MATHEMATICAL MODELS IN INSURANCE

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1. INTRODUCTION

1.1. This paper contains little which can be considered as new. It gives a survey of results which have been presented over the last 10-15 years. At one time these results seemed very promising, but in retrospect it is doubtful if they have fulfilled the expectations they raised. In this situation it may be useful to retrace one's steps and see if problems can be reformulated or if new approaches can be found.

1.2. Mathematical models have been used in insurance for a long time. One of the first was the Gompertz mortality law; a more recent model, which has been intensively studied is the Compound Poisson Distribution in Lundberg's risk theory.

When a model is introduced, one usually proceeds by stages. The first step is to see if the model appears acceptable on *a priori* reasons. If it does, the second step is to examine the implications of the model, to see if any of these are in obvious contradiction with observations. If the result of this examination is satisfactory, the third step is usually a statistical analysis to find out how well the model approximates the situation in real life, which one wants to analyse. If the model passes this second examination, the next and final step may be to estimate the parameters of the model, and use it in practice, i.e. to make decisions in the real world.

The advantage of working with a model is that it gives an overall purpose to the collection and analysis of data. A good model should tell us which data we need, and why.

1.3. A general model for decision making in insurance companies must necessarily be complicated, and it cannot be built in one day. We have to approach the goal gradually, proceeding from simple to slightly more complicated models. In this process we will, sooner or later, reach a stage when the implications of the model cannot be studied by reasonably simple analysis of neat closed expressions.

This means that we have got stuck at the second step, referred to in the preceding paragraph, and it makes little sense to proceed to the next step and test the model by proper statistical methods.

At this stage there will usually be two ways out

(i) We can retire into abstract mathematics and seek non-constructive existence proofs.

(ii) We can hand the problem over to the computer, and simulate.

Both ways are likely to be long, and expensive, in mental effort or in computer time. It may, therefore, be desirable to pause and think before making the choice. This paper presents some of my own reflections, before making the decision.

2. A Static Model

2.1. In the simplest possible model the situation of an insurance company can be described by two elements: The reserves R, and the claim distribution $F(x)$ of the company's portfolio of insurance contracts. Here $F(x)$ is the probability that claim payments under the contracts in the portfolio shall not exceed x.

The management of the company may be able to change a given situation—for instance by making a reinsurance arrangement. If the new situation is described by the elements R_k and $F_k(x)$, where k belongs to some set K, the problem is to determine the best available pair $(R_k, F_k(x))$. If the company's management has a consistent preference ordering over the set of all situations, the problem can be formulated as follows

$$\max_{k \varepsilon K} \int_0^\infty u(R_k - x)\, dF_k(x). \tag{1}$$

Here the "utility function" $u(x)$ represents the preference ordering, or the company's "attitude to risk".

As an illustration we can write $R = S + P$, where S stands for the company's "initial reserves", and P is the total amount of premiums which the company received by accepting liabilities for claims under the contracts in the portfolio. If only proportional reinsurance on original terms is available, the problem is to select the best, or most preferred, element in the set $(S + kP, F((1/k)\, x))$ where $k \varepsilon (0, 1)$.

2.2. The formula (1) illustrates how the so-called "Expected Utility Theorem" can be used to formulate decision problems in insurance in an operational manner. The class of models based on this theorem is very versatile. As another illustration we can consider an insurance company offering only one kind of insurance contracts, defined by the premium P and the claim distribution $F(x)$. Assume that the company can sell $n = n(s)$ such contracts, if it spends an amount s for sales promotion—for instance on advertising, or to provide incentives for the salesmen. If claims under different contracts are stochastically independent, the expenditure of s will give the company a portfolio with the claim distribution $F^{(n)}(x)$, i.e. the n-th convolution of $F(x)$ with itself. Hence the situation of the company can be described by the pair $(S + nP - s, F^{(n)}(x))$, and the problem is to determine the value of s, which leads to the best attainable situation. With the Expected Utility Theorem the problem can be formulated as an optimizing problem.

2.3. The two models we have sketched are completely static, and they cannot give a realistic representation of the decision problems which an insurance company has to solve in practice. The models do, however, in spite of their obvious oversimplification, seem to capture some of the essential elements of the situations in real life which we want to study. We shall just indicate two aspects which clearly will carry over in more complicated, and more realistic models.

Let us first note that the models show that a certain division of labour is natural

(i) The utility function $u(x)$ represents the company's attitude to risk, or more simply—its "policy". It will presumably be up to the top management to specify this function.

(ii) The claim distribution $F(x)$ is traditionally determined by the actuary.

(iii) The function $n(s)$ gives the market's response to an expenditure on sales promotion. It will usually be the task of a specialist on market analysis to determine this function.

On the other hand it is clear that the three tasks should not be completely separated, and be carried out in water-tight compartments. In practice the functions $F(x)$ and $n(s)$ must be estimated, more or less accurately, from statistical observations. We may then seek estimating methods which are "robust" in the sense that they will give good decisions for a wide class of utility functi/ns. It may, however, be more efficient to look for methods which give good estimates in the intervals which are important when a particular utility function is applied. This means in essence that statisticians in the actuarial and marketing departments of the company can do a better job if they know the general objectives of their top management.

2.4. A second, and more important aspect of the static model is that it gives some insight in the equilibrium of an insurance market. To illustrate this, we shall assume that there are n companies in the market.

Let the policy of company i be represented by the utility function $u_i(x)$, and let $F_i(x_i)$ be the claim distribution of its portfolio. $i = 1, 2, \ldots, n$.

The stochastic variable $z = x_1 + x_2 + \ldots + x_n$ represents the total amount of claims paid by all companies in the market. The most general reinsurance arrangement which these companies can make is defined by n functions $y_i(z)$ = the amount paid by company i if total claims are z. We must clearly have

$$y_1(z) + y_2(z) + \ldots y_n(z) = z. \tag{2}$$

If the n companies act rationally, they should reach an arrangement which is *Pareto optimal*, i.e. the arrangement must be such that no other arrangement will give *all* companies a higher utility. It has been proved in another paper [1] that the set of Pareto optimal arrangements is defined by the y-functions which satisfy (2) and the equations

$$u_i'(y_i(z)) = k_i u_1'(y_1(z)) \qquad i = 2, 3, \ldots n \tag{3}$$

where $k_2, k_3 \ldots k_n$ are arbitrary positive constants.

It is easy to see that a Pareto optimal arrangement can be reached through proportional reinsurance only if the functions defined by (2) and (3) are linear, i.e. if we have $y_i(z) = a_i z + b_i$ for all i. It can be proved [2] that the y-functions are linear if and only if all utility functions belong to one of the following three classes:

(i) $u_i(x) = k\, e^{a_i x}$

(ii) $u_i(x) = (x - c_i)^{\beta}$

(iii) $u_i(x) = \log (x - c_i)$.

If this should be the case, all companies have virtually the same attitude to risk. The functions in class (i) differ only by a scale factor, and those in the classes (ii) and (iii) allow only differences in attitude to risk which can be explained by differences in initial reserves.

2.5. The result in the preceding paragraph has some significance. To bring this out, let us first recall that proportional reinsurance is older than non-proportional. We must assume that non-proportional reinsurance was developed because all companies which participated in such arrangements found them more advantageous than the older proportional contracts. In other words, the introduction of non-proportional reinsurance made it possible to reach a general arrangement closer to Pareto optimality. This means, however, that the objectives or attitudes to risk of all companies are not so similar that they can be represented by utility functions belonging to one of the three classes.

One may argue that this conclusion is obvious, trivial, useless, or far too sweeping, as one's taste may be. The only point we have tried to prove is that the study of extremely simple models may give relatively deep insight into complicated situations.

3. Some Dynamic Models

3.1. In our discussion of the static model we assumed that the top management was able to spell out the company's objectives in an operational manner, i.e. so that they can be represented by a utility function. There are techniques, and even computer programs which can help management with this problem, and in literature on operational research the assumption is usually made without discussion. We shall approach this problem in a more old-fashioned way. Returning to the model of para 2.1, we shall write S_0 for the "initial reserve", and consider the "final reserve" represented by the stochastic variable $S_1 = S_0 + kP - kx$. The optimal value of k is the value which maximizes the expected utility of S_1.

The utility of "final reserves", or of "final wealth", to use the current term in the theory of finance, must in some way depend on the use one can make of this wealth. It is natural to assume that an insurance company primarily will look at the final reserve of one operating period as the initial reserve to be used in the following period. Thus we are led to study dynamic models in order to determine the utility function to be used in a static problem.

3.2. The considerations in the preceding paragraph lead us to write

$$S_{t+1} = S_t + k_t P_t - k_t x_t. \qquad t = 0, 1, 2 \ldots \tag{4}$$

An equation of this form is usually the starting point of stochastic control theory. In this theory the problem is to find a rule, or a "policy" for selecting the control variables k_t, which will give the most desirable of the attainable stochastic processes S_t. There is a large literature on such problems. A good and up to date survey of economic interpretations of the theory is given in a book by Burmeister and Dobell [4], which also contains a very good bibliography.

The first difficulty in control theory is to lay down a rule as to when one stochastic process shall be considered better than another. To make the rule operational, it must be formulated so that the problem consists in maximizing some "criterion" function. The most popular criteria seem to have been studied, not because they are realistic, but because they lead to mathematical problems which can be handled by familiar methods. A few examples will illustrate this.

(i) One can fix a horizon T, and seek a rule for selecting k_t, which will maximize the expectated value $E\{S_T\}$

(ii) One can fix a target $\bar{S}$, and seek the policy which minimizes the expected time required to reach the target.

(iii) One can seek the policy which minimizes some probability of ruin—or of termination, i.e. $Pr\ (S_t \leq 0)\ t \varepsilon T$.

The last of these criteria should be familiar to actuaries. It is, however, worth noting that it has found many applications outside the field of insurance.

3.3. None of the three criteria mentioned seem attractive in economic applications. In such applications S_t is usually interpreted as capital stock at the end of period t. Growth of capital can not—or should not—be considered as a goal in itself. Usually one is interested in the amounts which can be withdrawn and made available for consumption. The optimal growth path is generally defined as the path which maximizes the utility of the goods which are taken out of the production process and consumed.

When these ideas are applied to a company, they should lead us to consider the dividends which the company is able to pay. This brings us to a paper by De Finetti [5], which can be considered as the pioneering work in the contemporary actuarial theory of risk.

To present the main ideas in this paper, we introduce s_t = the dividend paid by the company at the end of period t. The equation (4) then takes the form

$$S_{t+1} = S_t - s_t + k_t P_t - k_t x_t \tag{5}$$

To complete the dynamic model, we must make assumptions about the future underwriting of the company. The simplest is obviously to assume that things do not change, i.e. that the company in each period receives an amount of premium P, and underwrites a portfolio with the claim distribution $F(x)$. The equation (5) can then be written

$$S_{t+1} = S_t - s_t + k_t P - k_t x \tag{6}$$

3.4. Equation (6) gives us a model which is almost operational. At the end of operating period t the company decides on the amount s_t which shall be paid as dividend, and on the quota of the portfolio k_t, which will be retained in the next operating period. The optimal decisions are those which maximize the utility of the dividend payments $s_0, s_1, s_2 \ldots$.

If the horizon is infinite, a fairly simple argument will show that the optimal decision at the end of period t will depend only on the "state" of the company at that time, i.e. on S_t alone, and not on the calender time t.

De Finetti assumed that if S_t becomes negative, the company is not allowed to operate, and must liquidate, so that no further dividends can be paid. An assumption of this kind is clearly necessary to give the model economic meaning. De Finetti's assumption may, however, be too strict to be realistic. This question has been discussed in another paper [3], and we shall not take it up here.

The sequences of dividend payments will be stochastic processes, and it is not easy to devise a general method for assigning utilities to such processes. Traditional actuarial thinking leads us to try the expected discounted sum, i.e. $\Sigma v^t E\{s_t\}$ as the first approach.

Let us now introduce the function $V(S)$ = the expected discounted sum of the dividend payments when the initial reserve is S, provided that the company follows an optimal policy.

It is easy to see that $V(S)$—if it exists—must satisfy the functional equation

$$V(S) = \max_{\substack{0 \leq k \leq 1 \\ 0 \leq s \leq S}} \{s + v \int V(S + kP - s - kx)\, dF(x)\}$$

3.5. Let us for the time being ignore the reinsurance, so that the functional equation is reduced to

$$V(S) = \max_{0 \leq s \leq S} \{s + v \int_0^{S+P-s} V(S + P - s - x)\, dF(x). \qquad (7)$$

If an internal maximum exists, the derivative of the expression in braces, with respect to s, must vanish for the optimal dividend payment. If a density $f(x) = F'(x)$ exists, the condition can be written

$$1 - V(0) f(S + P - s) - \int_0^{S+P-s} V'(S + P - s - x)\, dF(x) = 0 \qquad (8)$$

In this equation s occurs only in the expression $S + P - s$. Hence if the equation has a root in s, it must be of the form

$$s = S - Z,$$

where Z is a constant, which can be interpreted as the optimal reserve. Further considerations show that the optimal dividend policy can be described as follows:

If at the end of an operating period the reserve exceeds Z, the excess should be paid out as dividend immediately.

If the reserve is less than Z, no dividend should be paid

It is clear that a policy of this form will lead to considerable fluctuations in the dividend payments. In real life insurance companies seem anxious to maintain steady dividend payments, so we have here an example of the effect mentioned in para. 1.2. A model which appears reasonable at a cursory examination, may have implications which are contradicted by observations.

There are further problems connected with the model under consideration. The equation (8) may have more than one root. In this case the optimal dividend policy will be a "band strategy", to use the term introduced by Morill [8]. In the models studied by Morill, the claim distributions are discrete. It should be possible to construct models with continuous claim distributions, in which the optimal dividend policy is a band strategy, but so far no example seems to be available.

3.6. Let us now assume that an insurance company has fixed a reserve level Z—optimal or not—, and that excess reserves are paid out as dividend immediately. Let $V(S, Z)$ be the expected discounted sum of the dividends paid under this policy. The function $V(S, Z)$ must then satisfy the integral equation

$$V(S, Z) = v \int_0^{P+S} V(S + P - x, Z)\, dF(x)$$

in the interval $0 \leq S \leq Z$. The boundary conditions are

$$V(S, Z) = 0 \qquad \text{for } S < 0$$
$$V(S, Z) = S - Z + V(Z, Z) \qquad \text{for } Z < S$$

This equation can be solved, but the solution is complicated, and it is difficult to study how it depends on the given parameters P, v, etc. The question can obviously be studied by simulation, but as the model itself seems unrealistic, such an investment in computation may give a poor return.

We may get a more realistic model if we assume that the company seeks the dividend policy which maximizes a criterion of the type

$$\sum_{t=0}^{\infty} v^t E\{u(s_t)\}$$

i.e. a sum of the discounted expected utility of the payments. The functional equation (7) then takes the form

$$V(S) = max\,\{u(s) + v \int V(S + P - s - x)\, dF(x)\} \qquad (9)$$

This model has become popular in investment analysis and in the theory of optimal economic growth. Models of this kind have been studied in great detail by Hakansson [7] and others. One can show that an optimal policy exists under fairly general assumptions, and the policy may be fairly simple for some utility functions, i.a. $u(x) = x^{\alpha}$. It is also clear that these models will lead to smoother dividend sequences than the original De Finetti model.

The results of these studies do, however, not appear to be immediately applicable to insurance, because most authors find it necessary to assume away the event of ruin, which seems essential in any theory of insurance. The wealth of an unlucky investor may converge to zero, but it cannot disappear with a bang if he follows an optimal policy –for instance if he never risks more than half his money.

3.7. The results of Hakansson are complicated, and it is not easy to discuss how his solutions depend on the given parameters. It seems possible to obtain simpler expressions for the solutions if one works with continuous time, and makes use of the results in diffusion theory. Papers by Gerber [6] and others provide good examples of this. It is, however, clear that by continuous control of a process, one can avoid ruin, so that some of the relevance to insurance is lost.

3.8. The desire for simple solutions, expressed several times in this paper, should not be taken as a conclusive proof of the author's laziness. The results which so far have been obtained from the study of dynamic models, must be seen as tentative and preliminary. They all concern the decisions of a single insurance company in a given situation—usually studied under very special assumptions. Few, if any, attempts seem to have been made to study the interaction of the decisions made by all the companies in the market. One of the first goals of a dynamic theory should be to reach some results of the same generality as the theorem about optimal reinsurance arrangements mentioned in para 2.4. The chances of obtaining such results do, however, seem slim, as long as the problems of a single company appear so formidable as current research indicates.

References

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