1 are of class R_1. If $p(0, m)$ is known for $m = 0, 1, \ldots$, the bivariate pf $p(n, m)$ may therefore be calculated recursively from (3.1) even when R_2 does not belong to R_1. A similar remarks holds if R_1 does not belong to R_1. □

From the recurrence relation in Theorem 3.1 we easily obtain the following recursion for the corresponding bivariate compound distribution

Theorem 3.2 *Assume that condition B holds true Then*

$$g(0, 0) = \psi_0(f_1(0)f_2(0))\,\psi_1(f_1(0))\,\psi_2(f_2(0)), \tag{3.4}$$

where $\psi_j(z)$ *is the pgf for* R_j. *For* $x \leq 1$,

$$\begin{aligned} g(x, y) = {} & \sum_{u=0}^{1} \left(a_1 + \frac{b_1 u}{x}\right) f_1(u)\, g(x-u, y) \\ & + \sum_{u=0}^{1} \sum_{v=0}^{1} \left(a_0 + \frac{b_0 u}{x}\right) f_1(u) f_2(v)\, g(x-u, y-v) \\ & - \sum_{u=0}^{1} \sum_{v=0}^{1} \left(a_0 a_1 + \frac{(a_0 b_1 + b_0 a_1)u}{2x}\right) f_1^{*2}(u) f_2(v)\, g(x-u, y-v), \end{aligned} \tag{3.5}$$

and for $y \geq 1$,

$$\begin{aligned} g(x, y) = {} & \sum_{v=0}^{1} \left(a_2 + \frac{b_2 v}{y}\right) f_2(v)\, g(x, y-v) \\ & + \sum_{v=0}^{1} \sum_{u=0}^{1} \left(a_0 + \frac{b_0 v}{y}\right) f_2(v) f_1(u)\, g(x-u, y-v) \\ & - \sum_{v=0}^{1} \sum_{u=0}^{1} \left(a_0 a_2 + \frac{(a_0 b_2 + b_0 a_2)v}{2y}\right) f_2^{*2}(v) f_1(u)\, g(x-u, y-v) \end{aligned} \tag{3.6}$$

Proof. We have that

$$g(0, 0) = \sum_{n, m=0}^{\infty} p(n, m) f_1(0)^n f_2(0)^m = \varphi(f_1(0), f_2(0)),$$

where $\varphi(s, t)$ is the pgf for (N, M), and (3.4) therefore follows from (3.3)

Using (1.5) we have for $i \geq 1$ and $j \geq 0$ that

$$\sum_{n=i,\, m=j}^{\infty} \frac{1}{n} p(n-i, m-j) f_1^{*n}(x) f_2^{*m}(y) = \sum_{u=0}^{1} \sum_{v=0}^{1} \frac{u}{ix} f_1^{*i}(u) f_2^{*j}(v)\, g(x-u, y-v).$$

Multiplying (3.1) with $f_1^{*n}(x) f_2^{*m}(y)$ and summing over n, m then yields (3.5), and (3.6) follows analogously from (3.2). QED

Remark 2. Since (3.5) was obtained by use of (3.1), it follows from Remark 1 that $g(x, y)$ may be calculated recursively from (3.5) when $g(0, y)$ is known for $y = 0, 1, \ldots$, even if R_2 does not belong to R_1. A similar remark holds if R_1 does not belong to R_1.

It is seen from (3.5) and (3.6) that the number of multiplications involved with the calculation of $g(u, v)$ for $(u, v) \leq (x, y)$ is of order $O(x^2 y^2)$. If the severity

distributions f_1 and f_2 have bounded support, as will typically be the case in practical applications, the number of multiplications will be of order $O(xy)$

For the bivariate Poisson distribution corresponding to $a_j = 0$ and $b_j = \lambda_j$, the recursive formulas in Theorem 3.2 simplify substantially In this case we have,

$$g(x, y) = \frac{\lambda_1}{x} \sum_{u=1}^{x} u f_1(u) g(x-u, y)$$
$$+ \frac{\lambda_0}{x} \sum_{u=1}^{x} \sum_{v=0}^{y} u f_1(u) f_2(v) g(x-u, y-v), \quad x \geq 1,$$

$$g(x, y) = \frac{\lambda_2}{y} \sum_{v=1}^{y} v f_2(v) g(x, y-v)$$
$$+ \frac{\lambda_0}{y} \sum_{u=0}^{x} \sum_{v=1}^{y} v f_1(u) f_2(v) g(x-u, y-v), \quad y \geq 1$$

3.2. Marginal and conditional distributions

SUNDT (1992) considered the class R_k of counting distributions satisfying

$$q(n) = \sum_{i=1}^{k} \left(\alpha_i + \frac{\beta_i}{n} \right) q(n-i), \tag{3.7}$$

for suitable constants α_i and β_i, and showed that if $q_1 \in R_k$, $q_2 \in R_l$, then the convolution $q_1 * q_2$ belongs to R_{k+l} From this we observe that the marginal distributions of N and M belong to R_2 under condition B, and Theorem 3 2 may therefore be viewed as a bivariate extension of SUNDT (1992, Theorem 9) The representation (3.7) for the marginal distribution of N can be found in SUNDT (1992, Corollary 4), or may be obtained by summing (3.1) over m Similarly, a recursion for the marginal distribution of X can be obtained from SUNDT (1992, Theorem 9), or by summing (3 5) over y.

The following result gives a recursion for the auxiliary quantities $\tilde{\mu}_{l,x}$ Since the marginal pf $g(x)$ can be calculated recursively from SUNDT (1992, Theorem 9) we then have a recursive procedure for calculating the conditional moments $\mu_{l,x} = \tilde{\mu}_{l,x}/g(x)$. Theorem 3 3 follow by straightforward calculations, and the details can be found in Appendix A

Theorem 3.3 *Assume that condition B holds true. With*

$$c_j(l, i) = a_j \binom{l}{i} + b_j \binom{l-1}{i}, \quad j = 0, 2,$$

$$d(l, i) = a_0 c_2(l, i) + a_2 c_0(l, i),$$

It holds for $l \geq 1$ *that*

$$(1-a_2)(1-a_0 f_1(0))\tilde{\mu}_{l,\nu} = \sum_{i=0}^{l-1} c_2(l,i)\tilde{\mu}_{i,\nu} e_{l-i} + a_0(1-a_2)\sum_{u=1}^{\nu} f_1(u)\tilde{\mu}_{l,\nu-u}$$

$$+ \sum_{u=0}^{\nu} f_1(u) \sum_{i=0}^{l-1} \tilde{\mu}_{i,\nu-u}\left[c_0(l,i)e_{l-i} - \frac{1}{2}d(l,i)e_{l-i}^{*2}\right],$$

$e_j = \sum_{v=0}^{\infty} v^j f_2(v)$ *and* $e_j^{*2} = \sum_{v=0}^{\infty} v^j f_2^{*2}(v)$

Remark 3. The recursion in Theorem 3.3 is obtained from (3.6), and is according to Remarks 1 and 2 therefore also valid when R_1 does not belong to R_1. □

4 BIVARIATE MIXED POISSON DISTRIBUTIONS

Consider the following situation:

C. N and M are conditionally independent given $\theta = \vartheta$, with pf

$$p_\vartheta(n,m) = \frac{(\vartheta\lambda_1)^n}{n!} e^{-\vartheta\lambda_1} \frac{(\vartheta\lambda_2)^m}{m!} e^{-\vartheta\lambda_2} \tag{4.1}$$

The parameter $\theta \in (\sigma_1, \sigma_2)$, $0 \leq \sigma_1 < \sigma_2 \leq \infty$, has a density u which satisfies

$$\frac{d}{d\vartheta}\log u(\vartheta) = \frac{\sum_{i=0}^{k} a_i \vartheta^i}{\sum_{i=0}^{k} b_i \vartheta^i}, \tag{4.2}$$

for suitable constants a_i and b_i, and

$$\sum_{i=0}^{k} b_i \vartheta^i u(\vartheta) \to 0, \quad \vartheta \to \sigma_1, \sigma_2. \tag{4.3}$$

Let

$$g_\vartheta^{(j)}(x) = \sum_{n=0}^{\infty} \frac{(\vartheta\lambda_j)^n}{n!} e^{-\vartheta\lambda_j} f_j^{*n}(x), \quad j = 1, 2,$$

denote the conditional pf's of X and Y and let

$$g_\vartheta(x,y) = \sum_{n\ m=0}^{\infty} p_\vartheta(n,m) f_1^{*n}(x) f_2^{*m}(y) = g_\vartheta^{(1)}(x)\, g_\vartheta^{(2)}(y), \tag{4.4}$$

denote the joint conditional pf of (X, Y). Finally, introduce the auxiliary functions

$$h_i(x,y) = \int_{\sigma_1}^{\sigma_2} \vartheta^i g_\vartheta(x,y)\, u(\vartheta)\, d\vartheta, \tag{4.5}$$

and note that $g(x,y) = h_0(x,y)$

4.1. Bivariate distributions

The following recursion for the function $h_t(x, y)$, and hence the pf $g(x, y)$, is a bivariate extension of the recursion presented in HESSELAGER (1993).

Theorem 4.1 *Under condition C it holds that*

$$h_t(0, 0) = \int_{\sigma_1}^{\sigma_2} \vartheta^t e^{-\lambda(1-\tilde{f}(0))\vartheta} u(\vartheta)\, d\vartheta, \tag{4.6}$$

with $\lambda = \lambda_1 + \lambda_2$, *and* $\tilde{f}(0) = \dfrac{\lambda_1 f_1(0) + \lambda_2 f_2(0)}{\lambda_1 + \lambda_2}$. For $t = 0, \quad , k-1$,

$$h_t(x, y) = \lambda_1 \sum_{u=1}^{x} \frac{u}{x} f_1(u) h_{t+1}(x-u, y), \quad x \geq 1, \tag{4.7}$$

$$h_t(x, y) = \lambda_2 \sum_{v=1}^{y} \frac{v}{y} f_2(v) h_{t+1}(x, y-v), \quad y \geq 1, \tag{4.8}$$

and

$$c_k h_k(x, v) = \lambda_1 \sum_{u=1}^{x} f_1(u) \sum_{t=0}^{k} b_t h_t(x-u, y) + \lambda_2 \sum_{v=1}^{y} f_2(v) \sum_{t=0}^{k} b_t h_t(x, y-v) + \sum_{t=0}^{k-1} h_t(x, y)[(t+1) b_{t+1} - c_t], \tag{4.9}$$

where $c_t = \lambda \left(1 - \tilde{f}(0)\right) b_t - a_t$

With initial values $h_t(0, 0)$ one calculates $h_0(x, 0), \ldots, h_{k-1}(x, 0)$ from (4.7) and $h_k(x, 0)$ from (4.9) for $x = 1, 2, \quad$. For $y = 1, 2, \quad$ one calculates $h_0(0, y), \quad , h_{k-1}(0, y)$ from (4.8) and $h_k(0, y)$ from (4.9). For $(x, y) \geq (1, 1)$ one may then use either (4.7) or (4.8) together with (4.9) The recursion is seen to be of order $O(xy(x+y))$, reducing to $O(xy)$, when the supports of f_1 and f_2 are finite.

Proof of Theorem 4.1. Since $g_\vartheta^{(j)}$ is compound Poisson with parameter $\lambda_j \vartheta$ it holds that $g_\vartheta^{(j)}(0) = e^{-\vartheta\lambda_j(1-f_j(0))}$, and $g_\vartheta(0, 0) = g_\vartheta^{(1)}(0)\, g_\vartheta^{(2)}(0) = e^{-\vartheta\lambda(1-\tilde{f}(0))}$ with λ. and $\tilde{f}(0)$ as stated, and (4.6) then follows from the definition (4.5).

Under the Poisson assumption of condition C, the conditional pf $g_\vartheta^{(j)}(x)$ satisfies the Panjer recursion,

$$g_\vartheta^{(j)}(z) = \vartheta\lambda_j \sum_{u=1}^{z} \frac{u}{z} f_j(u)\, g_\vartheta^{(j)}(z-u), \quad z \geq 1, \; j = 1, 2,$$

and from (4.4) we than have that

$$g_\vartheta(x, y) = g_\vartheta^{(2)}(y)\,\vartheta\lambda_1 \sum_{u=1}^{x} \frac{u}{x} f_1(u)\, g_\vartheta^{(1)}(x-u)$$

$$= \vartheta\lambda_1 \sum_{u=1}^{x} \frac{u}{x} f_1(u)\, g_\vartheta(x-u, y),\quad x \geq 1,$$

and similarly,

$$g_\vartheta(x, y) = \vartheta\lambda_2 \sum_{v=1}^{y} \frac{v}{y} f_2(v)\, g_\vartheta(x, y-v),\quad y \geq 1.$$

By multiplying this expression with $\vartheta^t u(\vartheta)$, and integrating over ϑ we obtain (4 7) and (4 8).

Differentiating (4 1) wrt ϑ yields

$$\frac{d}{d\vartheta} p_\vartheta(n, m) = \lambda_1 p_\vartheta(n-1, m) + \lambda_2 p_\vartheta(n, m-1) - \lambda_\cdot p_\vartheta(n, m). \tag{4 10}$$

According to (4 2) it holds that $u(\vartheta)\sum_{t=0}^{k} a_t \vartheta^t = u'(\vartheta)\sum_{t=0}^{k} b_t \vartheta^t$, and partial integration using (4 10) yields

$$\int_{\sigma_1}^{\sigma_2} \sum_{t=0}^{k} a_t \vartheta^t p_\vartheta(n, m)\, u(\vartheta)\, d\vartheta = \int_{\sigma_1}^{\sigma_2} \sum_{t=0}^{k} b_t \vartheta^t p_\vartheta(n, m)\, u'(\vartheta)\, d\vartheta$$

$$= \left[\sum_{t=0}^{k} b_t \vartheta^t p_\vartheta(n, m)\, u(\vartheta) \right]_{\sigma_1}^{\sigma_2} - \int_{\sigma_1}^{\sigma_2} \sum_{t=1}^{k} t\vartheta^{t-1} b_t u(\vartheta) p_\vartheta(n, m)\, d\vartheta$$

$$- \int_{\sigma_1}^{\sigma_2} \sum_{t=0}^{k} b_t \vartheta^t u(\vartheta) [\lambda_1 p_\vartheta(n-1, m) + \lambda_2 p_\vartheta(n, m-1) - \lambda\, p_\vartheta(n, m)]\, d\vartheta$$

$$= \int_{\sigma_1}^{\sigma_2} \sum_{t=0}^{k-1} (t+1)\vartheta^t b_{t+1} u(\vartheta) p_\vartheta(n, m)\, d\vartheta$$

$$- \int_{\sigma_1}^{\sigma_2} \sum_{t=0}^{k} b_t \vartheta^t u(\vartheta) [\lambda_1 p_\vartheta(n-1, m) + \lambda_2 p_\vartheta(n, m-1) - \lambda_\cdot p_\vartheta(n, m)]\, d\vartheta,$$

where the last equality follows from (4.3). Multiply on both sides with $f_1^{*n}(x) f_2^{*m}(y)$ and sum over (n, m) to obtain

$$\sum_{t=0}^{k} a_t h_t(x, y) = -\sum_{t=0}^{k-1} (t+1) b_{t+1} h_t(x, y) - \lambda_1 \sum_{u=0}^{x} f_1(u) \sum_{t=0}^{k} b_t h_t(x-u, y) \tag{4 11}$$

$$- \lambda_2 \sum_{v=0}^{y} f_2(v) \sum_{t=0}^{k} b_t h_t(x, y-v) + \lambda \sum_{t=0}^{k} b_t h_t(x, y).$$

By rearranging terms we then arrive at (4 9) QED

The expression (4 6) for the initial values is analogous to the expression given by HESSELAGER (1993) for the univariate case, and the reader will in that paper find explicit formulas for (4.6) for a number of cases

It should be noted that the gamma mixing density satisfies (4 2), and the corresponding mixed Poisson distribution can be shown to be the bivariate negative binomial distribution treated in Section 2 For this case, the recursion in Theorem 2.2 is simpler than the one given in Theorem 4 1, although they are both of the same order.

The condition (4.3) may be dropped without serious consequences, and the identity (4.9) will in this case contain an additional term as in Theorem 1 of HESSELAGER (1993). In all of the examples considered in HESSELAGER (1993), (4 3) is however fulfilled

Univariate mixed Poisson distributions with a mixing density which satisfies (4 2) were studied by WILLMOT (1993) who obtained a recurrence relation for the counting distribution and also investigated a number of special cases where (4 2) holds true. A recursion for the bivariate counting distribution is obtaining by letting $f_1(1) = f_2(1) = 1$ in Theorem 4 1 In this case we may eliminate the auxiliary functions $h_i(n, m)$ for $i \geq 1$ and obtain a recurrence relation for the pf $p(n, m) = h_0(n, m)$, analogous to that of WILLMOT (1993):

Theorem 4.2 *Under condition C it holds for $n > k$ that*

$$(4\ 12)\quad \sum_{i=0}^{k+1} \lambda_1^i (n-i)^{\underline{i}} p(n-i, m) \{a_{k-i} - \lambda\, b_{k-i} + (n+m+1-i) b_{k+1-i}\} = 0,$$

and for $m > k$,

$$(4\ 13)\quad \sum_{i=0}^{k+1} \lambda_2^i (m-i)^{\underline{i}} p(n, m-i) \{a_{k-i} - \lambda\, b_{k-i} + (n+m+1-i) b_{k+1-i}\} = 0,$$

with the convention that $a_{-1} = b_{-1} = b_{k+1} = 0$

Proof. When $f_1(1) = f_2(1) = 1$, the relations (4 7), (4 8) become

$$(4\ 14)\quad h_i(n, m) = \frac{\lambda_1}{n} h_{i+1}(n-1, m), \quad h_i(n, m) = \frac{\lambda_2}{m} h_{i+1}(n, m-1),$$

Repeated use of the first relation gives $h_i(n, m) = \lambda_1^{-i} (n+i)^{(i)} p(n+i)$, where $(n+i)^{(i)} = (n+i)^{\underline{i}}/n^{\underline{i}}$. By inserting into (4.11), which in this case becomes

$$(4\ 15)\quad \sum_{i=0}^{k} a_i h_i(n, m) = -\sum_{i=0}^{k-1} (i+1) b_{i+1} h_i(n, m) - \lambda_1 \sum_{i=0}^{k} b_i h_i(n-1, m)$$
$$- \lambda_2 \sum_{i=0}^{k} b_i h_i(n, m-1) + \lambda \sum_{i=0}^{k} b_i h_i(n, m),$$

we then obtain after a little rearrangement that

$$\sum_{\iota=0}^{k} (a_{k-\iota} - \lambda\, b_{k-\iota}) \lambda_1^{\iota} (n-\iota)^{\downarrow \iota} p(n-\iota, m) =$$

$$- \sum_{\iota=1}^{k} (k-\iota+1) b_{k-\iota+1} \lambda_1^{\iota} (n-\iota)^{\downarrow \iota} p(n-\iota, m)$$

$$- \lambda_1 \sum_{\iota=0}^{k} b_{k-\iota} \lambda_1^{\iota} (n-k)(n-\iota-1)^{\downarrow \iota} p(n-\iota-1, m)$$

$$- \lambda_2 \sum_{\iota=0}^{k} b_{k-\iota} \lambda_1^{\iota} (n-\iota)^{\downarrow \iota} p(n-\iota, m-1) \tag{4 16}$$

From (4 14) it follows that

$$p(n-\iota, m-1) = \frac{\lambda_1}{n-\iota} h_1(n-\iota-1, m-1) = \frac{\lambda_1 m}{(n-\iota)\lambda_2} p(n-\iota-1, m),$$

which inserted into the last term in (4 16) leads to (4 12) after rearranging terms The relation (4 13) follows analogously. QED

The reader will notice from the proof of Theorem 4.2 that there is a whole variety of recurrence relations for the bivariate pf $p(n, m)$. From (4 14) we observe that

$$h_{\iota}(n, m) = \frac{(n+\jmath)^{(\jmath)}}{\lambda_1^{\jmath}} \frac{(m+\iota-\jmath)^{(\iota-\jmath)}}{\lambda_2^{\iota-\jmath}} p(n+\jmath, m+\iota-\jmath) \tag{4 17}$$

for arbitrary $\jmath = 0, \quad , \iota$, which together with (4.15) also will give a recurrence relation In particular (4 16) is such a relation, which in fact is the bivariate extension of WILLMOT's (1993) recurrence relation for the univariate case, as one may verify by summing (4 16) over $m \geq 0$.

4.2. Marginal and conditional distributions

The marginal distributions are in this case the mixed Poisson and the corresponding compound mixed Poisson distributions considered by WILLMOT (1993) and HESSELAGER (1993)

Since N and M are conditionally independent given $\theta = \vartheta$, it follows that

$$p(m \mid n) = \int \frac{(\lambda_2 \vartheta)^m}{m!} e^{-\lambda_2 \vartheta} u(\vartheta \mid n)\, d\vartheta,$$

where $u(\vartheta \mid n)$ is the posterior density for θ given $N = n$ When the prior density $u(\vartheta)$ satisfies (4 2), it was verified in HESSELAGER (1993) that $u(\vartheta \mid n)$ also satisfies

(4.2) with updated constants $\tilde{a}_l$, $\tilde{b}_l$. Thus, the conditional compound distribution of Y given $N=n$ may be calculated recursively by use of the recursion from HESSELAGER (1993).

For the conditional distribution of Y given $X=x$ we may multiply (4.8) and (4.9) by y^l and sum over y to obtain a recursive algorithm for the auxiliary function $\tilde{\mu}_{l,v}^{(t)} = \sum_y y^l h_t(v, y)$. This, in particular, will give a recursion for the function $\tilde{\mu}_{l,v} = \tilde{\mu}_{l,v}^{(0)}$, and hence for the conditional moments $\mu_{l,v} = \tilde{\mu}_{l,v}/g(x)$. We state the result without proof.

Theorem 4.3 *With* $c_t = \lambda_1 b_t(1 - f_1(0)) - a_t$ *and* $e_j = \sum_v v^j f_2(v)$ *it holds that*

$$\tilde{\mu}_{l,v}^{(t)} = \lambda_2 \sum_{j=0}^{l-1} \binom{l-1}{j} e_{l-j} \tilde{\mu}_{j,v}^{(t+1)}, \quad t = 0, \ldots, k-1,$$

$$c_k \tilde{\mu}_{l,v}^{(k)} = \lambda_2 \sum_{t=0}^{k} \sum_{j=0}^{k-1} \binom{l}{j} \tilde{\mu}_{j,v}^{(t)} e_{l-j} + \sum_{t=0}^{k-1} (b_{t+1}(t+1) - c_t) \tilde{\mu}_{l,v}^{(t)}.$$

APPENDIX A

Proof of Theorem 2.4. For $l \geq 1$, multiply (2.6) by y^l and sum over y to obtain

$$\begin{aligned}
\tilde{\mu}_{l,v} &= \sum_{y=1}^{\infty} y^l g(x, y) \\
&= \rho_2 \sum_{v=1}^{\infty} \sum_{v=0}^{v} (ay^l + bvy^{l-1}) f_2(v) g(x, y-v) \\
&\quad + a\rho_1 \sum_{v=1}^{\infty} \sum_{u=0}^{v} f_1(u) y^l g(x-u, y) \\
&= \rho_2 \sum_{v=0}^{\infty} \sum_{v=0}^{\infty} (a(v+v)^l + bv(y+v)^{l-1}) f_2(v) g(x, v) \\
&\quad + a\rho_1 \sum_{u=0}^{v} f_1(u) \tilde{\mu}_{l,v-u} \\
&= \rho_2 a \sum_{t=0}^{l} \binom{l}{t} \tilde{\mu}_{t,v} e_{l-t} + \rho_2 b \sum_{t=0}^{l-1} \binom{l-1}{t} \tilde{\mu}_{t,v} e_{l-t} + a\rho_1 \sum_{u=0}^{v} f_1(u) \tilde{\mu}_{l,v-u}
\end{aligned}$$

QED

Proof of Theorem 3 3. Multiply (3 6) by y^l and sum over y to obtain for $l \geq 1$ that

$$\begin{aligned}
\tilde{\mu}_{l,x} &= \sum_{y=1}^{\infty} y^l g(x, y) \\
&= \sum_{v=0}^{\infty} [a_2 (y+v)^l + b_2 v (v+y)^{l-1}] f_2(v) g(x, y) \\
&\quad + \sum_{u=0}^{x} \sum_{v, y=0}^{\infty} [a_0 (y+v)^l + b_0 v (v+y)^{l-1}] f_1(u) f_2(v) g(x-u, y) \\
&\quad - a_0 a_2 \sum_{u=0}^{x} \sum_{v,y=0}^{\infty} (v+v)^l f_2^{*2}(v) f_1(u) g(x-u, y) \\
&\quad - \frac{b_0 a_2 + a_0 b_2}{2} \sum_{u=0}^{x} \sum_{v, y=0}^{\infty} v (v+y)^{l-1} f_2^{*2}(v) f_1(u) g(x-u, y) \\
&= a_2 \sum_{i=0}^{l} \binom{l}{i} \tilde{\mu}_{i,x} e_{l-i} + b_2 \sum_{i=0}^{l-1} \binom{l-1}{i} \tilde{\mu}_{i,x} e_{l-i} \\
&\quad + \sum_{u=0}^{x} f_1(u) \left[a_0 \sum_{i=0}^{l} \binom{l}{i} \tilde{\mu}_{i,x-u} e_{l-i} + b_0 \sum_{i=0}^{l-1} \binom{l-1}{i} \tilde{\mu}_{i,x-u} e_{l-i} \right] \\
&\quad - a_0 a_2 \sum_{u=0}^{x} f_1(u) \sum_{i=0}^{l} \binom{l}{i} \tilde{\mu}_{i,x-u} e_{l-i}^{*2} \\
&\quad - \frac{(b_0 a_2 + a_0 b_2)}{2} \sum_{u=0}^{x} f_1(u) \sum_{i=0}^{l-1} \binom{l-1}{i} \tilde{\mu}_{i,x-u} e_{l-i}^{*2} \\
&= a_2 \tilde{\mu}_{l,x} + \sum_{i=0}^{l-1} c_2(l, i) \tilde{\mu}_{i,x} e_{l-i} + a_0 (l - a_2) \sum_{u=0}^{x} f_l(u) \tilde{\mu}_{l,x-u} \\
&\quad + \sum_{u=0}^{x} f_1(u) \sum_{i=0}^{l-1} \tilde{\mu}_{i,x-u} \left[c_0(l, i) e_{l-i} - \frac{1}{2} d(l, i) e_{l-i}^{\circ 2} \right],
\end{aligned}$$

and the result follows upon rearranging terms. QED

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