A GENERIC CLAIMS RESERVING MODEL

A FUNDAMENTAL RISK ANALYSIS

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In their diversity, insurance risks often require very different types of claims reserving models to describe them and to estimate the necessary reserves. The tractability of the chain ladder has contributed to its popularity. A brief analysis is given of its statistical basis and its implied limitations, of which the most important is its propensity to underestimate the reserves. This paper proposes a new paradigm for actuarial risk analysis where the reserving estimate is just one of the many results it delivers. The generic claims reserving model is consistent with the claims development process, and from it a rich family of claims reserving models can be constructed. Without loss of generality, the variance is simply defined to be a function of the mean response. No particular consideration is given to estimation procedures, as those will depend on the selected model structure.

1. INTRODUCTION

A typical claims array of a book of business is cross-referenced by underwriting year and delay period, and is additionally described by payment year defined as underwriting year plus delay period minus one. As an example of annualised claims data consider the following array:

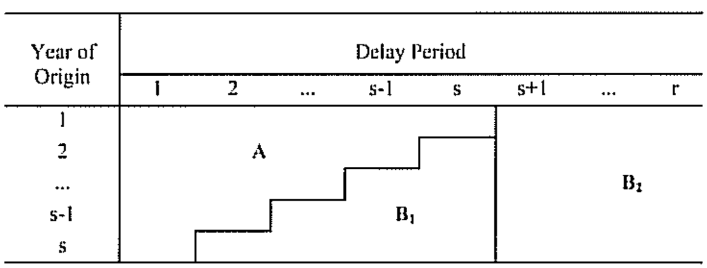

Fig.1 Example of a claims development array, A represents known claims, and B_1 and B_2 projected claims.

The array is divided into three regions: A for the cells with known claims. A typically consists of the upper left triangular region of the array. The last diagonal of A is s. It corresponds to the last payment year for which claims data are known. The triangular array right below A is denoted by $\mathbf{B}_1$ and the cells with unknown claims for delay periods beyond s by $\mathbf{B}_2$.

Variations in the shape of the data array represented by A are usual, and are well within the scope of the models discussed in this paper. They are normally the result of data exclusions from the latest delay periods, from the latest origin years or from the earliest payment periods. Truncation of the data in the underwriting year direction is a consequence of cessation of business or of changes in the underwritten risk.

The models that concern this paper have the capacity to predict future claims beyond delay period s. This is a distinctive advantage over models such as the chain ladder (Zehnwirth (1989)), or other models derived from it.

A very large variety of models has been developed to predict future claims for the lower triangle of the claims array $\mathbf{B}_1$. The chain ladder, being one of them, has been quoted and referenced frequently. Kremer (1982) proves the connection between the chain ladder and a two-way analysis of variance and the results are discussed and applied as the basis of further work by other authors. However, the use of the analysis of variance to provide a statistical justification to the chain ladder gives a hint of the limitations of this method and related models. The most important is that it excludes the tail factor beyond delay period s. This is acknowledged by Zehnwirth (1989), to which close reference is made in order to set the framework for the method this paper proposes. For consistency with the rest of the paper and without altering the conclusions that can be derived from it, the exposition of Zehnwirth (1989) made with reference to accident year, is summarised below with reference to underwriting year.

1.1 THE CHAIN LADDER METHOD

Denote the incremental paid claims in development year j and underwriting year w by $y^{\bullet}_{w,j}$. Then the cumulative claim amounts and development factors for underwriting year w at development year j can be defined by $c^{\bullet}_{w,j} = \sum_{h=1}^{j} y^{\bullet}_{w,h}$ and $D_{w,j} = \left(c^{\bullet}_{w,j-1}\right)^{-1} c^{\bullet}_{w,j}$. Zehnwirth (1989) summarises the chain ladder assumptions as follows:

ASSUMPTION 1: Each underwriting year has the same development factor, with an estimate defined as $\hat{D}_j = \left(\sum_{w=1}^{s-j+1} c^{\bullet}_{w,j-1}\right)^{-1} \sum_{w=1}^{s-j+1} c^{\bullet}_{w,j}, \forall j = 2,..,s$. Then, projections of $c^{\bullet}_{w,j}$ for $w = 2,...s$ and $j = s-w+2,...,s$ are $\hat{c}^{\bullet}_{w,s} = c^{\bullet}_{w,s-w+1} \prod_{k=s-w+2}^{s} \hat{D}_k$. For consistency let $\hat{c}^{\bullet}_{1,s} = c^{\bullet}_{1,s}$

ASSUMPTION 2: Each underwriting year has a parameter representing its level. For underwriting year w this is $c^{\bullet}_{w,s-w+1}$. The final underwriting year is represented by $c^{\bullet}_{s,1}$. Zehnwirth (1989) argues that when $c^{\bullet}_{s1}$ represents a single observation and all underwriting years are completely homogenous, it should be estimated by $\frac{1}{s}\sum_{w=1}^{s} c^{\bullet}_{w1}$.

To establish the statistical basis of the chain ladder, let

$$\hat{p}^{\bullet}_j = \begin{cases} \left(\sum_{w=1}^{s} c^{\bullet}_{w1}\right)\left(\sum_{w=1}^{s} \hat{c}^{\bullet}_{ws}\right)^{-1} & j = 1 \\ \left(\sum_{w=1}^{s-j+1} c^{\bullet}_{wj} - \sum_{w=1}^{s-j+1} c^{\bullet}_{w(j-1)}\right)\left(\sum_{w=1}^{s-j+1} \hat{c}^{\bullet}_{ws}\right)^{-1} & 2 < j < s \\ \left(c^{\bullet}_{1s} - c^{\bullet}_{1(s-1)}\right)\left(c^{\bullet}_{1s}\right)^{-1} & j = s \end{cases}$$

From assumption 1 $\sum_{w=1}^{s-j+1} \hat{c}_{ws}^{\bullet} = \left(\prod_{k=j+1}^{s} \hat{D}_k \right) \sum_{w=1}^{s-j+1} c_{wj}^{\bullet}$, then

$$\hat{p}_j^{\bullet} = \begin{cases} \left(1-\hat{D}_j^{-1}\right)\left(\prod_{k=j+1}^{s} \hat{D}_k\right)^{-1} & j = 2,..,s-1 \\ \left(1-\hat{D}_s^{-1}\right) & j = s \end{cases}$$

and

$$\hat{D}_j = \begin{cases} \left(1-\hat{p}_j^{\bullet}\prod_{k=j+1}^{s} \hat{D}_k\right)^{-1} & j = 2,..,s-1 \\ \left(1-\hat{p}_s^{\bullet}\right)^{-1} & j = s \end{cases}$$

Given the expression for $\hat{p}_j^{\bullet}$, $\sum_{j=1}^{s} \hat{p}_j^{\bullet} \sum_{w=1}^{s-j+1} \hat{c}_{ws}^{\bullet} = \sum_{j=1}^{s}\left(\sum_{w=1}^{s-j+1} c_{wj}^{\bullet} - \sum_{w=1}^{s-j+1} c_{w(j-1)}^{\bullet} \right) = \sum_{j=1}^{s}\left(\sum_{w=1}^{s-j+1} y_{wj}^{\bullet} \right)$ and the following can be concluded:

1. Based on the marginal estimates, the equality $\left(\sum_{j=1}^{s} \sum_{w=1}^{s-j+1} \hat{p}_j^{\bullet}\hat{c}_{w,s}^{\bullet} = \sum_{j=1}^{s} \sum_{w=1}^{s-j+1} y_{w,j}^{\bullet} \right)$ suggests the stochastic model proposed by Kremer (1982): $y_{w,j}^{\bullet} = \alpha_w^{'}\beta_j^{'}e_{w,j}^{'}$ where $\alpha_w^{'}$ and $\beta_j^{'}$ are the parameters for underwriting year w and development year j, and $e_{wj}^{'}$ the random error term. This model can be re-stated as a two-way analysis of variance $y_{w,j}^{\bullet\bullet} = \ln y_{w,j}^{\bullet} = \mu + \alpha_w + \beta_j + e_{w,j}$ such that $e_{w,j} \sim N\left(0,\sigma^2\right), \sum_{w=1}^{s} \alpha_w = \sum_{j=1}^{s} \beta_j = 0$ (Zehnwirth (1989)).

2. From assumptions 1 and 2, $\hat{c}_{1,s}^{\bullet} = c_{1,s}^{\bullet} = \sum_{h=1}^{s} y_{1,h}^{\bullet}$. Hence, $\left(\hat{c}_{1,s}^{\bullet} \sum_{j=1}^{s} \hat{p}_j^{\bullet} = \sum_{j=1}^{s} y_{1,j}^{\bullet} \right)$ implies $\left(\sum_{j=1}^{s} \hat{p}_j^{\bullet} = 1 \right)$.

3. Since $\sum_{w=1}^{s-j+1} \hat{c}_{w,j-1}^{\bullet} \equiv \left(\sum_{k=1}^{j-1} \exp(\beta_k) \right)\left(\sum_{w=1}^{s-j+1} \exp(\mu + \alpha_w) \right)$, the equivalent relationships emerging from points 1 and 2 are: $\hat{p}_j^{\bullet} \equiv \exp(\beta_j)\left(\sum_{k=1}^{s} \exp(\beta_k) \right)^{-1}$ and $\hat{D}_j \equiv 1 + \exp(\beta_j)\left(\sum_{k=1}^{j-1} \exp(\beta_k) \right)^{-1}$. The former gives a clear interpretation to $\hat{p}_j^{\bullet}$ as the percentage of the total claim amount to be paid in development year j. The definition of $\hat{p}_j^{\bullet}$ is restricted to the first s development years for which data exists. Hence, for $j > s$, $\hat{p}_j^{\bullet} = 0$.

Conclusions 1 and 2 show that the chain ladder understates the reserves for runoff triangles that at development year s are not fully developed. It is usual for practitioners who use the chain ladder to value their reserves to apply industry benchmarks in order to estimate future losses for the periods within the region $\mathbf{B_2}$ on the array. Particularly when they are generated from portfolios similar in claims experience and composition to those to which they are applied, benchmarks can certainly be a very useful source of reference to achieve safe reserves. Industry benchmarks could contain trends not yet apparent in the company's data. Nevertheless, in the context of conclusion 3, where the chain ladder is interpreted as the product of two marginal functions: the total claim amount and the percentage cash flow, the use of benchmarks shows that the chain ladder's implied assumptions are frequently adopted in error, and stresses the need for a more coherent reserving approach.

To construct the generic model the relationship $\left(\sum_{j=1}^{s}\sum_{w=2}^{s-j+1}\hat{p}_j^{\bullet}\hat{c}_{w,s}^{\bullet}=\sum_{j=1}^{s}\sum_{w=1}^{s-j+1}y_{w,j}^{\bullet}\right)$ should not be dismissed altogether, though more careful formulations of the marginal estimates $\hat{p}_j^{\bullet}$ and $\hat{c}_{w,s}^{\bullet}$ are required to ensure that $\hat{p}_j^{\bullet}$ is defined for all the possible periods of exposure, and $\hat{c}_{w,s}^{\bullet}$ is replaced by an estimate independent of s.

This paper is organised as follows. In section 2.1 the empirical data are defined. In sections 2.2 to 2.4 the components of the generic model are developed and the properties of the underlying function to the percentage cash flow are addressed. The mean square errors are derived in section 3 preceded by a reviewed version of the incremental claims reserving model. Behind the form of the generic model derived in section 2 is the assumption that the data of interest represents a single class of actuarial risk. This is relaxed in section 4, and a generic reserving model for claims data containing more than one class of actuarial risks is constructed.

2. A GENERIC MODEL

2.1 THE CUMULATIVE CLAIMS PROCESS

Let $C_w^{\bullet}$ be the ultimate loss incurred in underwriting year w during its entire settlement period.

For simplicity of notation, it will be assumed that claims are reported at regular periods, although in practice this needs not be a limitation of the models discussed. Let $t_0=0$, and consider the claims process for underwriting year w, reported at times $t_1,t_2,\ldots,t_e$, such that $0<t_1<t_2<\ldots<t_e$, and t_e is the time when the ultimate settlement is made. The number of partial or total claim payments by time t is defined by

$$N_w^{\bullet}(t)=\begin{cases}0 & t=t_0\\ N_w^{\bullet}(t_j) & t_{j-1}<t\le t_j,\quad 0<t_j<t_e\\ N_w & t_e\le t\end{cases}$$

where N_w is the total or ultimate number of claims from underwriting year w. Let the i_{th} incremental claim settlement for underwriting year w paid during time period $\left(t_{j-1},t_j\right]$ be denoted by $\left(X_{w,i,j}^{\bullet}\right)=\left\{i\in\left[1,\ldots,N_w^{\bullet}(t_j)-N_w^{\bullet}(t_{j-1})\right],j\in\left[1,e\right]\right\}$. Then the ultimate claim amount for underwriting year w is given by $C^{*}_{w} = \sum_{j=1}^{e} \sum_{i=1}^{\left[N^{*}_{w}(t_j) - N^{*}_{w}(t_{j-1})\right]} X^{*}_{w,i,j}$. Hence the aggregate claim amount for underwriting year w and delay period j and the corresponding percentage cash flow denoted respectively by $Y^{*}(w,j)$ and $P^{*}(w,j)$ can be defined as

$$Y^{*}(w,j) = \begin{cases} \sum_{k=1}^{j} \sum_{i=1}^{\left[N^{*}_{w}(t_k) - N^{*}_{w}(t_{k-1})\right]} X^{*}_{w,i,k} & 1 \le t_j < t_e \\ C^{*}_{w} & t_j \ge t_e \end{cases}$$
$$P^{*}(w,j) = \begin{cases} Y^{*}(w,j) C^{*-1}_{w} & 0 < t_j < t_e \\ 1 & t_j \ge t_e \end{cases} \tag{2.1}$$

Equation (2.1) shows that it is justifiable to represent the cumulative percentage cash flow by a continuous function, and this could be an integral. Its properties are defined below.

2.2 THE PERCENTAGE CASH FLOW

For theoretical purposes, the claims process is assumed to be continuous. We define a continuous function $P(w,j)$ for the percentage cash flow and a function C_w for the ultimate claim amount. Hence, for a random variable $Y(w,j)$

$$E\left(Y(w,j)\right) = C_w P(w,j) \tag{2.2}$$

To define the percentage cash flow function the properties of a probability density function with domanin $D \subset \mathbb{R}^{+}$.

DEF. If the underlying function of integral $\Pi(w,t)$ is denoted by $\pi(w,t)$, such that $t \in D$ then

$\pi(w,t) = \lim_{\partial t \to 0} \left(\frac{\Pi(w,t+\partial t) - \Pi(w,t)}{\partial t} \right) = \left(\frac{\partial \Pi(w,z)}{\partial z} \right)_{z=t}$, and the properties of $\pi(w,t)$ are as follows:

i. $\pi(w,t) \ge 0 \quad \forall t$

ii. $\Pi(w,t) = \int_{z=0}^{t} \pi(w,z) dz = \Pr[z \le t]$

iii. $\int_{z=0}^{\infty} \pi(w,z) dz = 1$

DEF. For a given probability density function $\pi(w,t)$, let

- $P(w,t_j) = \Pi(w,t_j)$
- $S(w,t_j) = \int_{z=t_j}^{t_e} \pi(w,z) dz = 1 - P(w,t_j)$

- $$p(w,j) = \int_{z=t_{j-1}}^{t_j} \pi(w,z)dz$$

where $p(w,j)$ denotes the percentage cash flow during the time interval $\left(t_{j-1},t_j\right]$ recorded at the end of development period j and $S(w,t_j)$ the percentage of ultimate losses to be paid after $t = t_j$.

Individual incurred claims could be negative as a consequence of the recording process of paid and outstanding claims. Incurred claims data, as the total of paid and outstanding claims, is subject to fluctuations in both data sets. Their fluctuations and adjustments often result in negative incremental incurred claims entries. However, a reserving model with a systematic component defined as (2.2) could deal with negative incremental adjustments, since those are normally corrections of earlier entries.

When the evaluation of integral $\Pi(w,t)$ cannot be obtained in terms of known functions, $p(w,j)$ and $P(w,j)$ need to be approximated. Numerical integration techniques can be used for this purpose. Newton-Cotes, Euler, Runge-Kutta and Simpson's Rule are computationally intensive. For simpler methods consider the following. If $\delta = \frac{x_2 - x_1}{K}$ and $\mathfrak{A}$ represents the area under a curve $\tilde{f}(.)$, $\mathfrak{A}$ can be approximated from below by $\mathfrak{A}_L(x_1,x_2) = \sum_{j=0}^{K-1} \tilde{f}(x_1 + j\delta)\delta$ and from above by $\mathfrak{A}_U(x_1,x_2) = \sum_{j=1}^{K} \tilde{f}(x_1 + j\delta)\delta$, such that $\mathfrak{A}_L \le \mathfrak{A} \le \mathfrak{A}_U$. The trapezoidal rule approximation between the interval (x_1,x_2) can be defined by

$$\mathfrak{A}_{tr}(x_1,x_2) = \delta\left(\frac{\tilde{f}(x_1)}{2} + \sum_{j=1}^{K-1} \tilde{f}(x_1 + j\delta) + \frac{\tilde{f}(x_2)}{2}\right)$$

Clearly as $K \to \infty$, $\mathfrak{A}_L, \mathfrak{A}_U$ and $\mathfrak{A}_{tr}$ tend to $\mathfrak{A}$, but $\mathfrak{A}_{tr}$ does so with greater accuracy than either $\mathfrak{A}_L$ or $\mathfrak{A}_U$.

If the area under curve $\pi(w,t)$ is segmented at discrete consecutive periods, such that $\delta = 1$, the above suggest three alternative approximations for $p(w,j)$ and $P(w,j)$:

$$\begin{aligned} P(w,j) &\approx \sum_{k=0}^{j-1} \pi(w,k) \\ p(w,j) &\approx \pi(w,j-1) \end{aligned} \tag{2.3}$$

$$\begin{aligned} P(w,j) &\approx \sum_{k=1}^{j} \pi(w,k) \\ p(w,j) &\approx \pi(w,j) \end{aligned} \tag{2.4}$$

$$\begin{aligned} P(w,j) &\approx \begin{cases} \dfrac{\pi(w,j)}{2} & j = 1 \\ \displaystyle\sum_{k=1}^{j-1} \pi(w,k) + \frac{\pi(w,j)}{2} & j > 1 \end{cases} \\ p(w,j) &\approx \frac{\pi(w,j-1) + \pi(w,j)}{2} \end{aligned} \tag{2.5}$$

Against the more accurate estimate of $P(w,j)$ and $p(w,j)$ that method (2.5) could possibly produce, is the neater construction of the cumulative and incremental claims reserving models that could be achieved by considering either of methods (2.3) and (2.4). The consequences of underestimating the percentage cash flow are more serious in underwriting years with claims known for only a few delay periods, as their proportion of the overall reserves is greater than the proportion represented by underwriting years with similar exposure and claims at a more advanced stage in their development. When method (2.4) is adopted,

$$S(w,j) = \int_j^{\infty} \pi(w,z)dz \approx \sum_{k=j+1}^{\infty} \pi(w,k) \quad \forall j \geq 0 \tag{2.6}$$

In cases where $P(w,j)$ and $p(w,j)$ need to be approximated, as the losses develop, the characteristics of the underwritten risk should be extracted from $\Pi(w,j)$. Those that could be made immediately available are briefly described in the next section.

$S(w,j)$ is the basis of the definition of the *IBNR* function given in section 2.3. It is possible to construct claims reserving models that fit the data well while $\int_{z=0}^{\infty} \pi(w,z)dz > 1$. However, when $\pi(w,t)$ is not a probability density function, the definitions of $S(w,j)$ and *IBNR* become meaningless.

2.3 CHARACTERISTICS OF THE ACTUARIAL RISK

The aggregation of claims data for the purpose of reserving assumes that the data in each claims array broadly follows a similar development. This assumption can and should be assessed by extracting and comparing the information contained in $\Pi(w,t)$.

From the expression of the cumulative percentage cash flow the hazard rate and its integral can be obtained:

$$h(w,t) = \frac{\left(\dfrac{\partial \Pi(w,z)}{\partial z}\right)_{z=t}}{1-\Pi(w,t)} = -\left(\frac{\partial\left(\log\left(1-\Pi(w,z)\right)\right)}{\partial z}\right)_{z=t} \tag{2.7}$$

$$H(w,t) = -\ln\left(1-\Pi(w,t)\right) \tag{2.8}$$

Kurtosis and skewness, as measures of shape, will also be available when the values of the reserving model parameters permit estimating the necessary moments.

Estimates of $S(w,t)$ and of future claims for the regions $\mathbf{B}_1$ and $\mathbf{B}_2$ of the array in Fig. 1, are essential for the various analytical tasks of a claims portfolio, such as evaluation of the solvency status, assessment of underwriting volumes versus *IBNR* projections by class, assessment of reinsurance cover, calculation of future premiums, etc. There are significant advantages when function $\pi(w,t)$ is a probability density function. In this case the claims reserving model makes available all the descriptive statistics of the density function as characteristics of the percentage cash flow function. At payment year s *IBNR* estimates for underwriting year w and for an entire claims array of u underwriting years are respectively:

$$IBNR_{(w,s-w+1)} = C_w S(w,s-w+1)$$

$$IBNR_s = \sum_{w=1}^{u} C_w S(w,s-w+1)$$

From section 2.2

$$\Pi(w,t) = \int_{z=0}^{t} \pi(w,z)dz \tag{2.9}$$

Consider the following. When $\pi(w,t)$ is a density function, $\pi(w,t) = \Pr[z=t]$. If $P(w,t) = \alpha$, then $t = \Pi^{-}(\alpha) = \Pi^{-}(P(w,t))$ such that Π^{-} is the inverse function of Π with respect to t. Hence it is possible to calculate the corresponding time period t for a given cumulative percentage cash flow value α and inverse function $\Pi^{-1}(\alpha)$. Π is used rather than P to allow for cases where P needs to be approximated.

In the same way, since

$$S(w,t) = \frac{IBNR_{(w,t)}}{C_w}$$

the value of time period t for a given value of future losses $\left(IBNR_{(w,t)}\right)$ can be determined as follows:

$$t = \Pi^{-}\left(1 - \frac{IBNR_{(w,t)}}{C_w}\right) = \Pi^{-}\left(P(w,t)\right)$$

The behaviour of the right tail of the probability distribution function is particularly relevant and important to the calculation of reserves and to subsequent analysis of the claims portfolio. Therefore, the selected distribution function should be a suitable description of the actuarial risk's percentage cash flow.

In portfolios on run-off, where all aspects of asset management are drastically simplified as the flow of premiums reduces, the importance of having readily available *IBNR* projections is more apparent, since those are essential to unable a company to formulate and update a coherent commutation strategy .

2.4 ESTIMATION TECHNIQUES AND THE GENERIC MODEL

Decisions on estimation techniques are determined by the overall structure of the model. If the percentage cash flow function can be linearized, generalized linear models could be considered (McCullagh and Nelder (1989)). For more complex claims reserving models, simulation techniques would be more useful.

Equation (2.2) gives the first part of the cumulative claims reserving model. For the mean response of the incremental models $P(w,t)$ should be replaced by $p(w,t)$. The selection of the variance function should depend on the data. With cumulative claims reserving models, particular attention is required to the possible presence of serial correlation. Where necessary, the models should be amended to allow for serial correlation. The literature available to help the practitioner explore and select the most suitable variance function is extensive.

The mean square errors for incremental claims reserving models for a single actuarial risk class are derived in section 3. To ensure that these can be immediately applied to different types of functions for $\pi(w,t)$, it is necessary to make the definition of $p(w,j)$ and C_w slightly more explicit. The final expression for the incremental claims reserving model is given in equation (3.1).

3. GENERAL INCREMENTAL CLAIMS RESERVING MODEL AND PREDICTED MEAN SQUARE ERRORS

Along the lines of Renshaw (1994), in this section the mean prediction errors for the incremental generic model are derived. In a claims array such as the one illustrated in Fig. 1, projected claims fall in the region defined by set $B = \bigcup_{i=1}^{2} B_i$. Renshaw (1994) assumes mutual independence between individual predictions in B, and independence between past and future claims. The main implication of these assumptions is that the results from Renshaw (1994) derived for GLM models, where the mean response is the product of marginal parameters, can be easily extended for more complex models. To illustrate this, the mean square errors are derived in sections 3.1 to 3.4.

For our purposes we classify probability distributions functions into those with or without a normalising function. An all-encompassing definition could be achieved by the following: $\pi(w,t) = g_w(.)G(w,\beta_w,t)$ where $g_w^{-1}(.) = \int_0^\infty G(w,\beta_w,t)dt$. $g_w(.)$ is a normalising function, independent of t and possibly dependent on β_w, where β_w is a parameter vector $\beta_w = \left(\beta_{w_1},..,\beta_{w_q}\right)$ and q is the number of parameters in $\pi(w,t)$. When the distribution does not have a normalising function $g_w(.) = 1$ and $\pi(w,t) = G(w,\beta_w,t)$. Hence, if $f(j,\beta_w) = \int_{t=j-1}^{t=j} G(w,\beta_w,t)dt$, to simplify the estimation procedures it is convenient to write $p(w,j) = f(j,\beta_w)g_w(.)$ and $C_w = \exp\left(\lambda_w\right)g_w^{-1}(.)$, where λ_w and β_w are the model parameters.

Hence, if $y\left(w,j\right)$ denotes the random variable representing the incremental claims data, then

$$E\left(y(w,j)\right)=\mu(w,j)$$
$$Var\left(y(w,j)\right)=\frac{\gamma}{\varpi_w}V\left(\mu(w,j)\right) \tag{3.1}$$

ϖ_w are prior weights. The mean can be alternatively expressed as

$$\mu(w,j)=\mu(j,\lambda_w,\beta_w)=\exp(\lambda_w)f(j,\beta_w) \tag{3.2}$$

Let $z(w,j)$ be the unknown future incremental claims in development year j. The estimates of individual predictions, future losses for underwriting year w alone and for a book of business of u underwriting years with know claims up to payment year s are respectively:

$$E\left(z(w,j)\right)=\exp(\lambda_w)\int_{j-1}^{j}G(w,\beta_w,t)dt$$
$$E\left(\sum_{j=s-w+2}^{\infty}z(w,j)\right)=\exp(\lambda_w)\sum_{j=s-w+2}^{\infty}\int_{j-1}^{j}G(w,\beta_w,t)dt=\exp(\lambda_w)\int_{s-w+1}^{\infty}G(w,\beta_w,t)dt=IBNR_{(w,s-w+1)}$$
$$E\left(\sum_{w=1}^{u}\sum_{j=s-w+2}^{\infty}z(w,j)\right)=\sum_{w=1}^{u}\exp(\lambda_w)\int_{s-w+1}^{\infty}G(w,\beta_w,t)dt=IBNR_s$$

When the definition of $\pi(w,t)$ includes a normalising function, this should be excluded from the estimation procedures. In general, a simpler model tends to converge more rapidly, and the calculations of the mean square error equations associated with it are also more transparent. By defining $\beta_w=\left(\beta_{w_1},..,\beta_{w_k}\right)$ the percentage cash flow function is allowed to change with underwriting year w.

In the sections that follow the prediction errors are estimated up to delay period ζ.

3.1 INDIVIDUAL PREDICTIONS AND ERRORS

For future incremental claims in development year j

$$E\left(z(w,j)\right)=\mu(j,\lambda_w,\beta_w)$$
$$Var\left(z(w,j)\right)=\gamma V\left(\mu(j,\lambda_w,\beta_w)\right) \tag{3.3}$$

For simplicity of notation denote $\mu_{wj}=\mu(j,\lambda_w,\beta_w)$, and let $\hat{\mu}_{wj}=\mu\left(j,\hat{\lambda}_w,\hat{\beta}_w\right)$ be the predictor of $z(w,j)$, such that $\hat{\lambda}_w$ and $\hat{\beta}_w$ are the parameter estimates. Then the first order Taylor series approximation of $\hat{\mu}_{wj}$ is

$$\hat{\mu}_{wj}\approx\mu_{wj}\left(1+\left(\hat{\lambda}_w-\lambda_w\right)+\sum_k\left(\hat{\beta}_{w_k}-\beta_{w_k}\right)\frac{\partial}{\partial\beta_{w_k}}\left(\ln\left(f(j,\beta_w)\right)\right)\right) \tag{3.4}$$

For $(w,j)\in B$ the mean square error associated with the predictor is given by $E\left(\left(z(w,j)-\hat{E}\left(z(w,j)\right)\right)^2\right)$.

However

$$\left(z(w,j)-\hat{E}\left(z(w,j)\right)\right)^2=\left(z(w,j)-\hat{\mu}_{wj}\right)^2=\left(\left(z(w,j)-\mu_{wj}\right)-\left(\hat{\mu}_{wj}-\mu_{wj}\right)\right)^2 \tag{3.5}$$

The expectations of the two square terms on the right hand side of (3.5) are

$$E\left(\left(z(w,j)-\mu_{wj}\right)^2\right)=Var\left(z(w,j)\right)$$

$$E\left(\left(\hat{\mu}_{wj}-\mu_{wj}\right)^2\right)\approx\mu_{wj}^2E\left(\left(\left(\hat{\lambda}_w-\lambda_w\right)+\sum_k\left(\hat{\beta}_{w_k}-\beta_{w_k}\right)\frac{\partial}{\partial\beta_{w_k}}\left(\ln\left(f\left(j,\beta_w\right)\right)\right)\right)^2\right) \tag{3.6}$$

Since $\hat{\mu}_{wj}$ is generated by past claims, owing to the assumptions of independence between past and future claims

$$E\left(\left(z(w,j)-\mu_{wj}\right)\left(\hat{\mu}_{wj}-\mu_{wj}\right)\right)=0$$

Hence

$$E\left(\left(z(w,j)-\hat{\mu}_{wj}\right)^2\right)\approx Var\left(z(w,j)\right)+\mu_{wj}^2E\left(\left(\left(\hat{\lambda}_w-\lambda_w\right)+\sum_k\left(\hat{\beta}_{w_k}-\beta_{w_k}\right)\frac{\partial}{\partial\beta_{w_k}}\left(\ln\left(f\left(j,\beta_w\right)\right)\right)\right)^2\right) \tag{3.7}$$

3. 2 PREDICTED ROW TOTALS AND THEIR MEAN SQUARE ERRORS

Let $\Im_w\subset B$, such that $\Im_1=\left(\bigcup_{j=s+1}^{\zeta}B_{2,(1,j)}\right)$ and $\Im_w=\bigcup\left(\bigcup_{j=s+2-w}^{s}B_{1,(w,j)}\right)\left(\bigcup_{j=s+1}^{\zeta}B_{2,(w,j)}\right), w>1$, represent in the matrix in Fig. 1 the periods in underwriting year w for which claims are yet unknown. For all, ζ is the maximum projection period. For $(w,j)\in\Im_w$ the unknown total claim amount, the mean response and the corresponding mean square errors associated with it can be defined as $Z_w=\sum_{(w,j)\in\Im_w}z(w,j)$, and

$$E\left(Z_w\right)=\sum_{(w,j)\in\Im_w}E\left(z(w,j)\right)=\sum_{(w,j)\in\Im_w}\mu_{wj} \tag{3.8}$$

$$E\left(\left(Z_w-\hat{E}\left(Z_w\right)\right)^2\right)=E\left(\left(\sum_{(w,j)\in\Im_w}\left(z(w,j)-\hat{\mu}_{wj}\right)\right)^2\right) \tag{3.9}$$

The right hand side of (3.9) gives rise to two types of terms:

i. When $i \neq j$:

$$\left(z(w,i)-\hat{\mu}_{wi}\right)\left(z(w,j)-\hat{\mu}_{wj}\right)=\left(z(w,i)-\mu_{wi}\right)\left(z(w,j)-\mu_{wj}\right)+\left(\hat{\mu}_{wi}-\mu_{wi}\right)\left(\hat{\mu}_{wj}-\mu_{wj}\right) \\ -\left(\hat{\mu}_{wi}-\mu_{wi}\right)\left(z(w,j)-\mu_{wj}\right)-\left(\hat{\mu}_{wj}-\mu_{wj}\right)\left(z(w,i)-\mu_{wi}\right) \tag{3.10}$$

From the assumptions of independence between $z(w,i)$ and $z(w,j)$ and between past and future claims

$$E\left(\left(z(w,i)-\mu_{wi}\right)\left(z(w,j)-\mu_{wj}\right)\right)=\operatorname{cov}\left(z(w,i),z(w,j)\right)=0$$

$$E\left(\left(\hat{\mu}_{wi}-\mu_{wi}\right)\left(z(w,j)-\mu_{wj}\right)\right)=E\left(\left(\hat{\mu}_{wj}-\mu_{wj}\right)\left(z(w,i)-\mu_{wi}\right)\right)=0$$

Finally, from equation (3.4)

$$E\left(\left(\hat{\mu}_{wi}-\mu_{wi}\right)\left(\hat{\mu}_{wj}-\mu_{wj}\right)\right)\approx\mu_{wj}\mu_{wi}Var\left(\hat{\lambda}_w\right)$$

$$+\mu_{wj}\mu_{wi}E\left(\left(\left(\hat{\lambda}_w-\lambda_w\right)\sum_k\left(\hat{\beta}_{w_k}-\beta_{w_k}\right)\left(\frac{\partial}{\beta_{w_k}}\left(\ln\left(f(i,\beta_w)*f(j,\beta_w)\right)\right)\right)\right)\right)$$

$$+\mu_{wj}\mu_{wi}E\left(\left(\sum_k\left(\hat{\beta}_{w_k}-\beta_{w_k}\right)\frac{\partial}{\beta_{w_k}}\left(\ln\left(f(i,\beta_w)\right)\right)\right)\left(\sum_k\left(\hat{\beta}_{w_k}-\beta_{w_k}\right)\frac{\partial}{\beta_{w_k}}\left(\ln\left(f(j,\beta_w)\right)\right)\right)\right)$$

ii. When $i=j$:

$$\sum_{(w,j)\in\Im_w}E\left(\left(z(w,j)-\hat{\mu}_{wj}\right)^2\right)\approx\sum_{(w,j)\in\Im_w}Var\left(z(w,j)\right) \\ +\sum_{(w,j)\in\Im_w}\mu_{wj}^2E\left(\left(\left(\hat{\lambda}_w-\lambda_w\right)+\sum_k\left(\hat{\beta}_{w_k}-\beta_{w_k}\right)\frac{\partial}{\partial\beta_k}\left(\ln\left(f(j,\beta_w)\right)\right)\right)^2\right) \tag{3.11}$$

Hence, the only expectations contributing to the mean square error (3.9) are (3.11) and

$$2\sum_{\substack{(w,j)\in\Im_w\\ i<j}}E\left(\left(\hat{\mu}_{wi}-\mu_{wi}\right)\left(\hat{\mu}_{wj}-\mu_{wj}\right)\right)$$

3.3 PREDICTED PAYMENT YEAR TOTALS AND THEIR MEAN SQUARE ERRORS

For a book of business of u underwriting years, let $\wp_\tau=\bigcup\left(\bigcup_{w=u}^{\tau-s+1}B_{1,(w,\tau-w+1)}\right)\left(\bigcup_{w=\tau-s}^{1}B_{2,(w,\tau-w+1)}\right)$, $\wp_\tau\subset B$ represents in the matrix in Fig. 1 the time periods in payment year τ, and let the unknown total claim amount for payment year τ be defined as $Z_{++\tau}=\sum_{(w,j)\in\wp_\tau}z(w,j)$. Then

$$E\left(Z_{++\tau}\right)=\sum_{(w,j)\in\wp_\tau}E\left(z(w,j)\right)=\sum_{(w,j)\in\wp_\tau}\mu_{wj} \tag{3.12}$$

$$E\left(\left(Z_{++\tau}-\hat{E}\left(Z_{++\tau}\right)\right)^2\right)=E\left(\left(\sum_{(w,j)\in\wp_\tau}\left(z(w,j)-\hat{\mu}_{wj}\right)\right)^2\right) \tag{3.13}$$

i. From (3.7) when $w_1 = w_2 = w$ the square terms of (3.13) are

$$E\left(\sum_{(w,j)\in\wp_r}\left(z(w,j)-\hat{\mu}_{wj}\right)^2\right)\approx\sum_{(w,j)\in\wp_r}Var\left(z(w,j)\right)+ \sum_{(w,j)\in\wp_r}\mu_{wj}^2 E\left(\left(\left(\hat{\lambda}_w-\lambda_w\right)+\sum_k\left(\hat{\beta}_{w_k}-\beta_{w_k}\right)\frac{\partial}{\partial\beta_{w_k}}\left(\ln\left(f\left(j,\beta_w\right)\right)\right)\right)^2\right) \tag{3.14}$$

ii. For $w_1 \neq w_2$ the combined terms in the right hand side summation of (3.13) can be written as follows:

$$\begin{aligned}\left(z(w_1,i)-\hat{\mu}_{w_1i}\right)\left(z(w_2,j)-\hat{\mu}_{w_2j}\right)&=\left(z(w_1,i)-\mu_{w_1i}\right)\left(z(w_2,j)-\mu_{w_2j}\right)+\left(\hat{\mu}_{w_1i}-\mu_{w_1i}\right)\left(\hat{\mu}_{w_2j}-\mu_{w_2j}\right)\\&-\left(\hat{\mu}_{w_1i}-\mu_{w_1i}\right)\left(z(w_2,j)-\mu_{w_2j}\right)-\left(\hat{\mu}_{w_1j}-\mu_{w_1j}\right)\left(z(w_2,i)-\mu_{w_2i}\right)\end{aligned} \tag{3.15}$$

From the assumption of independence between $z(w_1,i)$ and $z(w_2,j)$ and between past and future claims

$$E\left(\left(z(w_1,i)-\mu_{w_1i}\right)\left(z(w_2,j)-\mu_{w_2j}\right)\right)=\operatorname{cov}\left(z(w_1,i),z(w_2,j)\right)=0$$

$$E\left(\left(\hat{\mu}_{w_1i}-\mu_{w_1i}\right)\left(z(w_2,j)-\mu_{w_2j}\right)\right)=E\left(\left(\hat{\mu}_{w_2j}-\mu_{w_2j}\right)\left(z(w_1,i)-\mu_{w_1i}\right)\right)=0$$

Finally, from equation (3.4)

$$\begin{aligned}E\left(\left(\hat{\mu}_{w_1i}-\mu_{w_1i}\right)\left(\hat{\mu}_{w_2j}-\mu_{w_2j}\right)\right)&\approx\mu_{w_1i}\mu_{w_2j}Cov\left(\hat{\lambda}_{w_1}\hat{\lambda}_{w_2}\right)\\+\mu_{w_1i}\mu_{w_2j}E&\left(\left(\left(\hat{\lambda}_{w_1}-\lambda_{w_1}\right)\sum_k\left(\hat{\beta}_{w_{2k}}-\beta_{w_{2k}}\right)\frac{\partial}{\partial\beta_{w_{2k}}}\left(\ln\left(f\left(j,\beta_{w_2}\right)\right)\right)+\left(\hat{\lambda}_{w_2}-\lambda_{w_2}\right)\sum_k\left(\hat{\beta}_{w_{1k}}-\beta_{w_{1k}}\right)\frac{\partial}{\partial\beta_{w_{1k}}}\left(\ln\left(f\left(i,\beta_{w_1}\right)\right)\right)\right)\right)\\+\mu_{w_1i}\mu_{w_2j}E&\left(\left(\sum_k\left(\hat{\beta}_{w_{1k}}-\beta_{w_{1k}}\right)\frac{\partial}{\partial\beta_{w_{1k}}}\left(\ln\left(f\left(i,\beta_{w_1}\right)\right)\right)\right)\left(\sum_k\left(\hat{\beta}_{w_{2k}}-\beta_{w_{2k}}\right)\frac{\partial}{\partial\beta_{w_{2k}}}\left(\ln\left(f\left(j,\beta_{w_2}\right)\right)\right)\right)\right)\end{aligned}$$

Hence, the only expectations contributing to the mean square error (3.13) are (3.14) and

$$2\sum_{\substack{(w,j)\in\wp_r\\i<j}}E\left(\left(\hat{\mu}_{w_1i}-\mu_{w_1i}\right)\left(\hat{\mu}_{w_2j}-\mu_{w_2j}\right)\right)$$

3.4 PREDICTED OVERALL TOTAL AND ITS MEAN SQUARE ERROR

For $(w,j)\in B$ the unknown total claim amount can be defined as $Z=\sum_{(w,j)\in B}z(w,j)$ and

$$E(Z)=\sum_{(w,j)\in B}E\left(z(w,j)\right)=\sum_{(w,j)\in B}\mu_{wj} \tag{3.16}$$

$$E\left(\left(Z-\hat{E}(Z)\right)^2\right)=E\left(\left(\sum_{(w,j)\in B}\left(z(w,j)-\hat{\mu}_{wj}\right)\right)^2\right) \tag{3.17}$$

Equation (3.17) is equal to

$$\sum_{(w,j)\in B} E\left(\left(z(w,j)-\hat{\mu}_{wj}\right)^2\right)+2 \sum_{\substack{(w_1,i)\in B \\ (w_2,j)\in B \\ (w_1,i)\neq(w_2,j)}} E\left(\left(z(w_1,i)-\hat{\mu}_{w_1 i}\right)\left(z(w_2,j)-\hat{\mu}_{w_2 j}\right)\right) \tag{3.18}$$

$(w_1,j),(w_2,j)$ in (3.18) represent all the distinct combinations of paired elements in set B The left-hand term of (3.18) can be obtained from (3.7) and the right hand term from the results of (3.10) and (3.15).

4. CLAIMS RESERVING MODELS IN THE PRESENCE OF RISK DISTORTIONS

The generic models defined in sections 2 and 3 assume that a claims portfolio can be segmented into distinctive data sets, such that within each set there is a single underlying claims process. This assumption cannot be readily extended to a reinsurance claims portfolio, which generally contains contracts underwriting more than one type of actuarial risk, or reflect distortions resulting from portfolio transfers or excess of loss policies with different limits. Although the generic model would not be appropriate in such cases, it can still be used to construct more complex claims reserving models. As an example of the simplest possible case, consider a reinsurance book of business underwriting two distinct and independent actuarial risk groups. If the development of the losses emerging from each can be assumed to vary across underwriting years, with equivalent notation to (2.2), the systematic component of the reserving model would be given by

$$E\left(Y(w,j)\right)=C_{1w}P_1(w,j)+C_{2w}P_2(w,j) \tag{4.1}$$

where $P_1(w,j)$ and $P_2(w,j)$ are the percentage cash flow functions for each actuarial risk group, and C_{1w} and C_{2w} their ultimate claim amount functions. Equation (4.1) can be re-expressed as

$$E\left(Y(w,j)\right)=\left(C_{1w}+C_{2w}\right)\left(\frac{C_{1w}}{\left(C_{1w}+C_{2w}\right)}P_1(w,j)+\frac{C_{2w}}{\left(C_{1w}+C_{2w}\right)}P_2(w,j)\right) \tag{4.2}$$

The ultimate claim amount, the percentage cash flow and the hazard rate functions for the contract for underwriting year w and development period j are:

$$C_w=\sum_{k=1}^{2}C_{kw} \tag{4.3}$$

$$P(w,j)=\sum_{k=1}^{2}\vee_{w_k}P_k(w,j) \tag{4.4}$$

$$h(w,j)=\sum_{k=1}^{2}\wedge_{w_k}\left(\frac{\left.\frac{\partial P_k(w,z)}{\partial z}\right|_{z=j}}{(1-P_k(w,j))}\right)=\sum_{k=1}^{2}\wedge_{w_k}h_k(w,j) \tag{4.5}$$

where $\vee_{w_i}$ and $\wedge_{w_i}$ are weights defined as

$$\vee_{w_i}=\frac{C_{iw}}{\sum_{k=1}^{2}C_{kw}}$$

$$\wedge_{w_i}=\frac{C_{iw}(1-P_i(w,j))}{\sum_{k=1}^{2}C_{kw}(1-P_k(w,j))}=\frac{IBNR_{(w,j)i}}{\sum_{k=1}^{2}IBNR_{(w,j)i}}$$

Hence, consistently with (2.2)

$$E(Y(w,j))=C_wP(w,j)$$

The weights for the percentage cash flow and the hazard rate functions are intuitively obvious. The model can be easily generalized for a contract with n types of actuarial risks by replacing 2 by n in the above equations. When the percentage cash flow functions of individual actuarial risks all satisfy the criteria given in section 2.2, in general so will $P(w,j)$.

The estimation method selected to model reinsurance reserves for claims emerging from different types of actuarial risks will depend on the formulation of the claims reserving model. Equation (4.1) already excludes generalized linear modeling techniques.

5. CONCLUSION

Sections 2.2 and 2.3 show that the generic model brings to light and suggests innumerable types of claims reserving models: incremental and cumulative, hierarchical and non-hierarchical. When $\pi(w,j)$ is a probability density function $S(w,j)$ and

$$IBNR_{(w,s-w+1)}=C_wS(w,s-w+1)=\exp(\alpha_w)\int_{s-w+1}^{\infty}G(w,\lambda_w,z)dz$$

$$IBNR_s=\sum_{w=1}^{u}C_wS(w,s-w+1)=\sum_{w=1}^{u}\exp(\alpha_w)\int_{s-w+1}^{\infty}G(w,\lambda_w,z)dz$$

can be defined and the reserving models provides a sound statistical basis for the calculation of reserves and for subsequent portfolio analyses. Once the parameters of the reserving models have been estimated, when $\pi(w,j)$ is a standard probability density function, values of $S(w,j)$ and $P(w,j)$ would be readily available from most statistical packages. Hence, a limited amount of programming would be required to obtain the projected losses needed for portfolio modeling.

Although the upper limit in the *IBNR* integrals is ∞, the scope of the calculations of the mean square errors method in section 3 has to be defined. It is suggested that this should be limited to a delay period (ζ say), where the incremental percentage cash flow is negligible. However, the significance of any *IBNR* shortfall in underwriting year w should be assessed by calculating

$$IBNR_{(w,\zeta)} = \exp(\alpha_w)\int_{\zeta}^{\infty} G(w, \lambda_w, z)dz$$

The application of the results in section 3 is restricted to incremental claims reserving models, for which they are clearly intended, and for variance/covariance structures that can be easily extracted from the model. Bayesian Markov chain Monte Carlo methods, through the complete predictive distributions they can provide for the *IBNR* estimates, offer an alternative approach to estimate the mean square errors, capable of overcoming limitations imposed by the model's variance structure, and unrestricted in the scope of the model's predictions.

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