A REMARK ON WIENER PROCESS APPROXIMATION OF RISK PROCESSES

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In Bohman the following model is considered. Our notation follows Bohman.

Let $Z_1, Z_2, \ldots$ be a sequence of independent random variables with distribution function F and put

$$m = EZ$$
$$\sigma^2 = \operatorname{Var} Z$$

Put

$$S_n = \sum_{k=1}^{n} Z_k$$

and define X by

$$X = \inf\{n;\, S_n > U,\, S_k \leq U \text{ for } k = 1, \ldots, n-1\}.$$

Bohman shows that if $U \to \infty$ in such a way that $U/\sigma \to \infty$ and $\lim\limits_{U\to\infty} \frac{mU}{\sigma^2} = \alpha$ then

$$\lim_{U\to\infty} P\left(\frac{X\sigma^2}{U^2} \leq x\right) = G(\alpha, x)$$

where $G(\alpha, x)$ is the distribution function for the time when a Wiener process $X(t)$ with $EX(t) = \alpha t$ and $\operatorname{Var} X(t) = t$ first crosses the level 1.

Let N be an integer, which in a certain sense corresponds to "time", and consider $P(X \leq N)$. This is thus the probability of ruin before the N:*th* claim.

Consider the process $\left\{\frac{S_n}{U};\, n = 1, \ldots, N\right\}$. Obviously $\{X \leq N\} = \left\{\sup\limits_{n<N} \frac{S_n}{U} > 1\right\}$. A necessary (and in fact also sufficient) condition for making the approximation with the Wiener process meaningful is that $\frac{S_N}{U}$ tends in distribution towards a normal variable when U and N tend to infinity. We have

$$\frac{S_N}{U} = \frac{\sqrt{N\sigma^2}}{U} \cdot \frac{S_N - Nm}{\sqrt{N\sigma^2}} + \frac{Nm}{U}.$$

Thus $\frac{N\sigma^2}{U^2}$ and $\frac{Nm}{U}$ have to tend towards constants.

To avoid a trivial case $\lim_{N\to\infty} \frac{N\sigma^2}{U^2}$ is assumed to be strictly positive.

Thus $\frac{U}{\sigma} \to \infty$ and $\frac{mU}{\sigma^2} = \frac{Nm}{U} \cdot \frac{U^2}{N\sigma^2} \to$ constant.

Thus we have shown that the assumptions due to Bohman are not only sufficient but also necessary. This may serve as a warning for the use of a Wiener process approximation without great care. The most important reason for this warning seems to be that m ($-m$ corresponds to the safety loading) has to be of the same magnitude as $1/U$ if σ^2 tends to a positive constant. Since U has to be large this implies that the approximation is meaningful only if the safety loading is very small. Finally it may be noted that "time" has been measured as the actual number of claims and not as the expected number of claims. In the model studied here it is, however, natural to assume that the claims follow a renewal process. For such a process it is easy to show that the limit results are independent of the measurement of "time" since $\frac{N}{T} \to 1$ in probability when $T \to \infty$ if N is the number of claims up to time T.

Reference

Bohman, H. (1972). Risk theory and Wiener processes.