A MULTIVARIATE GENERALIZATION OF THE GENERALIZED POISSON DISTRIBUTION

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ABSTRACT

This paper proposes a multivariate generalization of the generalized Poisson distribution. Its definition and main properties are given. The parameters are estimated by the method of moments.

KEYWORDS

Multivariate generalized Poisson distribution (MGP_m); generalized Poisson distribution (GPD); bivariate generalized Poisson distribution (BGPD).

1. INTRODUCTION

The univariate generalized Poisson distribution (GPD), introduced by CONSUL and JAIN (1973), is a well-studied alternative to the standard Poisson distribution. CONSUL (1989) provided a guide to the current state of modeling with the GPD at that time, and documented many real life examples. GPD has also been making appearances in the actuarial literature (see GERBER, 1990; GOOVAERTS and KAAS, 1991; KLING and GOOVAERTS, 1993; AMBAGASPITIYA and BALAKRISHNAN, 1994 etc.). A bivariate generalization was developed by VERNIC (1997) and was applied in the insurance field.

The multivariate generalization that we present in this paper is derived from the GPD in a similar way with the BGPD. In consequence, the BGPD can be obtained from the MGP_m for $m = 2$. In section 2 we present some properties of the MGP_m. The method of moments is used in section 3 for the estimation of the parameters. In section 4 the particular case of the BGPD is considered together with its application in the insurance field, based on the paper of VERNIC (1997) and illustrated with a numerical example. Since the BGPD is well fitted to the aggregate amount of claims for a compound class of policies submitted to claims of two kinds whose yearly frequencies are a priori dependent, it is natural to consider that the MGP_m is a good candidate for the aggregate amount of claims for a class of policies submitted to claims of m kinds.

2. The multivariate generalized Poisson distribution

2.1. Development of the distribution

If $N \sim GPD(\lambda, \theta)$, then its probability function (p.f.) is given by (Consul and Shoukri, 1985)

$$f(n) = P(N = n) = \begin{cases} \frac{1}{n!}\lambda(\lambda + n\theta)^{n-1}\exp\{-\lambda - n\theta\}, & n = 0, 1, \dots \\ 0, \quad \text{for } n > q \text{ when } \theta < 0 \end{cases}, \quad (2.1)$$

and zero otherwise, where $\lambda > 0$, $\max(-1, -\lambda/q) \leq \theta < 1$ and $q \geq 4$ is the largest positive integer for which $\lambda + \theta q > 0$ when $\theta < 0$.

Vernic (1997) used the trivariate reduction method to construct the BGPD in the following way: let N_i, $i = 1, 2, 3$, be independent generalized Poisson random variables (r.v.), $N_i \sim GPD(\lambda_i, \theta_i)$, $i = 1, 2, 3$, and let $X = N_1 + N_3$ and $Y = N_2 + N_3$. Then $(X, Y) \sim BGPD(\lambda_i, \theta_i;\ i = 1, 2, 3)$.

Similarly, we obtain the m-dimensional generalized Poisson distribution by taking $(m+1)$ independent generalized Poisson random variables, $N_i \sim GPD(\lambda_i, \theta_i)$, $i = 0, \dots, m$, and considering $X_1 = N_1 + N_0, \dots, X_m = N_m + N_0$. Then $(X_1, \dots, X_m) \sim MGP_m(\Lambda, \Theta)$, where $\Lambda = (\lambda_0, \lambda_1, \dots, \lambda_m)$ and $\Theta = (\theta_0, \theta_1, \dots, \theta_m)$. This method can be called **the multivariate reduction method**, as an extension of the trivariate reduction method.

It is easy to see that the joint p.f. of $(X_1, \dots, X_m)$ reads

$$p(x_1, \dots, x_m) = P(X_1 = x_1, \dots, X_m = x_m) = \\ = \sum_{k=0}^{\min\{x_1,\dots,x_m\}} f_1(x_1 - k) \cdot \dots \cdot f_m(x_m - k) f_0(k), \quad (2.2)$$

where f_i is the p.f. of the r.v. N_i.

Using (2.1) in (2.2) we get

$$p(x_1, \dots, x_m) = \left(\prod_{j=0}^{m} \lambda_j\right) \exp\left\{-\lambda - \sum_{j=1}^{m} x_j\theta_j\right\} \cdot \\ \cdot \sum_{k=0}^{\min\{x_1,\dots,x_m\}} \left(\prod_{j=1}^{m} \frac{[\lambda_j + (x_j - k)\theta_j]^{x_j-k-1}}{(x_j - k)!}\right) \cdot \\ \cdot \frac{(\lambda_0 + k\theta_0)^{k-1}}{k!} \exp\left\{k\left(\sum_{j=1}^{m} \theta_j - \theta_0\right)\right\}, \quad (2.3)$$

$$x_1, \dots, x_m = 0, 1, 2, \dots,$$

where $\lambda = \sum_{j=0}^{m} \lambda_j$ and $0! = 1$.

2.2. Properties of the distribution

We will first make some remarks on the GPD.

The GPD reduces to the Poisson distribution when $\theta = 0$ and it possesses the twin properties of over-dispersion and under-dispersion according as $\theta > 0$ or $\theta < 0$. When θ is negative, the GPD model includes a truncation due to the fact that $f(n) = 0$ for all $n > q$ (see 2.1). In the following, the moments expressions and the other formulas for the GPD are valid only for the case $\lambda > 0, 0 \leq \theta < 1$ and $q = \infty$, as discussed in SCOLLNIK (1998). This is a point frequently misrepresented in the literature.

In conclusion, we will assume for simplicity that $\theta > 0$. From AMBGASPITIYA and BALAKRISHNAN (1994) we have the following formulas for $N \sim GPD(\lambda, \theta)$:

– the probability generating function (p.g.f.)

$$\Pi_N(t) = \exp\left\{-\frac{\lambda}{\theta}[W(-\theta t \exp\{-\theta\}) + \theta]\right\}. \tag{2.4}$$

– the moment generating function (m.g.f.)

$$M_N(t) = \exp\left\{-\frac{\lambda}{\theta}[W(-\theta \exp\{-\theta + t\}) + \theta]\right\}, \tag{2.5}$$

where the Lambert W function is defined as $W(x)\exp\{W(x)\} = x$. For more details about this function see CORLESS et al. (1996).

– the first four central moments

$$\left\{\begin{array}{ll} E(N) = \mu_1 = \lambda M\,; & Var(N) = \mu_2 = \lambda M^3 \\ \mu_3 = \lambda(3M-2)M^4\,; & \mu_4 = 3\lambda^2 M^6 + \lambda(15M^2 - 20M + 6)M^5 \end{array}\right\}, \tag{2.6}$$

where $M = (1-\theta)^{-1}$.

The probability generating function of the MGP$_m$

Let now $\Pi_i(t)$ denote the p.g.f. of the r.v. N_i, $i = 0, \ldots, m$. Then the joint p.g.f. of $(X_1, \ldots, X_m)$ is

$$\begin{aligned}\Pi(t_1, \ldots, t_m) &= E\left(t_1^{X_1} \cdot \ldots \cdot t_m^{X_m}\right) = E\left(t_1^{N_1} \cdot \ldots \cdot t_m^{N_m}(t_1 \cdot \ldots \cdot t_m)^{N_0}\right) = \\ &= \Pi_1(t_1) \cdot \ldots \cdot \Pi_m(t_m)\Pi_0(t_1 \cdot \ldots \cdot t_m).\end{aligned} \tag{2.7}$$

Using (2.4) in (2.7) and assuming that $\theta_i > 0$, $i = 0, \ldots, m$, we have

$$\Pi(t_1, \ldots, t_m) = \exp\left\{-\sum_{i=1}^{m}\frac{\lambda_i}{\theta_i}W(-\theta_i t_i e^{-\theta_i}) - \frac{\lambda_0}{\theta_0}W\left(-\theta_0 e^{-\theta_0}\prod_{i=1}^{m} t_i\right) - \lambda\right\}.$$

The moment generating function of the MGP$_m$

If the m.g.f. of N_i is $M_i(t)$, $i = 0, ..., m$, then the m.g.f. of $(X_1, ..., X_m)$ is

$$M(t_1, ..., t_m) = E(\exp\{t_1X_1 + ... + t_mX_m\}) = E\left(e^{t_1N_1} \cdot ... \cdot e^{t_mN_m}e^{(t_1+...+t_m)N_0}\right)$$
$$= M_1(t_1) \cdot ... \cdot M_m(t_m)M_0(t_1 + ... + t_m). \qquad (2.8)$$

Using (2.5) in (2.8), the joint m.g.f. is given for $\theta_i > 0$, $i = 0, ..., m$, by

$$M(t_1, ..., t_m) =$$
$$\exp\left\{-\sum_{i=1}^{m}\frac{\lambda_i}{\theta_i}W(-\theta_i\exp\{-\theta_i + t_i\}) - \frac{\lambda_0}{\theta_0}W\left(-\theta_0\exp\left\{-\theta_0 + \sum_{i=1}^{m}t_i\right\}\right) - \lambda\right\}.$$

Moments

Let $\mu_{r_1,...,r_m} = E\left(\prod_{j=1}^{m}(X_j - EX_j)^{r_j}\right)$ be the $(r_1, ..., r_m)^{th}$ central moment of $(X_1, ..., X_m)$. The equation for $\mu_{r_1,...,r_m}$ given $\mu_k^{(j)}$ the k^{th} central moment of N_j, $j = 0, ..., m$, results as follows

$$\mu_{r_1,...,r_m} = E\left[\prod_{j=1}^{m}(N_j - EN_j + N_0 - EN_0)^{r_j}\right] =$$
$$= E\left[\prod_{j=1}^{m}\sum_{i_j=0}^{r_j}\binom{r_j}{i_j}(N_j - EN_j)^{i_j}(N_0 - EN_0)^{r_j-i_j}\right] =$$
$$= \sum_{(i_1,...,i_m)=(0,...,0)}^{(r_1,...,r_m)}\left(\prod_{j=1}^{m}\binom{r_j}{i_j}\mu_{i_j}^{(j)}\right)\mu^{(0)}_{\sum_{j=1}^{m}(r_j-i_j)}. \qquad (2.9)$$

From (2.6) and the independence of $N_j, j = 0, ..., m$, we also have for $\theta_i > 0$, $i = 0, ..., m$,

$$\begin{cases} EX_i = \lambda_iM_i + \lambda_0M_0 \\ Var(X_i) = \lambda_iM_i^3 + \lambda_0M_0^3 \end{cases}, \quad i = 1, ..., m, \qquad (2.10)$$

and from (2.9) we have, for example

$$\begin{cases} \mu_{110...0} = \mu_{0...010...010...0} = \mu_2^{(0)} = \lambda_0M_0^3 \\ \mu_{1110...0} = \mu_{0...010...010...010...0} = \mu_3^{(0)} = \lambda_0(3M_0 - 2)M_0^4 \\ \cdot \\ \cdot \\ \cdot \\ \mu_{11...1} = \mu_m^{(0)} \end{cases}. \qquad (2.11)$$

Marginal distributions
The marginal distributions are

$$P(X_i = r) = P(N_i + N_0 = r) = \lambda_0 \lambda_i \exp\{-(\lambda_0 + \lambda_i) - r\theta_0\} \cdot$$
$$\cdot \sum_{j=0}^{r} \frac{1}{j!(r-j)!} (\lambda_i + j\theta_i)^{j-1} (\lambda_0 + (r-j)\theta_0)^{r-j-1} \exp\{-j(\theta_i - \theta_0)\}\,, \quad i = 1, \ldots, m.$$

In particular, if $\theta_i = \theta_0 = \theta$, this reduces to $X_i \sim GPD(\lambda_i + \lambda_0, \theta)$. Elsewhere, X_i is not a GPD.

Remark. From the development of the MGP_m, it is easy to see that if $(X_1, \ldots, X_m) \sim MGP_m(\Lambda, \Theta)$, then for any $\{i_1, \ldots, i_k\} \subset \{1, \ldots, m\}$ with $2 \leq k < m$, $(X_{i_1}, \ldots, X_{i_k}) \sim MGP_k(\Lambda', \Theta')$, where $\Lambda' = (\lambda_0, \lambda_{i_1}, \ldots, \lambda_{i_k})$ and $\Theta' = (\theta_0, \theta_{i_1}, \ldots, \theta_{i_k})$.

For $k = 1$ the remark is not always true. But if we consider the particular case $\theta_0 = \theta_1 = \ldots = \theta_m = \theta$, then from $(X_1, \ldots, X_m) \sim MGP_m(\Lambda, \Theta)$ it follows that $X_i \sim GPD(\lambda_i + \lambda_0, \theta)$, $i = 1, \ldots, m$.

Recurrence relations
The marginal p.f. can be computed using the univariate generalized Poisson distribution, as it is seen from

$$p(0, \ldots, 0) = \exp\{-\lambda\}$$

$$p(0, \ldots, 0, \underset{j}{x_j}, 0, \ldots, 0) = f_j(x_j) \left(\prod_{\substack{i=1 \\ i \neq j}}^{m} f_i(0) \right) f_0(0) =$$
$$= f_j(x_j) \exp\{-(\lambda - \lambda_j)\}, \; j = 1, \ldots, m, \quad x_j > 0.$$

Given these probabilities, for $x_j > 0$, $j = 1, \ldots, m$, we have the following recurrence relation

$$p(x_1, \ldots, x_m) = \lambda_0 \exp\{(m-1)\lambda\} \sum_{k=0}^{\min\{x_1, \ldots, x_m\}} \left(\prod_{j=1}^{m} p(0, \ldots, 0, x_j - k, 0, \ldots, 0) \right) \cdot$$
$$\cdot \frac{(\lambda_0 + k\theta_0)^{k-1}}{k!} \exp\{-k\theta_0\}.$$

3. ESTIMATION OF THE PARAMETERS: METHOD OF MOMENTS

Let $(x_{1i}, \ldots, x_{mi})$, $i = 1, \ldots, n$ be a random sample of size n from the population. We will assume that the frequency of the m-tuple $(s_1, \ldots, s_m)$ is $n_{s_1,\ldots,s_m}$ for $s_1, \ldots, s_m = 0, 1, \ldots$. We recall that $\sum_{s_1,\ldots,s_m} n_{s_1,\ldots,s_m} = n$. Also

$$\begin{cases} n_{+\ldots+s_j+\ldots+} = \sum_{\{s_k | k=1,\ldots,m,\ k\neq j\}} n_{s_1,\ldots,s_m} \\ n_{+\ldots+s_i+\ldots+s_j+\ldots+} = \sum_{\{s_k | k=1,\ldots,m,\ k\neq j,\ k\neq i\}} n_{s_1,\ldots,s_m}, \quad i<j \end{cases}. \tag{3.1}$$

We denote

$$\begin{cases} \bar{x}_j = \frac{1}{n} \sum_{s_j} s_j n_{+\ldots+s_j+\ldots+} \\ \hat{\sigma}_j^2 = \frac{1}{n} \sum_{s_j} (s_j - \bar{x}_j)^2 n_{+\ldots+s_j+\ldots+} \end{cases}, \quad j = 1, \ldots, m, \tag{3.2}$$

and, with the notations in (3.1)

$$\begin{cases} \overline{x_i x_j} = \frac{1}{n} \sum_{s_i,s_j} s_i s_j n_{+\ldots+s_i+\ldots+s_j+\ldots+}, \quad i<j \\ \overline{x_i x_j x_k} = \frac{1}{n} \sum_{s_i,s_j,s_k} s_i s_j s_k n_{+\ldots+s_i+\ldots+s_j+\ldots+s_k+\ldots+}, \quad i<j<k \end{cases}.$$

It is easy to see that

$$\begin{cases} \mu_{0\ldots 0\underset{i}{1}0\ldots 0\underset{j}{1}0\ldots 0} = E(X_i X_j) - E(X_i)E(X_j), \quad i<j \\ \mu_{0\ldots 0\underset{i}{1}0\ldots 0\underset{j}{1}0\ldots 0\underset{k}{1}0\ldots 0} = E(X_i X_j X_k) - E(X_i X_j)E(X_k) - E(X_j X_k)E(X_i) - \\ \qquad -E(X_i X_k)E(X_j) + 2E(X_i)E(X_j)E(X_k), \quad i<j<k \end{cases},$$

so we can use the sample moments

$$\begin{cases} \hat{\mu}_{0\ldots 0\underset{i}{1}0\ldots 0\underset{j}{1}0\ldots 0} = \overline{x_i x_j} - \bar{x}_i \bar{x}_j, \quad i<j \\ \hat{\mu}_{0\ldots 0\underset{i}{1}0\ldots 0\underset{j}{1}0\ldots 0\underset{k}{1}0\ldots 0} = \overline{x_i x_j x_k} - \overline{x_i x_j}\,\bar{x}_k - \overline{x_i x_k}\,\bar{x}_j - \overline{x_j x_k}\,\bar{x}_i + \\ \qquad +2\bar{x}_i \bar{x}_j \bar{x}_k, \quad i<j<k \end{cases} \tag{3.3}$$

The general method

The classical method of moments consists of equating the sample moments to their populations equivalents, expressed in terms of the parameters. The number of moments required is equal to the number of parameters which equals $2(m+1)$. For example, using (2.10), (2.11), (3.2) and (3.3), we can choose the following $2(m+1)$ equations

$$\begin{cases} \bar{x}_j = \lambda_j M_j + \lambda_0 M_0 \\ \hat{\sigma}_j^2 = \lambda_j M_j^3 + \lambda_0 M_0^3 \\ \hat{\mu}_{110\ldots0} = \lambda_0 M_0^3 \\ \hat{\mu}_{1110\ldots0} = \lambda_0(3M_0 - 2)M_0^4 \end{cases}, \quad j = 1, \ldots, m.$$

Denoting $a = \dfrac{\hat{\mu}_{1110\ldots0}}{\hat{\mu}_{110\ldots0}}$, the solution of the system is

$$\begin{cases} M_0 = \dfrac{1 + \sqrt{1 + 3a}}{3} \\ \lambda_0 = \dfrac{\hat{\mu}_{110\ldots0}}{M_0^3} \\ M_j = \sqrt{\dfrac{\hat{\sigma}_j^2 - \hat{\mu}_{110\ldots0}}{\bar{x}_j - \lambda_0 M_0}} \\ \lambda_j = \dfrac{\bar{x}_j - \lambda_0 M_0}{M_j} \end{cases}, \quad j = 1, \ldots, m. \tag{3.4}$$

We used the fact that $\theta < 1$, so $M = \frac{1}{1-\theta} > 0$.

Particular case: $\theta_0 = \theta_1 = \ldots = \theta_m = \theta$, so $M_0 = M_1 = \ldots = M_m = M$.

Method I. The number of parameters is now $(m+2)$: $\lambda_0, \ldots, \lambda_m$ and M, so we can use the following equations:

$$\begin{cases} \bar{x}_j = (\lambda_j + \lambda_0)M \\ \hat{\mu}_{110\ldots0} = \lambda_0 M^3 \\ \hat{\mu}_{1110\ldots0} = \lambda_0(3M - 2)M^4 \end{cases} \text{, with the solution } \begin{cases} M = \dfrac{1 + \sqrt{1+3a}}{3} \\ \lambda_0 = \dfrac{\hat{\mu}_{110\ldots0}}{M^3} \\ \lambda_j = \dfrac{\bar{x}_j}{M} - \lambda_0 \end{cases}, \quad j = 1, \ldots, m.$$

Method II. Another possibility is to use the method of moments in combination with the zero cell frequency method. If we denote by $f_{0\ldots0} = \dfrac{n_{0\ldots0}}{n}$ the frequency of the cell (0, ..., 0), we can consider the system

$$\begin{cases} I. \quad f_{0\ldots0} = \exp\{-(\lambda_0 + \ldots + \lambda_m)\} \\ II. \quad \bar{x}_j = (\lambda_j + \lambda_0)M \\ III. \quad \hat{\sigma}_j^2 = (\lambda_j + \lambda_0)M^3 \end{cases}, \quad j = 1, \ldots, m.$$

We have here $(2m+1)$ equations. By summing equations I and II separately, we get

$$\begin{cases} IV.\ \sum_{j=1}^{m} \bar{x}_j = \left(\sum_{j=1}^{m} \lambda_j + m\lambda_0\right) M \\ V.\ \sum_{j=1}^{m} \hat{\sigma}_j^2 = \left(\sum_{j=1}^{m} \lambda_j + m\lambda_0\right) M^3 \end{cases}, \quad j = 1, \ldots, m.$$

Dividing the two relations gives $M^2 = \left(\sum_{j=1}^{m} \hat{\sigma}_j^2\right)\left(\sum_{j=1}^{m} \bar{x}_j\right)^{-1}$, hence the solution

$$M = \sqrt{\left(\sum_{j=1}^{m} \hat{\sigma}_j^2\right)\left(\sum_{j=1}^{m} \bar{x}_j\right)^{-1}} . \tag{3.5}$$

From equation I we have

$$-\ln f_{0\ldots0} = \lambda_0 + \sum_{j=1}^{m} \lambda_j ,$$

and using equation IV we are lead to

$$-\ln f_{0\ldots0} = \lambda_0 + \frac{1}{M}\sum_{j=1}^{m} \bar{x}_j - m\lambda_0 ,$$

so that

$$\lambda_0 = \frac{1}{m-1}\left(\frac{1}{M}\sum_{j=1}^{m} \bar{x}_j + \ln f_{0\ldots0}\right) . \tag{3.6}$$

Then, from equation II we have

$$\lambda_j = \frac{1}{M}\bar{x}_j - \lambda_0 ,\ j = 1, \ldots, m. \tag{3.7}$$

Finally, the solution $(M, \lambda_0, \lambda_j, j = 1, \ldots, m)$ is given by (3.5), (3.6) and (3.7).

Remark. In method II, the estimation of M is based on the empirical moments from all m variables, while in method I only three variables are taken into consideration by $\hat{\mu}_{1110\ldots0}$.

4. Particular case: bivariate generalized Poisson distribution (BGPD)

Considering $m = 2$, the multivariate generalized Poisson distribution reduces to the bivariate generalized Poisson distribution. The BGPD was introduced by VERNIC (1997) and was applied in the insurance field. The distribution was fitted to the aggregate amount of claims for a compound class of policies submitted to claims of two kinds whose yearly frequencies are a priori dependent. A comparative study with the classical bivariate Poisson distribution and with two bivariate mixed Poisson distributions has been carried out, based on two sets of data concerning natural events insurance in the U.S.A. and third party liability automobile insurance in France. The conclusion, after applying the χ^2 goodness-of-fit test, is that the BGPD fits better to the data, so it can be considered as a valid alternative to the usual bivariate Poisson or mixed Poisson distributions. For more details see VERNIC (1997).

In the following, we will consider another example, based on the accident data of CRESSWELL and FROGATT (1963), with X_1 as the accidents in the first period and X_2 as the accidents in the second period. The data are given in table 1, first row in each cell.

The summary statistics for these data are:

$$\bar{x}_1 = 1.0014, \quad \bar{x}_2 = 1.291, \quad \hat{\sigma}_1^2 = 1.1935, \quad \hat{\sigma}_2^2 = 1.5961,$$
$$\hat{\mu}_{11} = 0.3258, \quad \hat{\mu}_{21} = 0.365.$$

Under the hypothesis $(X_1, X_2) \sim BGPD(\lambda_0, \lambda_1, \lambda_2; \theta_0, \theta_1, \theta_2)$, we have from (3.4)

$$\left\{ \begin{array}{lll} \theta_0 = 0.0286, & \theta_1 = 0.1057, & \theta_2 = 0.1200 \\ \lambda_0 = 0.2987, & \lambda_1 = 0.6206, & \lambda_2 = 0.8653 \end{array} \right\}.$$

The theoretical frequencies in this case are given in table 1, second row in each cell. After grouping in 32 categories: $(i,j)_{i=0..4;\, j=0..5}$; (0..4, 6 and above); (5 and above, 0 and above), we obtain $\chi^2_{obs} = \sum (obs - th)^2/th = 25.935$ and a significance level (P-value) verifying $0.45 \leq \hat{\alpha} \leq 0.75$. So the distribution is adequate.

We will now consider the particular case $\theta_0 = \theta_1 = \theta_2 = \theta$, so that we have the hypothesis $(X_1, X_2) \sim BGPD(\lambda_0, \lambda_1, \lambda_2; \theta)$. From (3.5), (3.6) and (3.7) we have $\theta = 0.0935$, $\lambda_0 = 0.2778$, $\lambda_1 = 0.63$, $\lambda_2 = 0.8925$, and the theoretical frequencies are given in table 1, last row in each cell. For the same categories we have $\chi^2_{obs} = 23.6082$ and $0.7 \leq \hat{\alpha} \leq 0.85$, so this particular distribution fits even better than the general one.

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TABLE 1

COMPARISON OF OBSERVED AND THEORETICAL FREQUENCIES

X_2 / X_1	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	Σ
0	117	96	55	19	2	2	0	0	291
	118.843	91.204	44.710	17.959	6.460	2.171	0.697	0.217	282.261
	117	95.100	46.748	18.088	6.081	1.865	0.537	0.148	285.567
1	61	69	47	27	8	5	1	0	218
	66.356	85.419	51.437	23.005	8.820	3.087	1.019	0.324	239.467
	67.132	84.165	50.881	22.205	8.065	2.608	0.780	0.220	236.056
2	34	42	31	13	7	2	3	0	132
	24.834	38.319	30.090	15.577	6.505	2.402	0.822	0.267	118.816
	24.976	37.584	30.048	15.739	6.427	2.249	0.711	0.209	117.943
3	7	15	16	7	3	1	0	0	49
	7.871	13.249	12.124	7.260	3.386	1.341	0.480	0.161	45.872
	7.694	12.602	12.004	7.911	3.849	1.516	0.520	0.162	46.258
4	3	3	1	1	2	1	1	1	13
	2.287	4.040	3.860	2.610	1.676	0.616	0.226	0.079	15.394
	2.138	3.685	3.774	2.927	1.799	0.844	0.327	0.111	15.605
5	2	1	0	0	0	0	0	0	3
	0.632	1.149	1.142	0.816	0.464	0.220	0.090	0.033	4.546
	0.558	0.995	1.075	0.910	0.647	0.382	0.176	0.068	4.811
6	0	0	0	0	1	0	0	0	1
	0.169	0.313	0.319	0.236	0.140	0.071	0.031	0.012	1.291
	0.140	0.255	0.285	0.255	0.198	0.136	0.079	0.036	1.384
7	0	0	0	1	0	0	0	0	1
	0.044	0.083	0.086	0.065	0.040	0.021	0.010	0.004	0.353
	0.034	0.063	0.072	0.067	0.055	0.041	0.028	0.016	0.376
	224	226	150	68	23	11	5	1	
Σ	221.036	233.776	143.768	67.528	27.491	9.929	3.375	1.097	708
	219.672	234.449	144.887	68.102	27.121	9.641	3.158	0.970	

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