INTEGRATION OF THE NORMAL POWER APPROXIMATION

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1. Consider the set of functions

$$\pi_j(x) = \int_x^\infty (t - x)^j \, dF(t), \; j = 0, 1, \ldots \tag{1}$$

Obviously, $\pi_1(x)$ represents the net premium of the excess cover over the priority x, and $\sigma^2(x) = \pi_2(x) - \pi_1^2(x)$ the variance thereof.

If a distribution function $F(x) = 1 - \pi_0(x)$ is given, the set (1) can be generated by means of the recursion formulae

$$\pi_j'(x) = -j \, \pi_{j-1}(x), \; j = 1, 2, \ldots \tag{2}$$

2. Let us study the special class of d.fs. $F(x)$ which satisfy

$$F(x) = \Phi(y) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{1}{2}t^2} \, dt, \tag{3a}$$

where

$$x = \Delta(y) \equiv \beta_0 + \beta_1 y + \ldots + \beta_k y^k. \tag{3b}$$

If these conditions are met, the integrals (1) have the solution:

$$\pi_j(x) = A_j(y) \cdot (1 - \Phi(y)) + B_j(y) \cdot \Phi'(y). \tag{4}$$

$A_j(y)$ and $B_j(y)$, respectively, are polynomials of rank jk and $jk - 1$. Their coefficients are determined by the equations:

$$\left.\begin{aligned} A_j'(y) &= -j \, \Delta'(y) \cdot A_{j-1}(y), \\ B_j'(y) &= -j \, \Delta'(y) \cdot B_{j-1}(y) + A_j(y) + yB_j(y), \\ A_0(y) &= 1, \\ B_0(y) &= 0. \end{aligned}\right\} \tag{5}$$

The system (5) is obtained by differentiation of (4) with respect to y, and observing (2).

3. The idea behind the normal power expansion is to apply (3a) as approximation, subject to a transformation $x = \Delta(y)$. Preferably the parameters of $\Delta(y)$ should not depend on the particular choice of y or x, but only on general characteristics of the d.f. $F(x)$, such as $E = \pi_1(0)$, $\sigma = \sigma(0)$, γ_1 = skewness and γ_2 = excess.

Kauppi and Ojantakanen [1] have tackled the problem to define functions $x = \Delta(y)$, which make (3a) a reasonable approximation. They found three suitable expressions $\Delta(y)$, one of them—credited to Loimaranta—has the form (3b) and this one became known as the normal power expansion. Under this method (see Beard-Pentikaeinen-Pesonen [2]) the coefficients β_i of (3b) are determined by reversion of the Edgeworth expansion as follows:

$$\frac{x - E}{\sigma} = y + \frac{\gamma_1}{6}(y^2 - 1) + \frac{\gamma_2}{24}(y^3 - 3y) - \left(\frac{\gamma_1}{6}\right)^2 (2y^3 - 5y) + \dots \tag{6}$$

We may denote by NPk the normal power approximation, which uses the first k terms of (6). Then, NP1 corresponds to the well known normal approximation.

NP2 uses the first line of (6) only, and NP3 everything which is written out. Thus, NP3 requires the solution of a cubic equation.

4. The methods NP2 and NP3 were programmed in APL. This required about 20 lines, including the subprograms to solve (5), the quadratic or cubic equation (6), and to determine $\Phi(y)$ and $\Phi'(y)$.

The cubic equation for NP3 has in some relevant cases 3 real roots. It is necessary therefore to program rules to select the meaningful of several real roots y.

On an IBM 370, the CPU time needed to calculate $\pi_0(x)$, $\pi_1(x)$ and $\sigma(x)$ for a set of 6 values x was 1 second for NP2, and 2.4 seconds for NP3.

5. The NP approximations were applied first to a life insurance distribution similar to the one used by Ammeter [3]. The result is

shown in *Table 1*. The exact values were obtained by another APL program, the CPU time needed was:

28 seconds for $t = 100$
3.2 seconds for $t = 10$
1 seconds for $t = 1$

Thus, the approximation technique makes economical sense only, if the number t of expected claims is at least 10 or more.

As another example, the non-industrial fire distribution from the work of Bohman-Escher [4] was chosen. *Table 2* shows a comparison with correct values from [4], *Table 3* some additional comparisons with numerical results from Seal [5].

6. The comparisons contained in the Tables 1 to 3 point out the following suggestions:

a) The integration does not seem to enlarge the error margin. Thus, the NP technique can be applied to estimate stop loss gross premiums.

b) NP2 yields quite reasonable results, if $\gamma_1 \leq 2$. This corresponds with previous experience.

c) NP3 does not generally produce better results than NP2. It appears that NP3 is preferable only for lower values of x (say $x \leq E + 2\sigma$).

d) NP3 yields reasonable results even in the Life case with $\gamma_1 = 4.3$, but not in the Fire cases with $\gamma_1 = 3.5$ and 3.8 (not even in the vicinity of $x = E$). It may be that not only γ_1, but also the relation $E\gamma_1/\sigma$ is a criterion of goodness of fit.

References

[1] KAUPPI, OJANTAKANEN (1969): "Approximations of the generalized Poisson function"; Astin Bulletin.
[2] BEARD. PENTIKAEINEN, PESONEN (1969): "Risk Theory"; Methuen, London.
[3] Ammeter (1955). "The calculation of premium rates for excess of loss and stop loss reassurance treaties"; Arithbel, Brussels.
[4] BOHMAN, ESCHER (1964): "Studies in Risk Theory..."; Skand. Aktu. Tidskr.
[5] SEAL (1971): "Numerical calculation of the Bohman-Escher family convolution-mixed negative binomial distribution functions"; MVSM.

Table 1

Life insurance distribution

			$\pi = \pi_1(x) / E$					$\sigma = \sqrt{\pi_2(x) - \pi_1^2(x)} / E$				
t	h_0	x/E	Exact	NP2	NP3	NP2 %	NP3 %	Exact	NP2	NP3	NP2 %	NP3 %
100	∞	1.0	0.06968	0.07005	0.06969	100.5	100.0	0.11056	0.11103	0.11052	100.4	100.0
		1.1	3316	3344	3314	100.8	↓	7769	7814	7765	100.6	100.0
		1.2	1385	1402	1384	101.2	↓	4972	5012	4970	100.8	100.0
		1.3	511	520	511	101.8	↓	2944	2976	2944	101.1	100.0
		1.4	168	172	168	102.4	↓	1635	1659	1637	101.5	100.1
		1.5	50	51	50	102	↓	860	877	863	102.0	100.3
10	∞	1.0	0.21136	0.22636	0.21520	107.1	101.8	0.40408	0.41512	0.39934	102.7	98.8
		1.1	17530	18796	17735	107.2	101.2	37239	38328	36766	102.9	98.7
		1.2	14516	15530	14545	107.0	100.2	34116	35196	33658	103.2	98.7
		1.3	11991	12773	11875	106.5	99.0	31090	32165	30660	103.5	98.6
		1.4	9861	10460	9653	106.0	97.8	28198	29269	27808	104.0	98.6
		1.5	8073	8531	7816	105.7	96.8	25465	26533	25125	104.2	98.7
1	∞	1.0	0.49726	0.83629	0.53941	168	110	1.54529	2.00249	1.53883	130	99.6
		1.2	45075	77818	49481	173	110	49747	1.94449	48574	130	99.2
		1.4	40424	72427	45430	179	112	45299	88724	43394	130	98.7
		1.6	36609	67422	41744	184	114	40964	83088	38351	130	98.1
		1.8	33251	62773	38383	189	115	36778	77549	33449	130	97.6
		2.0	30119	58454	35316	194	117	32827	72115	28692	130	96.9

t	h	γ_1	CPU — time in seconds		
100	∞	.43	28	1.0	2.4
10	∞	1.35	3.2	1.0	2.4
1	∞	4.27	1.0	1.0	2.4

Table 2

Non-industrial fire distribution

$x = E + \xi . \sigma$			$\pi_0(x) = 1 - F(x)$					$\pi = \pi_1(x) / E$				
t	h_0	ξ	Exact (Bohman-Escher)	NP2	NP2	NP3 %	NP3 %	Exact (Bohman-Escher)	NP2	NP3	NP2 %	NP3 %
1000	∞	0	0.4265	0.4228	0.4131	99	97	0.0823	0.0888	0.0830	108	101
		1	1364	1587	1425	116	104	260	289	269	111	103
		2	04523	04938	4497	109	99	815	817	835	100	102
		3	01401	01348	1387	96	99	222	209	258	94	116
		4	00352	00333	428	95	121	55	49	81	89	147
		6	000219	00164	422	75	193	31	22	81	71	370
	20	0	0.4476	0.4472	0.4444	100	99	0.1220	0.1257	0.1221	103	100
		1	1502	1587	1509	106	100	345	362	345	105	100
		2	03968	04179	400	105	100	823	831	845	101	102
		3	00892	00881	920	99	103	171	159	185	93	108
		4	00177	00157	195	89	110	32	26	38	81	119
		6	000053	000034	78	64	147	9	000005	15	60	167
100	∞	0	0.3743	0.3129	0.1641	84	44	0.2191	0.3206	0.2054	146	94
		1	947	1587	827	168	87	800	1643	1251	205	156
		2	3450	8152	4827	236	140	4024	8438	8129	210	202
		3	1709	4195	3016	245	176	2358	4329	5484	184	233
		4	893	2156	1967	241	220	1483	2216	3796	149	256
		6	3780	565	908	149	240	6826	576	1920	84	281
	20	0	0.3801	0.3226	0.1795	85	47	0.2364	0.3302	0.2191	140	93
		1	1006	1587	0827	158	82	845	1629	1289	193	153
		2	3521	7856	488	223	139	4070	8027	816	197	200
		3	1680	3880	298	231	177	2311	3939	538	170	233
		4	855	1907	1897	223	222	1431	1924	364	134	254
		6	3649	454	843	124	231	6296	4529	178	72	283

Table 3

Non-industrial fire distribution

$x = E + \xi \cdot \sigma$				$\pi_0(x) = 1 - F(x)$				
t	h_0	x/E	ξ	Exact (Seal)	NP2	NP3	NP2 %	NP3 %
1000	1	.5		0.59778	0.5645	0.5825	94	97
		1.0	0	36710	3805	3593	104	98
			1	13531	1587	1347	117	100
			3	1839	229	194	124	106
			5	250	28	29	113	117
100	1	.5		0.5470	0.4905	0.4846	90	89
		1.0	0	3448	3540	3040	103	88
			1	1226	1587	1189	129	97
			3	198	297	238	150	120
			5	46	51	56	111	122

The total claim distributions being tested have these statistical measures:

t	h_0	σ/E	γ_1	γ_2
Life:				
100	∞	.175	.427	.246
10	∞	.554	1.351	2.459
1	∞	1.751	4.271	24.590
Non-industrial Fire:				
1000	∞	.218	1.214	2.624
	20	.312	.811	1.153
	1	1.024	2.010	6.045
100	∞	.690	3.839	26.234
	20	.725	3.505	22.577
	1	1.215	2.614	10.729