SOME APPLICATIONS OF LÉVY PROCESSES TO STOCHASTIC INVESTMENT MODELS FOR ACTUARIAL USE

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ABSTRACT

This paper presents a continuous time version of a stochastic investment model originally due to Wilkie. The model is constructed via stochastic differential equations. Explicit distributions are obtained in the case where the SDEs are driven by Brownian motion, which is the continuous time analogue of the time series with white noise residuals considered by Wilkie In addition, the cases where the driving "noise" are stable processes and Gamma processes are considered.

KEYWORDS

Lévy process; Brownian motion; stochastic investment model

1. INTRODUCTION

Wilkie (1986) presented an investment model based on time series, which has since been updated and extended in Wilkie (1995) This paper presents some continuous time variants of Wilkie's original model using stochastic differential equations driven by appropriate Lévy processes. There is no single correct continuous time equivalent to the model in Wilkie (1986), the aim of this paper is to suggest some possible ways of constructing the analogous continuous time models and to analyse these mathematically. It seems that whatever one takes to be the "right" continuous time equivalent of the Wilkie model, similar methods to those presented here can be used to analyse it.

One reason one might be interested in a continuous time model is that in a continuous time setting one is free to choose any unit of time and to model the state of the various investment variables at any time, not just at discrete instants. However, the main attraction of continuous time models is their

mathematical tractability; whereas the Wilkie model is mainly intended for computer simulations, in the continuous time setting here many questions admit explicit answers which can be obtained in a simple way. Here, we concentrate on obtaining explicit distributions but other questions can undoubtedly be answered

The model introduced in Wilkie (1986) only makes use of Gaussian (white noise) series, for our model the driving noises are allowed to have other distributions.

2. DESCRIPTION OF THE MODEL

In many ways, the model described here is the most direct and obvious continuous-time version of the model in Wilkie (1986), although some modifications are necessitated by the transition to a continuous time scale. We do not make any special claims about its appropriateness to practical situations beyond pointing out its similarity to the original Wilkie model which has by now gained wide acceptance, at least in the world of insurance. The two main guiding principles behind the construction of the continuous time model presented here are firstly the analogy with the corresponding time series and secondly the similarities between certain features of the Wilkie model and other models which feature widely in different areas of financial modelling, occasionally we shall depart from an exact analogy with the time series to emphasise these similarities because the qualitative features common to all these models are of potentially greater interest Thus, it would be more appropriate to refer to the continuous time model presented here as inspired by the Wilkie model, rather than "the continuous time Wilkie model" The model should be treated as a "first draft" rather than a final version. As with the original Wilkie model, the model here is based on four processes (although these are not exactly the same as the ones in Wilkie (1986)) and we describe each of these in turn.

Let Z_1, Z_2, Z_3 and Z_4 be four independent (not necessarily continuous) processes. Exactly what kind of processes are the Z_i will be discussed later.

1.1. Retail prices index and inflation

Consider first a retail prices index, $Q_t \equiv \exp\{P_t\}$ We use an Ornstein-Uhlenbeck type model for the process P.

$$
\begin{aligned}
dP_t &= R_t dt \\
dR_t &= -a_1 R_t dt + \phi(t) dt + \sigma_1 dZ_1(t)
\end{aligned}
\tag{1 1}
$$

where $a_1 > 0$, $\sigma_1 \in \mathbb{R}$ and ϕ is a (deterministic) positive periodic function with period $h > 0$. Here the process R plays the role of the continuous force of inflation. A direct translation of Wilkie's model would have $\phi \equiv$ constant, but in passing to continuous time it may be desirable to take into account the seasonal fluctuations in inflation over a year. The period h here corresponds

to a year in our units of time (see Remark (ii) below) (To spell things out in a little more detail, supposing $\phi \equiv$ constant, the process R_t in (1 1) corresponds to Wilkie's $\nabla \log Q(t)$, the parameter a_1 corresponds to the parameter Wilkie calls $1 - QA$, ϕ corresponds to QMU$(1 - QA)$ in Wilkie's original paper and σ_1 plays the role of QSD).

Because (1.1) is a linear equation, it is easy to solve explicitly, whatever our choice of driving noise Z_1. The general version of Itô's formula for discontinuous semimartingales X states that if f is a continuous function with the necessary derivatives,

$$f(X_t, t) - f(X_0, 0) = \int_0^t \dot{f}(X_s, s)ds + \int_0^t f'(X_{s-}, s)dX_s + \frac{1}{2}\int_{0+}^t f''(X_{s-}, s)d[X]_s$$
$$+ \sum_{0 < s \leq t} \left(f(X_s, s) - f(X_{s-}, s) - f'(X_{s-}, s)\Delta X_s - \frac{1}{2}f''(X_{s-}, s)(\Delta X_s)^2 \right)$$

where $\Delta X_s = X_s - X_{s-}$, $\dot{f} \equiv \partial f / \partial t$, $f' \equiv \partial f / \partial x$ etc. For this and other aspects of the general theory of stochastic integration with respect to semimartingales, we refer the reader to Protter (1990) and Roger and Williams (1987), which approach the subject in different ways. (Note that X is assumed to be right-continuous and can only have countably many jumps, so the sum above is actually a sum over countably many values of s). Consider now the case that $f(x, t) = e^{a_1 t} X_t$ We have $f'' \equiv 0$ and $f'(X_{s-}, s)\Delta X_s = f(X_s, s) - f(X_{s-}, s)$, so the terms involving the jumps of X in Itô's formula all vanish Therefore applying Itô's formula to $e^{a_1 t}R_t$, we obtain an explicit formula for R_t:

$$R_t = e^{-a_1 t}R_0 + \int_0^t e^{-a_1(t-s)}\phi(s)ds + \int_0^t \sigma_1 e^{-a_1(t-s)}dZ_1(s). \qquad (1\ 2)$$

From (1.2), we can find $P_t = P_0 + \int_0^t R_s ds$ and the resulting double integrals can be handled by interchanging the order of integration (e g. see Lemma 3 1 in the sequel)

1.2. Share yield process

Wilkie (1986) next considers two inter-related processes: an index of share dividends and the dividend yield process. Let Y_t denote the share dividend yield The continuous time analogue of Wilkie's model would be

$$Y_t = Y_* \exp\{X_t + \zeta R_t\},$$
$$\text{where} \quad dX_t = -a_2 X_t dt + b_1 dt + \sigma_2 dZ_2(t) \qquad (1\ 3)$$

(Here, $Y_* = Y_0 e^{-(X_0 + \zeta R_0)}$. In the sequel, this notation will be frequently used to denote this kind of "modified initial condition".) Equation (1.3) admits an explicit solution similar to (1.2), namely

$$X_t = X_0 e^{-a_2 t} + b_1 \left(\frac{1 - e^{-a_2 t}}{a_2} \right) + \int_0^t \sigma_2 e^{-a_2(t-s)} dZ_2(s). \qquad (1\,4)$$

1.3. Share dividend process

We next turn to the index of share dividends, D_t Our model follows Wilkie in using an exponentially discounted "sum of inflation effects".

$$d(\log D_t) = \left(b_2 + \beta\lambda \int_0^t e^{-\lambda s} R_{t-s} ds + \gamma R_t \right) dt + \eta_2 dZ_2(t) + \eta_3 dZ_3(t). \quad (1.5)$$

In Wilkie's time series model, the noise has a simultaneous as well as a lagged effect which is captured by moving average in the noise. There is no sensible equivalent in the continuous time context for such a moving average. Another feature of the model (inherited from Wilkie) is the mixing of the driving noises for Y_t and D_t.

The share price S_t is related to the dividends and the yield by $S_t = D_t / Y_t$ It is interesting to note that the process S_t satisfies an equation of the form

$$dS_t = c_t S_t dt + S_t (\delta_1 dZ_1(t) + \delta_2 dZ_2(t) + \delta_3 dZ_3(t)),$$

which has exactly the same form as the ubiquitous geometric Brownian motion model of share prices, except that the coefficient c_t here takes a rather complicated form which involves the whole path of the force of inflation R up to time t, as well as the usual constant drift terms

Interchanging the order of integration, it is easy to see that

$$\lambda \int_0^t \int_0^s e^{-\lambda u} R_{s-u} du\, ds = \int_0^t (1 - e^{-\lambda(t-u)}) R_u du ,$$

therefore from (1 5) we have

$$D_t = D_* \exp\left\{ \eta_2 Z_2(t) + \eta_3 Z_3(t) + \beta \int_0^t (1 - e^{-\lambda(t-u)}) R_u du + \gamma \int_0^t R_u du + b_2 t \right\}.$$
$$(1.6)$$

where D_* is a constant determined by D_0 and R_0 in a similar manner to Y_* (see Remark (iv) in §1.5 below).

1.4. Consol yield process

Finally, we have the yield on consols C_t,

$$C_t = \xi\rho \int_0^t e^{-\rho s} R_{t-s} ds + C_* e^{V_t},$$

$$dV_t = -a_4 V_t dt + \sigma_4 dZ_4(t), \quad V_0 = v \tag{1.7}$$

The equation for V in (1.7) admits an explicit solution for the same form as (1 2)

1.5. General remarks on the model

(i) We do not claim that the method in (1.1) is the most appropriate way to model seasonal effects in inflation – it is *one* simple and obvious way to do it without destroying the most attractive features of the Ornstein-Uhlenbeck process but we could equally plausibly let σ_1 be a periodic function as well and we would still be able to obtain an explicit solution as before

(ii) Some remarks on the time scale of the continuous time processes here and their relationship with their discrete-time counterparts in Wilkie (1986) might be useful Typically these continuous time processes run at a much faster speed than their discrete-time equivalents: for example, if the unit of time in Wilkie (1986) is years, the unit of time here might be centuries, so that $h = 0.01$ would correspond to a year. This is essentially an artifact of the discretization in passing from continuous time to discrete time. If we were to discretize (1.1) in multiples of h using first-order Euler approximation together with the approximation $P_t - P_{t-h} = \int_{t-h}^t R_s \, ds \approx h R_{t-h}$ and noting that $\phi(t) = \phi(t - h) = \phi = $ constant, we would recover the Wilkie model provided we rescale time by defining $\tilde{R}_t := R_{ht}$. For example, assuming that Z_1 is Brownian motion for simplicity, the first-order Euler discretization of (1.1) is

$$R_t - R_{t-h} = -a_1 h R_{t-h} + \phi h + \sigma_1(Z_1(t) - Z_1(t - h)),$$

which can be rewritten as

$$\begin{aligned} R_t &= (1 - a_1 h) R_{t-h} + \phi h + \sigma_1 \sqrt{h}\, W_t \\ &= \mu + a(R_{t-h} - \mu) + \sigma_1 \sqrt{h}\, W_t, \end{aligned} \tag{1.8}$$

where we have put $a = 1 - a_1 h$, $\mu = \phi/a_1$ and $W_t = (Z_1(t) - Z_1(t - h))/\sqrt{h}$. Note that W_h, W_{2h}, W_{3h}, ... are i.i.d. standard Gaussian random variables. Defining $\tilde{R}_t := R_{ht}$, we obtain from (1.8) the AR(1) time-series model of Wilkie:

$$\tilde{R}_t = \mu + a(\tilde{R}_{t-1} - \mu) + \tilde{\sigma}_1 W_t, \tag{1.9}$$

where $\tilde{\sigma}_1 = \sigma_1 \sqrt{h}$. The calculations at (2.5) and (2.6) below and the subsequent discussion illustrate this point in greater detail Observe that the corresponding parameters in (1.1) are rescaled in the appropriate way with this time change: $\tilde{a}_1 = ha_1$ and $\tilde{\sigma}_1 = \sigma_1\sqrt{h}$ Because the parameters are automatically scaled accordingly once a time scale has been chosen, such comparisons with the discrete time-series are usually irrelevant from a practical point of view; in practice, one would choose a suitable time scale and then fit the model to data directly without reference to any discrete-time model and if one wished to do simulation, one would choose a discretization for its numerical efficiency rather than for its consistency with another discrete-time model. The same comment applies to all the other processes discussed above

(iii) For our choices of Z_1, the process R will have a stationary distribution. Throughout this paper, we assume that the initial condition R_0 is some fixed number as in (1.2) However, it is also possible to let R_0 be a random variable with the stationary distribution, in which case R would be a stationary process. The same can be said of all the other processes which have stationary distributions.

(iv) Because the processes X and R in (1.3) are not spatially homogeneous, the initial values X_0 and R_0 cannot be absorbed into Y_* and so separate parameters for the initial values are needed The same applies to the processes D and C. Also, Wilkie (1986) has an extra drift term of the form $c\,dt$ appearing in the equation for V_t in (1.7) but we have omitted it here because it is clear from the explicit formula for V_t that c can be absorbed into the two parameters v and C_*, and so serves no additional purpose

1.6. Lévy processes

We are mainly interested in the case where the "noise" processes Z_t are symmetric Lévy processes, that is processes with stationary independent increments ("Symmetric" in this context just means that Z and $-Z$ have the same law.) We end this section by briefly recalling some results about Lévy processes which we shall need in the sequel. Let Z be a (symmetric) Lévy process. Since Z has stationary independent increments, its characteristic function must take the form $\mathbb{E}\left[e^{i\theta Z_t}\right] = e^{t\psi(\theta)}$ for some function ψ, called the *Lévy exponent* of Z. The Lévy-Khintchine formula says that

$$\psi(\theta) = \frac{\sigma^2}{2}\theta^2 + ia\theta + \int_{\{|x|<1\}}(1 - e^{i\theta x} - i\theta x)\nu(dx) + \int_{\{|x|\geq 1\}}(1 - e^{i\theta x})\nu(dx) \quad (1\ 10)$$

for $a, \sigma \in \mathbb{R}$ (if Z is symmetric, $a = 0$) and for some σ-finite measure ν on $\mathbb{R} \setminus \{0\}$ satisfying $\int \min(1, x^2)\nu(dx) < \infty$. The measure ν is called the *Lévy measure* of Z. (To put readers on more familiar ground, consider the situation when $\sigma = a = 0$ and suppose that the total mass λ of ν, $\lambda = \int_{\mathbb{R}}\nu(dx)$ is finite. Then the Lévy process Z with such a Lévy measure is just an ordinary compound Poisson process which jumps occurring as a Poisson process of rate λ and whose jump-size distribution is $\lambda^{-1}\nu(dx)$. In the case that the integral of ν diverges near 0, Z will have infinitely many small jumps in a finite time-interval At the other extreme, if $\nu \equiv 0$, there are no jumps so we just have Brownian motion and ψ is the same as the exponent for a normal distribution)

From the Lévy-Khintchine formula we can deduce the exact form Z must take. it turns out that Z must be a linear combination of a Brownian motion (the continuous part) and a pure-jump process independent of the Brownian part. Specifically, let $Q(dt, dx)$ be a Poisson measure on $(0, \infty) \times \mathbb{R} \{0\}$ with expectation measure $dt \times \nu$ (here dt denotes Lebesgue measure), then (assuming $a = 0$ in (1.10)) we have the Lévy decomposition

$$Z_t = \sigma B_t + J_t + A_t \tag{1.11}$$

where, corresponding to each of the three terms in (1.10) respectively, B is a Brownian motion, J is the pure-jump martingale $J_t = \int_{|x|<1} x(Q((0,t],dx) - t\nu(dx))$ and A is the finite-variation jump process $A_t = \int_{\{|x|\geq 1\}} xQ((0,t], dx)$. The processes B, J and A are independent A more detailed treatment can be found in Protter (1990) and Rogers and Williams (1987) also contains a nice direct construction of (1.11). Because of independence, we lose no generality in treating separately the cases where Z is a Brownian motion and where Z is a pure-jump process We do this in the next two sections

3 Explicit distributions in the Brownian case

If the Z_t are all Brownian motions, all the processes described in the previous section are either Gaussian processes or exponentials of Gaussian processes Since in order to specify the law of a Guassian process one only has to specify the mean and the covariance, the results of this section are essentially trivial.

Recall that for a Brownian motion W, $\int_0^t f(s)dW_s = B\left(\int_0^t f(s)^2 ds\right)$ where B is some other Brownian motion. Applying this result to (1.2) gives

$$R_t = e^{-a_1 t}R_0 + \int_0^t e^{-a_1(t-s)}\phi(s)ds + \sigma_1 e^{-a_1 t}B_1\left(\frac{e^{2a_1 t} - 1}{2a_1}\right) \tag{2.1}$$

where B_1 is a Brownian motion. Hence, R_t has Gaussian distribution with mean

$$\mu_R(t) = e^{-a_1 t} R_0 + \int_0^t e^{-a_1(t-s)} \phi(s) ds \qquad (2.2a)$$

and variance

$$v_R(t) = \sigma_1^2 \left(\frac{1 - e^{-2a_1 t}}{2a_1} \right). \qquad (2.2b)$$

(In 2.2a,b) we have used the fact that B_t is Gaussian with mean 0 and variance t). Similar results hold for the other Ornstein-Uhlenbeck type processes X and V introduced in Section 1.

From (1.6) and (1.7), it is clear that the key to finding the distributions of D_t and C_t lies in obtaining the distribution of $\int_0^t f(s) R_s \, ds$ for suitable (deterministic) functions f. Since R is a Gaussian process, so is $t \mapsto \int_0^t f(s) R_s \, ds$ and so all we need to do is work out the mean and variance of $\int_0^t f(s) R_s \, ds$. The mean is trivial: by interchanging the order of integration it is easy to see that the mean is just $\int_0^t f(s) \mu_R(s) ds$ We now turn to the variance Since the mean is irrelevant here, the variance is simply given by

$$\mathbb{E}^0 \left[\left(\int_0^t f(s) H_s ds \right)^2 \right] = \mathbb{E}^0 \left[\int_0^t \int_0^t f(s) f(u) H_s H_u \, du \, ds \right]$$

where we have put

$$H_t = \sigma_1 e^{-a_1 t} B_1 \left(\frac{e^{2a_1 t} - 1}{2a_1} \right)$$

and we use the superscript in $\mathbb{E}^0$ to emphasise that $H_0 = B_1(0) = 0$. Using the covariance of Brownian motion $\mathbb{E}(B_s B_u) = \min(s, u)$ and interchanging the order of integration, we get

$$\mathbb{E}^0 \left[\int_0^t \int_0^t f(s) f(u) H_s H_u \, du \, ds \right]$$

$$= 2 \int_0^t f(s) \int_0^s f(u) E(H_s H_u) \, du \, ds$$

$$= 2\sigma_1^2 \int_0^t f(s) e^{-a_1 s} \int_0^s f(u) e^{-a_1 u} \left(\frac{e^{2a_1 u} - 1}{2a_1} \right) du \, ds \qquad (2.3)$$

Putting $f \equiv 1$ in (2.3) gives the variance of $\int_0^t R_s \, ds$ to be

$$\frac{\sigma_1^2}{a_1} \left(\frac{t}{a_1} + \frac{2e^{a_1 t}}{a_1^2} - \frac{e^{-2a_1 t}}{2a_1^2} - \frac{3}{2a_1^2} \right). \qquad (2.4)$$

At this point, it may be instructive to compare these results with the analogous ones for the AR(1) time series (1.9) The mean and variance of $\sum_{i=1}^{t} \tilde{R}_i$ has been obtained by Hurlimann (1992) and Wilkie (1995). Keeping to our notation established in (1.9), the mean of the accumulated force of inflation $\sum_{i=1}^{t} \tilde{R}_i$ is

$$\mu t + (\tilde{R}_0 - \mu) \frac{a(1 - a^t)}{1 - a} \tag{2 5}$$

while in the continuous model the mean $\int_0^t R_s \, ds$ is (assuming $\phi = $ const.)

$$\int_0^t \mu_R(s) ds = \mu t + \frac{R_0 - \mu}{a_1} [1 - e^{-a_1 t}], \tag{2 6}$$

where $\mu = \phi/a_1$ as before. We see immediately that (2 5) and (2.6) have the same form. To check that they in fact agree, recall that to obtain the time-series (1.9) from (1.1), we discretized time into steps of size h Therefore $\sum_{i=1}^{t/h} h\tilde{R}_i$ is precisely the Riemann-sum approximation to $\int_0^t R_s ds$. According to the formula (2.5), the mean of $\sum_{i=1}^{t/h} h\tilde{R}_i$ is

$$\mu t + (\tilde{R}_0 - \mu) h \frac{a(1 - a^{t/h})}{1 - a}$$

$$= \mu t + (R_0 - \mu) \left(\frac{1 - a_1 h}{a_1} \right) \left[1 - (1 - a_1 h)^{t/h} \right]$$

$$\rightarrow \mu t + \frac{R_0 - \mu}{a_1} [1 - e^{-a_1 t}]$$

as $h \rightarrow 0$, which is precisely the mean of $\int_0^t R_s ds$ given by (2 6). Similarly, Hurlimann (1992) gives the variance of $\sum_{i=1}^{t} \tilde{R}_i$ as

$$\frac{\tilde{\sigma}_1^2}{(1 - a)^2} \left[t - \frac{2a(1 - a^t)}{1 - a} + \frac{a^2(1 - a^{2t})}{1 - a^2} \right],$$

which has the same form as (2 4).

It is just as easy to obtain the distributions of the other processes in our model Putting $f(s) = \rho e^{-\rho(t-s)}$ in (2 3) we get that $\int_0^t e^{-\rho(t-s)} R_s \, ds$ has Gaussian distribution with mean

$$\int_0^t \rho e^{-\rho(t-s)} \mu_R(s) \, ds \tag{2.7a}$$

and variance

$$\frac{\rho \sigma_1^2}{a_1} \left(\frac{(a_1 - \rho)^2 - (a_1 + \rho)(\rho e^{-2a_1 t} + a_1 e^{-2\rho t}) + 4 a_1 \rho e^{-(a_1 + \rho)t}}{2(a_1 - \rho)^2 (a_1 + \rho)} \right) \tag{2.7b}$$

Putting $f = \beta + \gamma - \beta e^{-\lambda(t-s)}$ also gives an explicit expression for the variance of $\beta \int_0^t (1 - e^{-\lambda(t-s)}) R_s\, ds + \gamma \int_0^t R_s\, ds$, although this is too messy to write down here – the formula is simplified somewhat by choosing $\gamma = 0$ and simplified considerably by choosing $\gamma = -\beta$, for this would then reduce to (2.7b). The full covariance structure of the process $t \mapsto \int_0^t f(s) R_s\, ds$ can also be obtained in this way

Armed with these results, we can now state the distributions of interest. We have already found the distributions of R_t and $P_t = \int_0^t R_s\, ds$ (see (2.2) and (2.4)). Applying the results (2 2) to the process X, we get from (1.3) and (1.4) that log Y_t has Gaussian distribution with mean

$$\log Y_* + X_0 e^{-a_2 t} + b_1\left(\frac{1 - e^{-a_2 t}}{a_2}\right) + \zeta \mu_R(t)$$

and variance

$$\zeta^2 v_R(t) + \sigma_2^2\left(\frac{1 - e^{-2a_2 t}}{2a_2}\right).$$

For the dividend index D, the result (2.3), with $f(s) = \beta + \gamma - \beta e^{-\lambda(t-s)}$, together with the analogous results (2.2) for U give that log D_t has Gaussian distribution with mean

$$\log D_* + b_2 t + \int_0^t (\beta + \gamma - \beta e^{-\lambda(t-s)}) \mu_R(s)\, ds$$

and variance

$$(\eta_2^2 + \eta_3^2)t$$
$$+ 2\sigma_1^2 \int_0^t (\beta + \gamma - \beta e^{-\lambda(t-s)}) e^{-a_1 s} \int_0^s (\beta + \gamma - \beta e^{-\lambda(t-u)})\left(\frac{e^{a_1 u} - e^{-a_1 u}}{2a_1}\right) du\, ds.$$

Applying (2 2) to V_t shows that it is Gaussian with mean

$$\mu_V(t) = v e^{-a_4 t}$$

and variance

$$v_V(t) = \sigma_4^2\left(\frac{1 - e^{-2a_4 t}}{2a_4}\right)$$

The distribution of C_t is the convolution of normal and log normal distributions and the results (2.2) and (2 7) show that C_t has mean

$$\xi \int_0^t \rho e^{-\rho(t-s)} \mu_R(s)\, ds + C_* e^{\mu_V(t) + v_V(t)/2}$$

and variance

$$\frac{\rho\xi^2\sigma_1^2}{a_1}\left(\frac{(a_1-\rho)^2-(a_1+\rho)(\rho e^{-2a_1 t}+a_1 e^{-2\rho t})+4a_1\rho e^{-(a_1+\rho)t}}{2(a_1-\rho)^2(a_1+\rho)}\right)$$
$$+C_*^2 e^{2\mu_V(t)+v_V(t)}(e^{v_V(t)}-1).$$

It is also possible to specify the full multivariate structure of R, Y and D using the methods here. Since R, Y and D are either Gaussian or log Gaussian, their joint law is specified once we have the covariances $\mathrm{Cov}(R_t, \log Y_t)$, $\mathrm{Cov}(R_t, \log D_t)$ and $\mathrm{Cov}(\log Y_t, \log D_t)$. For the most part, we only need to know the covariance stucture of the process $t \mapsto R_t$, which is given by

$$\mathbb{E}[R_t R_s] = \mu_R(t)\mu_R(s) + \mathbb{E}[H_s H_u] = \mu_R(t)\mu_R(s) + \sigma_1^2 e^{-a_1 t}\left(\frac{e^{a_1 s}-1}{2a_1}\right)$$

if $s < t$. Thus, for example,

$$\mathbb{E}[R_t \log D_t] = \mu_R(t)(\log D_* + b_2 t) + \int_0^t (\beta + \gamma - \beta e^{-\lambda(t-s)})\mathbb{E}[R_t R_s]\, ds$$

and we can then substitute the relevant previous results into the above expression. In addition, we also need the covariance of X_t and $Z_2(t)$, which is given by

$$\mathbb{E}^0[X_t Z_2(t)] = \sigma_2 e^{-a_2 t}\min\left(\frac{e^{a_2 t}-1}{2a_2},\ t\right)$$

using the covariance of Brownian motion. The detailed computations of the covariances are left to the reader.

4. EXPLICIT DISTRIBUTIONS IN THE DISCONTINUOUS CASE

There have been some suggestions that Gaussian noise terms are not entirely appropriate for these models and that more realistically, the noise should have jumps In this section, we perform the same analysis as in Section 2 on the assumption that the Z_t are symmetric pure-jump Lévy processes.

From the analysis in Section 2, it is clear that once we know what the law of $\int_0^t f(s)Z_s\, ds$ is for fixed t (where $f(s)$ or $f(t, s)$ is a suitable function and Z is a generic Lévy process), we can obtain the necessary explicit distributions. It all turns out to rest on the following simple lemma allowing the interchange of order of integration.

LEMMA 3.1: *Let f and g be Riemann-integrable functions. Then the laws of*

$$\int_0^t f(s)\int_0^s g(u)\, dZ_u\, ds \quad and \quad \int_0^t g(u)\int_u^t f(s)\, ds\, dZ_u$$

are the same for each fixed t and the common law is given by

$$\mathbb{E}\left[\exp\left\{-i\theta\int_0^t g(u)\int_u^t f(s)\,ds\,dZ_u\right\}\right]=\exp\left\{-\int_0^t \psi(\theta g(u)[F(t)-F(u)])\,du\right\}$$

$$(3.1)$$

where ψ is given by the Lévy-Khintchine formula (1.10) and $F(u) = \int_0^u f(s)\,ds$

The proof, although not very pretty, uses only well-known standard results in the theory of stochastic integration and Lévy processes and is presented in the Appendix.

Remarks
(ı) The above lemma is trivial if Z has finite variation, for then the integral $\int_0^t g(s)\,dZ_s$ exists as an ordinary Riemann-Stieltjes integral. Changing the order of integration as for ordinary integrals, we actually have the much stronger result that

$$\mathbb{P}\left(\int_0^t f(s)\int_0^s g(u)\,dZ_u\,ds = \int_0^t g(u)\int_u^t f(s)\,ds\,dZ_u \quad \forall t\right) = 1.$$

When Z has infinite variation, the integral with respect to Z is a "genuine" stochastic integral. In this case, we have to emphasise that Lemma 3.1 holds only for fixed t; the two integrals clearly cannot have the same law as processes since the former is a process of finite variation while the latter has infinite variation.

(ıı) Since t is a fixed parameter in the present context, Lemma 3.1 holds equally if we allow f and g to also depend on t, which we need to do for some of the processes considered earlier.

(ııı) Note that a simple special case of (3.1) is that

$$\mathbb{E}\left[\exp\left\{-i\theta\int_0^t G(t,u)\,dZ_u\right\}\right] = \exp\left\{-\int_0^t \psi(\theta G(t,u))\,du\right\} \qquad (3\,2)$$

for any (Riemann-integrable) function G.

Consider now the model described in Section 1 where the Z_t are symmetric Lévy processes with jumps. From the explicit formula (1.2) for R, we see that to find the law of R_t we can apply (3.2) with $G(t,u) = \sigma_1 e^{-a_1(t-u)}$, in which case we obtain

$$\mathbb{E}[e^{-i\theta R_t}] = \exp\left\{-i\theta\mu_R(t) - \int_0^t \psi(\theta\sigma_1 e^{-a_1(t-u)})\,du\right\},$$

where $\mu_R(t)$ is as defined by (2.2a). In a similar way we can obtain the laws of the processes X, U and V introduced in Section 1. For the law of $\int_0^t R_s\,ds$, we can apply Lemma 3.1 with $g(t,u) = \sigma_1 e^{-a_1(t-u)}$, $f \equiv 1$ and for the law of $\int_0^t \rho e^{-\rho(t-s)} R_s\,ds$ we can take $g(t,u) = \sigma_1 e^{-a_1(t-u)}$, $f(t,s) = \rho e^{-\rho(t-s)}$. In this

way, we obtain the distributions of R_t, S_t, C_t and $\log D_t$ in a similar manner to Section 2 However, the joint distribution is much more difficult to obtain

We end this section with a brief word on some specific examples of Lévy processes one might choose to use in these models. We just mention two commonly used Lévy processes. One is the symmetric α-stable process, whose Lévy exponent is $\psi(\theta) = |\theta|^\alpha$ and whose Lévy measure is

$$\nu(dx) = \frac{C_\alpha}{|x|^{1+\alpha}} dx, \quad x \neq 0,$$

where $C_\alpha = \pi^{-1}\Gamma(1+\alpha)\sin(\pi\alpha/2)$. (Here $0 < \alpha < 2$; $\alpha = 2$ corresponds to the Gaussian distribution and $\alpha = 1$ gives the Cauchy distribution). Stable distributions are examples of so-called heavy-tailed distributions One of the disadvantages of stable processes is that they do not have higher order moments than 1 (for $\alpha \leq 1$ they do not even have a first moment) which may cause awkward problems, for example, when we take exponentials of stable processes as we are frequently doing in these models

Another commonly used class of Lévy processes which overcomes this problem is the Gamma process. A Lévy process Y is said to be Gamma with parameters $(\alpha, \beta$ where $\alpha, \beta > 0$ if $\mathbb{P}(Y_1 \leq x) = \Gamma(\alpha)^{-1}\beta^\alpha \int_0^x y^{\alpha-1}e^{-\beta y}dy$. Hence

$$\mathbb{E}[e^{-i\theta Y_t}] = \left(\frac{\beta}{\beta+i\theta}\right)^{\alpha t} = \exp\left\{-\alpha t \log\left(1 + \frac{i\theta}{\beta}\right)\right\}.$$

Note that such a process is non-decreasing, so to obtain a symmetric process, we simply take two independent copies Y and $\tilde{Y}$ and define $Z = Y - \tilde{Y}$. The process Z is therefore a symmetric Lévy process with Lévy exponent $\psi(\theta) = \alpha\log(1 + \theta^2/\beta^2)$ and Lévy measure $\nu(dx) = \alpha|x|^{-1}e^{-\beta|x|}dx$. Looking at the Lévy decomposition, since $\int_{\{|x|<1\}}|x|\nu(dx) < \infty$, we see that Z has finite variation and since $\int_{\{|x|\geq 1\}}|x|^n\nu(dx) < \infty$, Z_t has finite moments of all orders.

Applying Lemma 3 1 we obtain (replacing $i\theta$ with θ for convenience)

$$\mathbb{E}\left[\exp\left\{-\theta\int_0^t f(t-s)\,dZ_s\right\}\right] = \exp\left\{-\alpha\int_0^t \log\left(1 - \frac{\theta^2 f(t-s)^2}{\beta^2}\right)ds\right\} \quad (3\ 3)$$

and

$$\mathbb{E}\left[\exp\left\{-\theta\int_0^t g(u)\int_u^t f(s)\,ds\,dZ_u\right\}\right]$$

$$= \exp\left\{-\alpha\int_0^t \log\left(1 - \frac{\theta^2 g(u)^2}{\beta^2}\left[\int_u^t f(s)\,ds\right]^2\right)du\right\} \quad (3.4)$$

Although in general it is not possible to give explicit formulae for the integrals in (3.3) and (3 4) for our choices of f and g as in the preceding two sections, the Laplace transforms (3 3-4) do give relatively simple expressions for the moments, involving integrals which can be readily evaluated by numerical means.

5. Concluding remarks

We have concentrated here on obtaining explicit formulae, both in the case where our SDEs are driven by Brownian motions and in the case where they are driven by symmetric Lévy processes with jumps. Of course, many other questions – which we have not considered – do not admit explicit answers and one must then resort to numerical solutions. It is not our intention here to give a detailed quantitative analysis of numerical simulations of the models presented in the preceding sections, as this could well constitute a paper in its own right. We simply present some examples of numerical simulations to give a feel for what these processes look like In the case of SDEs driven by Brownian motion, great advances have been made in recent years in numerical methods for solving them. For a comprehensive survey of these techniques as well as an extensive bibliography on the subject, we refer the reader to Kloeden and Platen (1992) By contrast, numerical methods for SDEs driven by processes with jumps, such as stable processes, have received far less attention until recently and the literature on this subject is more limited: a systematic treatment in book form can be found in Janicki and Weron (1993).

For simplicity, we present some simulations for the inflation process R_t only since of the four components, this is closest to the time-series model of Wilkie Figure 1 shows three trajectories of the process R_t, in the case where the noise Z_t is Brownian motion. The scaling used is such that the time interval [0, 1] corresponds to a period of 50 years Specifically, in the context of Remark (ii) in Section 1, we have used $h = 0.02$ and in equation (1.2) our choice of ϕ is $\phi(t) = b + c \cos(2\pi t/h)$. Since the picture is only intended to give a qualitative indication of how the process behaves, the actual numerical values on the vertical axes are not of any great importance: the parameter values in Wilkie (1986) are used as a rough guide to the sort of values which might be appropriate for the parameters here – in particular, the parameter values of Wilkie are rescaled in the manner discussed in Remark (ii) of Section 1

Throughout, we have taken the various parameters in our models as given quantities and we have said nothing about the problems of their estimation. There is some discussion of this question in §6.4 and §13.2 of Kloeden and Platen (1992) which is especially relevant to the linear equations which appear repeatedly in our models.
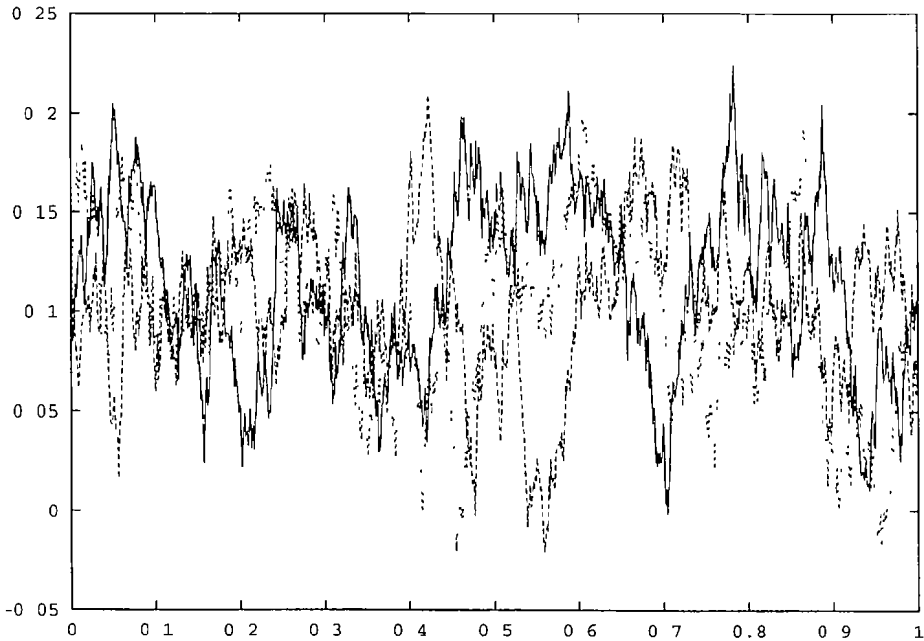
FIGURE 1 SAMPLE PATH REALIZATIONS OF THE FORCE OF INFLATION PROCESS R_t

APPENDIX PROOF OF LEMMA 3 1

Consider first the integral $I(s) = \int_0^s g(u)\,dZ_u$ Take a sequence of partitions $\left(u_k^{(n)}, u_{k+1}^{(n)}\right]$ of the interval $[0, t]$, such that $\sup_k |u_{k+1}^{(n)} - u_k^{(n)}| \to 0$ *as* $n \to \infty$. It is known that, as $n \to \infty$,

$$I_n(s) = \sum_{u_k^{(n)} \le s} g(u_k^{(n)}) \left(Z(u_{k+1}^{(n)}) - Z(u_k^{(n)}) \right) \to \int_0^s g(u)\,dZ_u$$

in probability uniformly in s over the time interval $[0, t]$ (see Protter (1990)). Therefore, there is a subsequence (n_i) such that $I_{n_i}(s) \to I(s)$ almost surely as $i \to \infty$ and without loss of generality we can assume that $I_n(s) \to I(s)$ almost surely. Next, take a different sequence of successively refining partitions of $[0, t]$ and call this $\left[s_j^{(m)}, s_{j+1}^{(m)}\right)$. Put

$$F_m(u_k^{(n)}, t) = \sum_{s_j^{(m)} \ge u_k^{(n)}} f\left(s_j^{(m)}\right)\left(s_{j+1}^{(m)} - s_j^{(m)}\right)$$

(Of course, $F_m(u_k^{(n)}, t) \to F(t) - F(u_k^{(n)})$ as $n \to \infty$.) We then have

$$\lim_{m \to \infty} \lim_{n \to \infty} \sum_{s_j^{(m)} \le t} f\left(s_j^{(m)}\right)\left(s_{j+1}^{(m)} - s_j^{(m)}\right) \sum_{u_k^{(n)} \le s_j^{(m)}} g\left(u_k^{(n)}\right)\left(Z(u_{k+1}^{(n)}) - Z(u_k^{(n)})\right)$$

$$= \int_0^t f(s) \int_0^s g(u)\, dZ_u\, ds$$

and so for fixed n,

$$\mathbf{E}\left[\exp\left\{-\iota\theta \sum_{s_j^{(m)} \le t} f\left(s_j^{(m)}\right)\left(s_{j+1}^{(m)} - s_j^{(m)}\right) \sum_{u_k^{(n)} \le s_j^{(m)}} g\left(u_k^{(n)}\right)\left(Z(u_{k+1}^{(n)}) - Z(u_k^{(n)})\right)\right\}\right]$$

$$= \mathbf{E}\left[\exp\left\{-\iota\theta \sum_{u_k^{(n)} \le t} g\left(u_k^{(n)}\right)\left(Z(u_{k+1}^{(n)}) - Z(u_k^{(n)})\right) \sum_{s_j^{(m)} \ge u_k^{(n)}} f\left(s_j^{(m)}\right)\left(s_{j+1}^{(m)} - s_j^{(m)}\right)\right\}\right]$$

$$= \prod_{u_k^{(n)} \le t} \exp\left\{-\left(u_{k+1}^{(n)} - u_k^{(n)}\right)\psi\left[\theta g(u_k^{(n)})F_m(u_k^{(n)}, t)\right]\right\}$$

$$= \exp\left\{-\sum_{u_k^{(n)} \le t}\left(u_{k+1}^{(n)} - u_k^{(n)}\right)\psi\left[\theta g(u_k^{(n)})F_m(u_k^{(n)}, t)\right]\right\}$$

$$\to \exp\left\{-\sum_{u_k^{(n)} \le t}\left(u_{k+1}^{(n)} - u_k^{(n)}\right)\psi\left[\theta g(u_k^{(n)})\left(F(t) - F(u_k^{(n)})\right)\right]\right\} \tag{A1}$$

as $m \to \infty$. In the above calculation, we have used the stationary independent increments property of Z and the fact that $\mathbf{E}\left[e^{-\iota\theta(Z_t - Z_s)}\right] = e^{-(t-s)\psi(\theta)}$. Letting $n \to \infty$ in (A1) then gives the right-hand side of (3.1).

For the integral $\int_0^t g(u) \int_u^t f(s)\, ds\, dZ_u$, we know that

$$\sum_k g(u_k^{(n)})\left(F(t) - F(u_k^{(n)})\right)\left(Z(u_{k+1}^{(n)} - Z(u_k^{(n)})\right) \to \int_0^t g(u) \int_u^t f(s)\, ds\, dZ_u$$

almost surely as $n \to \infty$ (passing to a subsequence if necessary) A similar calculation as in (A1) easily yields the identity (3.1).

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