ON BOUNDS FOR THE DIFFERENCE BETWEEN THE STOP-LOSS TRANSFORMS OF TWO COMPOUND DISTRIBUTIONS

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ABSTRACT

In the present note we deduce a class of bounds for the difference between the stop-loss transforms of two compound distributions with the same severity distribution. The class contains bounds of any degree of accuracy in the sense that the bounds can be chosen as close to the exact value as desired; the time required to compute the bounds increases with the accuracy.

1. INTRODUCTION

During the last twenty years, there has grown up a large literature on approximations and inequalities for stop-loss premiums under various assumptions. One way of approximation is to approximate the original distribution with another distribution that makes the evaluation simpler. In such cases it is useful to have bounds for the difference between the exact stop-loss premium and the approximation, that is, it is of interest to have bounds for the difference between the stop-loss transforms of two distributions.

When approximating the stop-loss transform of a compound distribution, it is sometimes convenient to replace the counting distribution with another distribution, e.g. a Bernoulli distribution or a Poisson distribution, and keep the severity distribution unchanged. Such approximations are discussed by i.a. Dhaene & Sundt (1996).

In the present note we deduce classes of bounds for the difference between the stop-loss transforms of two compound distributions with the same severity distribution. The classes contain bounds of any degree of accuracy in the sense that the bounds can be chosen as close to the exact value as desired; the time required to compute the bounds increases with the accuracy.

2. NOTATION AND CONVENTIONS

Let $\mathcal{R}_+$ and $\mathcal{Z}_+$ denote the sets of respectively the non-negative real numbers and the non-negative integers, and $\mathcal{P}_{\mathcal{R}_+}$ and $\mathcal{P}_{\mathcal{Z}_+}$ the classes of probability distributions with

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finite mean on respectively $\mathcal{R}_+$ and Z_+. For distributions in $\mathcal{P}_{\mathcal{R}_+}$, we shall denote the cumulative distribution function with a capital letter and, for distributions in $\mathcal{P}_{Z_+}$, the discrete density function with the corresponding lower case letter. The stop-loss transform of a distribution will be denoted by a horizontal bar on the top of the symbol of the distribution, that is, for a distribution $F \in \mathcal{P}_{\mathcal{R}_+}$, we have the stop-loss transform $\overline{F}$ given by

$$\overline{F}(x) = \int_x^\infty (y-x)dF(y) = \int_x^\infty (1-F(y))dy. \qquad (x \geq 0)$$

The mean of F is denoted by μ_F, that is,

$$\mu_F = \overline{F}(0) = \int_0^\infty y dF(y) = \int_0^\infty (1-F(y))dy.$$

We shall denote a compound distribution with counting distribution $P \in \mathcal{P}_{Z_+}$ and severity distribution $H \in \mathcal{P}_{\mathcal{R}_+}$ by $P \vee H$, that is,

$$P \vee H = \sum_{n=0}^{\infty} p(n) H^{n*},$$

where H^{n*} denotes the n-fold convolution of H.

For $F \in \mathcal{P}_{\mathcal{R}_+}$ and $r \in \mathcal{R}_+$, we define the approximation F_r by

$$F_r(x) = \begin{cases} F(x) & (0 \leq x < r) \\ 1. & (x \geq r) \end{cases}$$

This approximation can be interpreted as the distribution obtained by setting all observations greater than r equal to r. The limiting cases $r = 0$ and $r = \infty$ correspond to respectively the distribution concentrated in zero and the original distribution F.

We shall interpret $\sum_{i=a}^{b} v_i = 0$ when $b < a$.

3. RESULTS

3.1. The following lemma is proved as formula (38) in De Pril & Dhaene (1992) for the special case $r = 1$; the proof is easily extended to the general case.

Lemma 1 *For $H \in \mathcal{P}_{\mathcal{R}_+}$, $r, m \in Z_+$ such that $r \leq m$, and $x \in \mathcal{R}_+$, we have*

$$(m-r)\overline{H}(x) \leq \overline{H^{m*}}(x) - \overline{H^{r*}}(x) \leq (m-r)\mu_H.$$

Lemma 2 *For $P \in \mathcal{P}_{Z_+}$, $H \in \mathcal{P}_{\mathcal{R}_+}$, $r \in Z_+$, and $x \in \mathcal{R}_+$, we have*

$$\overline{H}(x)\overline{P}(r) \leq \overline{P \vee H}(x) - \overline{P_r \vee H}(x) \leq \mu_H \overline{P}(r). \qquad (1)$$

Proof. We have

$$\overline{P \vee H}(x) - \overline{P_r \vee H}(x) = \sum_{n=r}^{\infty} p(n)\ (\overline{H^{n*}}(x) - \overline{H^{r*}}(x)).$$

Application of Lemma 1 gives

$$\sum_{n=r}^{\infty} p(n)\ (n-r)\overline{H}(x) \le \overline{P \vee H}(x) - \overline{P_r \vee H}(x) \le \sum_{n=r}^{\infty} p(n)\ (n-r)\mu_H,$$

from which we obtain (1). Q.E.D

The second inequality in (1) was proved under more general assumptions by Sundt (1991), who also showed that $0 \le \overline{P \vee H}(x) - \overline{P_r \vee H}(x)$, which is weaker than the first inequality in (1).

If $P_r = P$, that is, $P(r) = 1$, then the bounds in (1) become equal to zero.

Lemma 1 appears as a special case of Lemma 2 by letting P be the distribution concentrated in m.

3.2. For $P, Q \in \mathcal{P}_{Z_+}$, $H \in \mathcal{P}_{\mathcal{R}_+}$, $r \in Z_+$, and $x \in \mathcal{R}_+$, we introduce

$$B_r(x; P, Q, H) = \overline{P_r \vee H}(x) - \overline{Q_r \vee H}(x) + \mu_H \overline{P}(r) - \overline{H}(x)\overline{Q}(r), \tag{2}$$

which can also be written as

$$B_r(x; P, Q, H) = \sum_{n=1}^{r-1} (p(n) - q(n))\overline{H^{n*}}(x) -$$

$$\big(P(r-1) - Q(r-1)\big)\overline{H^{r*}}(x) + \mu_H \overline{P}(r) - \overline{H}(x)\overline{Q}(r). \tag{3}$$

Theorem 1 *For $P, Q \in \mathcal{P}_{Z_+}$, $H \in \mathcal{P}_{\mathcal{R}_+}$, $r \in Z_+$, and $x \in \mathcal{R}_+$, we have*

$$-B_r(x; Q, P, H) \le \overline{P \vee H}(x) - \overline{Q \vee H}(x) \le B_r(x; P, Q, H). \tag{4}$$

Proof. Application of Lemma 2 gives

$$\overline{P \vee H}(x) - \overline{Q \vee H}(x) \le \overline{P_r \vee H}(x) + \mu_H \overline{P}(r) - \overline{Q_r \vee H}(x) - \overline{H}(x)\overline{Q}(r) = B_r(x; P, Q, H),$$

which proves the second inequality in (4). The first inequality follows by symmetry.

This completes the proof of Theorem 1. Q.E.D.

We shall look at some special cases of Theorem 1:

1. As

$$B_r(x; P, P_r, H) = \mu_H \overline{P}(r) \qquad B_r(x; P_r, P, H) = -\overline{H}(x)\overline{P}(r),$$

we see that Lemma 2 (and thus also Lemma 1) is a special case of Theorem 1.

2. From (3) we obtain

$$B_1(x;P,Q,H) = -\big(p(0)-q(0)\big)\overline{H}(x) + \mu_H \overline{P}(1) - \overline{H}(x)\overline{Q}(1) = \Big(\mu_H - \overline{H}(x)\Big)\overline{P}(1) + \overline{H}(x)\big(\mu_P - \mu_Q\big).$$

For $H \in \mathcal{P}_{z_+}$ this case is discussed in Dhaene & Sundt (1996).

3. From (2) we obtain

$$B_0(x;P,Q,H) = \mu_H\mu_P - \overline{H}(x)\mu_Q. \tag{5}$$

4. If $P(r) = Q(r) = 1$, then $P_r = P$ *and* $Q_r = Q$, and from (2) we obtain

$$B_r(x;P,Q,H) = -B_r(x;Q,P,H) = \overline{P \vee H}(x) - \overline{Q \vee H}(x),$$

that is, in this case Theorem 1 becomes trivial.

5. From (2) we obtain

$$B_r(x;P,P,H) = \Big(\mu_H - \overline{H}(x)\Big)\overline{P}(r), \tag{6}$$

that is, unfortunately the bounds in Theorem 1 do not in general become equal to zero when comparing two identical compound distributions.

3.3. Let $D_r(x;P,Q,H)$ denote the difference between the upper and lower bound in Theorem 1, that is,

$$D_r(x;P,Q,H) = B_r(x;P,Q,H) + B_r(x;Q,P,H). \tag{7}$$

Then

$$D_r(x;P,Q,H) = \Big(\mu_H - \overline{H}(x)\Big)\Big(\overline{P}(r) + \overline{Q}(r)\Big). \tag{8}$$

We see that $D_r(x;P,Q,H)$ decreases to zero when r increases to infinity, that is, we can make the difference between the upper and lower bound in Theorem 1 as small as desired by making r sufficiently large.

We see that $D_r(x;P,Q,H)$ increases from zero to $\mu_H\Big(\overline{P}(r) + \overline{Q}(r)\Big)$ when x increases from zero to infinity. Thus our bounds are most accurate for low values of x. Furthermore, if for some $\varepsilon > 0$ we choose r such that

$$\overline{P}(r) + \overline{Q}(r) < \frac{\varepsilon}{\mu_H},$$

then $D_r(x;P,Q,H) < \varepsilon$ for all $x \in \mathcal{R}_+$.

3.4. Let

$$b_r(x;P,Q,H) = B_r(x;P,Q,H) - B_{r+1}(x;P,Q,H).$$

From (3) and trivial calculus we obtain

$$b_r(x;P,Q,H) = \big(P(r)-Q(r)\big)\Big(\overline{H^{(r+1)*}}(x) - \overline{H^{r*}}(x)\Big) + \mu_H\big(1-P(r)\big) - \overline{H}(x)\big(1-Q(r)\big). \tag{9}$$

By rewriting (9) as

$$b_r(x;P,Q,H) = \big(1-P(r)\big)\Big[\mu_h + \overline{H^{r*}}(x) - \overline{H^{(r+1)*}}(x)\Big] + \big(1-Q(r)\big)\Big[\overline{H^{(r+1)*}}(x) - \overline{H^{r*}}(x) - \overline{H}(x)\Big]$$

and application of Lemma 1, we see that $b_r(x;P,Q,H) \geq 0$. Thus $B_r(x;P,Q,H)$ is non-increasing in r. This implies that in (4), the upper bound is non-increasing and the lower bound is non-decreasing in r, and as $D_r(x;P,Q,H)$ goes to zero when r goes to infinity, both bounds converge to $\overline{P \vee H}(x) - \overline{Q \vee H}(x)$.

Formula (9) can be applied for recursive evaluation of $B_r(x;P,Q,H)$. Furthermore, when we have found $B_r(x;P,Q,H)$, we easily obtain $B_r(x;Q,P,H)$ from (7) and (8).

3.5. The main purpose of the present subsection is to deduce an improvement of the bounds in Theorem 1. For doing that, we shall need the following lemma.

Lemma 3 *For $P, Q \in \mathcal{P}_{Z_+}$, $H \in \mathcal{P}_{\mathcal{R}_+}$, $r \in Z_+$ and $x \in \mathcal{R}_+$, we have*

$$b_r(x;P,Q,H) \geq \big(\mu_H - \overline{H}(x)\big)\big(1 - max\big(P(r),Q(r)\big)\big). \tag{10}$$

Proof. We apply Lemma 1 in (9). If $P(r) \geq Q(r)$, then

$$b_r(x;P,Q,H) \geq \big(P(r)-Q(r)\big)\overline{H}(x) + \mu_H\big(1-P(r)\big) - \overline{H}(x)\big(1-Q(r)\big),$$

that is,

$$b_r(x;P,Q,H) \geq \big(\mu_H - \overline{H}(x)\big)\big(1-P(r)\big). \tag{11}$$

Analogously, if $P(r) < Q(r)$, then

$$b_r(x;P,Q,H) \geq \big(P(r)-Q(r)\big)\mu_H + \mu_H\big(1-P(r)\big) - \overline{H}(x)\big(1-Q(r)\big),$$

that is,

$$b_r(x;P,Q,H) \geq \big(\mu_h - \overline{H}(x)\big)\big(1-Q(r)\big). \tag{12}$$

From (11) and (12) we obtain (10). Q.E.D.

Theorem 2 *For $P, Q \in \mathcal{P}_{Z_+}$, $H \in \mathcal{P}_{\mathcal{R}_+}$, $r \in Z_+$, and $x \in \mathcal{R}_+$, we have*

$$B_r(x;Q,P,H)+\left(\mu_H-\overline{H}(x)\right)\sum_{k=r}^{\infty}\left(1-max\left(P(k),Q(k)\right)\right)\leq$$
$$\overline{P\vee H}(x)-\overline{Q\vee H}(x)\leq$$
$$B_r(x;P,Q,H)-\left(\mu_H-\overline{H}(x)\right)\sum_{k=r}^{\infty}\left(1-max\left(P(k),Q(k)\right)\right). \tag{13}$$

Proof. For $s \in Z_+$ such that $s \geq r$, we obtain by applying successively Theorem 1 and Lemma 3

$$\overline{P\vee H}(x)-\overline{Q\vee H}(x)\leq B_s(x;P,Q,H)=$$
$$B_r(x;P,Q,H)-\sum_{k=r}^{s-1}b_k(x;P,Q,H)\leq$$
$$B_r(x;P,Q,H)-\left(\mu_H-\overline{H}(x)\right)\sum_{k=r}^{s-1}\left(1-max(P(k),Q(k))\right).$$

By letting s go to infinity, we obtain the second inequality in (13); the first inequality follows by symmetry.

This completes the proof of Theorem 2. Q.E.D.

We see that in (13), like in (4), the lower bound is non-decreasing in r and the upper bound is non-increasing in r.

The infinite summation in (13) may seem complicated. However, as all the terms are non-negative, we obtain weaker bounds by including only a finite number of terms. Furthermore, if P or Q has a finite support, then only a finite number of terms are non-zero.

In the following corollary we consider another case where the summation obtains a particularly simple form.

Corollary 1 *Let $P, Q \in \mathcal{P}_{Z_+}$, $H \in \mathcal{P}_{\mathcal{R}_+}$, and $x \in \mathcal{R}_+$. If there exists a non-negative integer s (possibly equal to infinity) such that*

$$\left.\begin{array}{ll} Q(y)\leq P(y) & (y=0,1,\ldots,s-1)\\ Q(y)\geq P(y), & (y=s,s+1,\ldots)\end{array}\right\} \tag{14}$$

then

$$\begin{aligned} -B_r(x;Q,P,H)+\left(\mu_H-\overline{H}(x)\right)\left(\overline{P}(r)-\overline{P}(s)+\overline{Q}(s)\right) &\leq \\ \overline{P\vee H}(x)-\overline{Q\vee H}(x) &\leq \\ B_r(x;P,Q,H)-\left(\mu_H-\overline{H}(x)\right)\left(\overline{P}(r)-\overline{P}(s)+\overline{Q}(s)\right) & \\ (r=0,1,\ldots,s-1) & \end{aligned} \tag{15}$$

$$-B_r(x;Q,P,H)+\left(\mu_H-\overline{H}(x)\right)\overline{Q}(r)\le\overline{\overline{P\vee H}}(x)-\overline{\overline{Q\vee H}}(x)\le$$
$$Br(x;P,Q,H)-\left(\mu_H-\overline{H}(x)\right)\overline{Q}(r). \qquad (r=s,s+1,\ldots) \qquad (16)$$

Proof. For $r = 0, 1, \ldots, s-1$, we have

$$\sum_{k=r}^{\infty}\left(1-max\left(P(k),Q(k)\right)\right)=\sum_{k=r}^{s-1}\left(1-P(k)\right)+\sum_{k=s}^{\infty}\left(1-Q(k)\right)=$$
$$\overline{P}(r)-\overline{P}(s)+\overline{Q}(s),$$

and insertion in (13) gives (15). The inequalities (16) are proved analogously.

This completes the proof of Corollary 1. Q.E.D.

If we in addition to (14) assume that $\mu_Q \le \mu_P$, then we have the stop-loss orderings $\overline{Q} \le \overline{P}$ and $\overline{\overline{Q\vee H}} \le \overline{\overline{P\vee H}}$; for proofs cf. e.g. Goovaerts, Kaas, van Heerwaarden, & Bauwelinckx (1990).

At the end of subsection 3.2. we pointed out that unfortunately the bounds in Theorem 1 do not become equal to zero when $Q = P$. From (6) and (15) we see that the improved bounds of Theorem 2 do not have this deficiency.

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