The Cost of Conditional Risk Financing

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Title

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Abstract

This paper develops a risk pricing procedure by examining the role of capital in an insurance transaction. An insurance transaction differs from an investment in that an insurer uses capital at the time a claim is settled rather than when the policy is issued, and only if the damages exceed the premium for the exposure. The premium for the risk transfer, based on treating the use of the insurer's capital as a loan to the policyholder, is the amount required to ensure that the insurer's risk and return are in balance, with the expected loan payment representing the risk margin in the premium. The insurer's cost for providing these loans depends on the returns available on alternate investment opportunities. Risk diversification within a market segment is assumed to benefit the policyholders through reduced prices, while risk diversification across market segments is assumed to primarily benefit the insurer through a reduction in its risk. The specific form on the insurer's risk pricing function can be determined provided that the insurer operates under a capital preservation criterion in which its losses in one market segment are financed out of the profits earned in other market segments. The paper extends the model to consider expenses, federal income taxes, investment income, supply and demand, the competitive market price, and the time value of money.

The Cost of Conditional Risk Financing

Introduction

In "Pricing for Systematic Risk," the author examined the effect of risk diversification on price within a portfolio. This led to the development of a model for pricing insurance exposures based solely on their contribution to the systematic risk of the portfolio. Unlike financial pricing methods, the resulting prices were determined without regard to the insurer's capital, capital allocation techniques, risk adjusted rates of return, or the insurer's cost of capital. The model also placed no restrictions on the size of the portfolio. The portfolio might correspond to a single market segment for a single insurer or to the entire book of business for the insurance industry. In addition, the risk margin for each exposure was determined by allocating the risk margin for the portfolio based on the exposure's systematic risk. The model was not able to provide any information regarding the proper risk margin for the portfolio.

The purpose for this paper is to continue the examination of risk diversification and the insurer's price. Feldblum (1992) and D'Arcy and Dyer (1997) discuss a variety of financial methods that can be used to evaluate insurance risk margins. The Capital Asset Pricing Model has already been considered in "Pricing for Systematic Risk." Discounted cash flow methods, including net present value and internal rate of return models, are also frequently used for this purpose. One problem with discounted cash flow models is that the risk margin is based on the timing of the expected cash flows rather than on the uncertainty of the potential damages. As noted in Mango (2003), this type of approach evaluates the loading for risk "in an essentially deterministic framework." In addition, discounted cash flow models treat an insurance transaction as an investment decision. Capital is allocated to an insurance policy at the time the policy is issued, but unlike an actual investment, the capital is not actually spent. The reason this is misleading is that an insurer uses its capital at the time a claim is paid rather than at the time the policy is issued, and only if the damages exceed the premium for the policy. This differs from an investment in that the amount of capital required is not a fixed amount determined in advance. In order to provide a proper examination of the role of capital in support of an insurance transaction, the analysis needs to consider the gain or loss for each potential outcome. This approach is developed in the following section.

The Role of Capital in an Insurance Transaction

In order to examine the use of the insurer's capital in support of an insurance transaction, the following discussion will assume that all damages occur and are paid at time 0 and that the policy has no transaction expenses. Unless otherwise specified, the policyholder is assumed to purchase full coverage so that the indemnity payment equals the damages incurred. Each exposure will be considered in isolation from all other exposures so that the effect of risk diversification can be ignored. The notation P_X or $P(X)$ will be used to represent the insurer's premium for exposure X. Since the insurer requires an opportunity to earn a profit, the premium must be no less than the expected damages, $P_X >= E(X)$. In addition, the premium should not be so large that the insurer has no risk of incurring a loss, so that $P_X <= max(X)$. The insurer's use of its own capital depends on the actual outcome for the exposure. Whenever the damages x are less than the premium, the insurer would earn a profit of $P_X - x$, but whenever the damages exceed the premium, the insurer would need to contribute an amount $x - P_X$ of its own capital in order to settle the claim.

In order to illustrate the role of the insurer's capital in support of an insurance transaction, let X be an exposure having three outcomes of \$0, \$500, and \$3000 with probabilities of .25, .50, and .25, respectively. Table 1 shows the insurer's results for each outcome based on the selection of a premium identical to the expected damages of \$1000. The first four columns of Table 1 are self-explanatory. The column labeled "Return" represents the insurer's profit or loss for each outcome. Since the premium is equal to the expected damages of \$1000, the expected return shown in the final row is \$0. The final column, labeled "Deficit," shows how much capital the insurer contributes in order to settle the damages for each outcome.

Consider each of the outcomes for this transaction. If outcome A or B were to occur, the insurer would earn a profit of \$1000 or \$500, respectively. The insurer can use these profits in any manner it chooses, such as by adding these gains to its capital account, paying bonuses or stockholder dividends, or spending the gains in some other fashion. The only limitation on these funds is that they are not returned to the policyholder. The reason for this restriction is to prevent the policyholder from diversifying its risk over time. Otherwise, if the policyholder could offset the losses in one year with the profits in another year, its premium would be equal to its expected damages. As a result, the insurer would have no opportunity to earn a profit and insurance would not exist.

Table 1
Insurer Financial Outcomes Based on a Premium of \$1000

Outcome	Premium	Damages	Probability	Return	Deficit
A	1000	0	.250	1000	0
B	1000	500	.500	500	0
C	1000	3000	.250	-2000	2000
Expected	1000	1000	1.000	0	500

Next, consider the insurer's results if outcome C occurs. In this situation, the insurer would contribute \$2000 of its own capital in order to fund the deficit. In order to recover its loss on the transaction, the insurer could treat the \$2000 as a loan that the policyholder would repay in future years. In order for the insurer to replenish its capital funds by the time of the next expected occurrence of outcome C, the loan would need to be repaid within four years in accordance with its annual probability of .25. Disregarding interest charges, the annual payment on the loan would be equal to the expected deficit of \$500. In the subsequent year, the premium would be \$1500, the sum of the expected damages of \$1000 and the loan payment of \$500. This amount would be paid each year until the next occurrence of outcome C or until the loan is fully repaid, whichever came first. Since more than one loan may be outstanding at the same time, the policyholder's premium may go up or down over time in response to the actual damages incurred. Since the insurer's funds are recovered in the years following the capital contribution, the procedure illustrated in Table 1 can be considered to determine a retroactive premium for the exposure.

From the insurer's perspective, the problem with the pricing procedure described in Table 1 is that it requires the insurer to recover its capital contributions from the policyholder in the years following the occurrence of outcome C. Since the policyholder can purchase the next year's policy from another provider, the insurer has no assurance of recovering its capital contribution. One way the insurer can address this problem by charging for the anticipated loan payment in advance of when the capital contribution is needed, rather than for the years following the occurrence of outcome C. Since the expected loan payment in Table 1 is $500, this suggests that the initial premium for the transaction should be $1500 rather than $1000. Table 2 evaluates the impact on the insurer's financial results using the revised premium of $1500. The revision to the premium changes the insurer's return and deficit for each outcome. For example, outcome C of Table 2 now shows that the insurer needs to contribute only $1500 of its capital in order to fund the deficit. This scenario also differs from the previous example in that the insurer expects to earn a profit of $500 in each year. Since the expected profit exceeds the loan payment (i.e., the expected deficit) of $375, the insurer expects to recover its capital contribution in three years rather than four. Consequently, the premium of $1500 can be considered to consist of the expected damages of $1000, the expected loan payment of $375, and an additional charge of $125.

Table 2
Insurer Financial Outcomes Based on a Premium of $1500

Outcome	Premium	Damages	Probability	Return	Deficit
A	1500	0	.250	1500	0
B	1500	500	.500	1000	0
C	1500	3000	.250	-1500	1500
Expected	1500	1000	1.000	500	375

Since the premium of $1500 in Table 2 overcharges the policyholder for the expected damages and the expected deficit, another scenario can be tested using an indicated premium of $1000 + $375, or $1375. This process can be continued for several more iterations until the process converges to a premium of $1400. At this point, the insurer's expected return (i.e., its expected profit) is equal to its expected deficit, as shown in Table 3.

Table 3
Insurer Financial Outcomes Based on a Premium of $1400

Outcome	Premium	Damages	Probability	Return	Deficit
A	1400	0	.250	1400	0
B	1400	500	.500	900	0
C	1400	3000	.250	-1600	1600
Expected	1400	1000	1.000	400	400

This process determines what might be described as the optimal retroactive premium for the exposure. Since the insurer charges the expected rather than the actual deficit, the premium of $1400 in Table 3 can also be interpreted as being the insurer's prospective premium for the transaction. At this price, the profits earned in years with favorable outcomes are just sufficient to pay for the insurer's capital contributions in the unprofitable years. Over a four-year period, the policy in Table 3 would be expected to generate a profit of $1600, which is just enough to fund the insurer's expected capital contribution of $1600 over the same period. The premium, which consists of the expected damages of $1000 and the expected loan payment of $400, corresponds to a loss cost multiplier of 1.40. Even though the insurer can still lose money over the short term, the premium should be satisfactory over the long term since the policyholder pays for the expected use of the insurer's capital.

The Insurer's Cost for Providing Capital

The pricing procedure developed in the previous section allowed the policyholder to borrow the insurer's capital without paying interest on the loan. It also considered a payment in the future as being equivalent to a payment in the present. The procedure can be made more realistic by including interest on the borrowed funds and by discounting the future payments to present value at the risk-free rate. For example, suppose that the insurer charges interest at the risk-free rate of 3%. Since the interest rate and the discount rate are identical, this has no effect on the premium of $1400 shown in Table 3. For outcome C, the insurer would contribute $1600 of its capital, which the policyholder would repay over four years with an annual payment of $430.44. Since the present value of this set of payments is $1600, the expected risk and return in Table 3 would be unchanged.

Under normal conditions, the interest rate charged on any loan should be expected to be greater than the risk-free rate. Suppose that the insurer obtains the necessary funds to provide the loan from its maturing investments. Since the insurer has the option of reinvesting the funds rather than providing the policyholder loan, the interest rate on the loan should account for the lost investment income potential on the insurer's capital. As a result, the interest rate being charged should be competitive with the returns available on other investments whose term is similar to the term of the loan. The risk of the various investment opportunities may also need to be taken into account in making this decision.

The example shown in Table 4 evaluates the required premium for the same exposure X considered in Table 3, but now charges 8% interest on the borrowed funds. The present value of the cash flows is based on a risk-free rate of 3%. In order for the transaction to be self-supporting over the long term in the same manner as the premium in Table 3, the expected return and the present value of the future loan payments should be in balance. Given the occurrence of outcome C, the insurer would contribute $1561.81 of its capital in order to settle the claim. At an 8% interest rate, the annual payment on the loan would be $471.54. Based on a risk-free rate of 3%, the present value of the four loan payments is $1752.77, a surcharge of roughly 12% over the amount of the loan. Since outcome C occurs with probability .25, the expected present value of these payments is $438.19, which matches the insurer's expected return. The premium of $1428.19 is an increase of $38.19 over the premium in Table 3.

Table 4
Cost of Conditional Risk Financing at an 8% Interest Rate

Outcome	Premium	Damages	Probability	Return	Deficit	Annual Payment	PV of Payments
A	1438.19	0	.250	1438.19	0	0	0
B	1438.19	500	.500	938.19	0	0	0
C	1438.19	3000	.250	-1561.81	1561.81	471.54	1752.77

Expected	1438.19	1000	1.000	438.19	390.45	---	438.19

This result can also be expressed algebraically. Let P represent an arbitrarily selected premium for the transaction. Given this premium, the insurer's return for outcome i is $P - x_i$ so that its expected return is $P - E(X)$. Whenever the damages x_i exceed the premium P, the insurer contributes capital of $x_i - P$ to settle the claim. Since the loan should be repaid by the time of the next expected occurrence of x_i, the term of the loan is $1/p_i$ years, where p_i is the probability of the outcome. A surcharge is included on each loan in order to recognize the insurer's cost of providing the funds. In lieu of an interest rate, the surcharge will be represented by a factor s_i that depends on the term of the loan. While this point is not essential for this discussion, it can be assumed that $s_i > s_j$ whenever $p_i < p_j$. Using this notation, the policyholder's cost of borrowing capital of $x_i - P$ is $(x_i - P)s_i$, so that its expected cost is $\sum_{xi>P} (x_i - P)s_ip_i$. Based on the procedure described above, the required premium P_X for the transaction is the value at which the premium is just sufficient to cover the expected damages plus the expected cost of the loan. That is, P_X is the solution to:

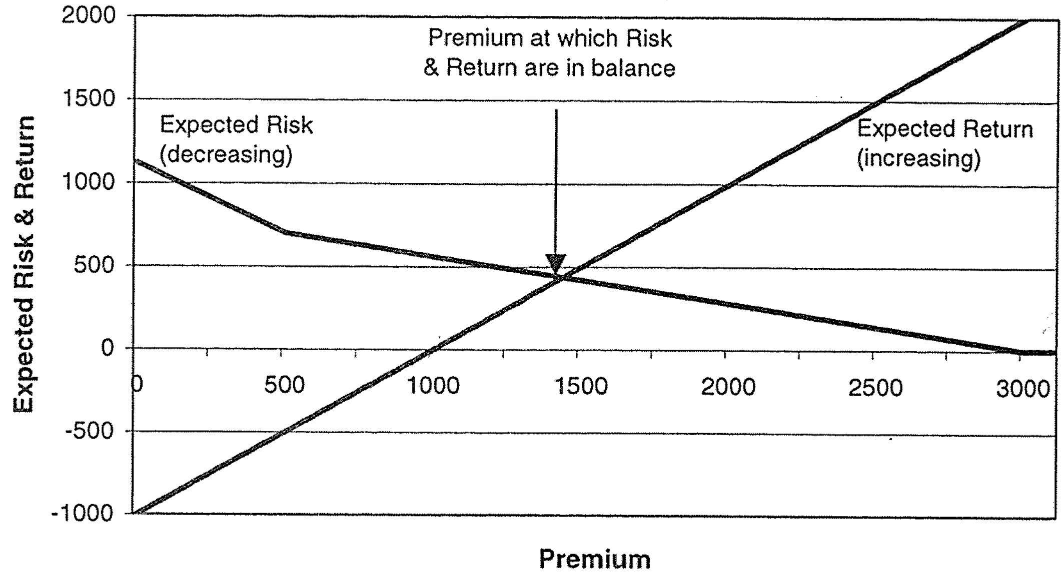
$$(1) \qquad P - E(X) = \sum_{xi>P} (x_i - P)s_ip_i$$

This result can be expressed more concisely by defining $(x_i - P)s_i$ as the risk for any adverse outcome x_i, that is, those outcomes for which $x_i > P$. With this definition, the right hand side of equation (1) represents the expected risk for the transaction while the left hand side represents the expected return. On this basis, P_X is the premium for which:

$$(2) \qquad \textit{Expected Risk} = \textit{Expected Return}$$

Figure 1 provides an illustration of the use of equation (1) to determine the premium for an exposure. The calculations are based on the three-outcome scenario in Table 4. Notice that the expected return is an increasing function while the expected risk is a decreasing function of P. At $P = 0$, the expected return is negative while the expected risk is positive. At $P = max(X)$, the expected return is positive while the expected risk is 0. Since the expected risk and the expected return are continuous functions of P, the two curves intersect at a single point. The intersection point determines the premium P_X where the expected risk and expected return are in balance.

Figure 1
Risk vs. Return for a Single Exposure



From the insurer's perspective, the disadvantage to equation (1) arises from its use of a different surcharge factor s_i for each probability p_i. The pricing procedure can be made more practical by replacing the insurer's surcharge factors s_i with a single surcharge factor α that represents its average cost of conditional risk financing. Since each s_i factor represents a surcharge, the minimum permissible value for α is 1. With this modification, the insurer's pricing equation for a discrete valued random variable becomes:

(3) $$P - E(X) = \alpha \sum_{xi>P} (x_i - P)p_i$$

The general pricing formula for any random variable X is:

(4) $$P - E(X) = \alpha \int_{x>P} (x - P)\, dF(x)$$

Equation (4) will be referred to as the risk pricing model in the remainder of this paper. The equation indicates that the model determines the risk margin based solely on the uncertainty of the damages for the exposure. Other risks, such as liquidity or reserving risks, have no bearing on the indicated price. This means that pricing risks, such as changes in rate adequacy across an entire insurance market pricing cycle, also have no effect on the price for the exposure. This prevents an insurer from being rewarded with a higher profit margin simply due to its earlier pricing decisions. The model also makes no differentiation between the profitability of stock and mutual companies. In addition, the model isn't concerned with the amount of capital held by the insurer or its target return on equity. Instead, capital is taken into account only through the amount of capital consumed for each outcome.

Self-Insurance

In the previous discussion, the insurance transaction focused exclusively on the insurer's point of view. By considering this transaction from the policyholder's perspective, the policyholder's self-insurance premium for the exposure can be determined. In place of a premium, the self-insurance program would have an out-of-pocket limit that would represent the maximum annual amount that the self-insurer would pay to reimburse any damages. The funds to pay for damages in excess of the out-of-pocket maximum would need to be obtained from other sources. The cost of these funds could differ depending on their source. For instance, relatives or friends of the self-insurer could provide the funds as a gift rather than as a loan. Another possibility is that they might loan the money to the self-insurer at below market rates. Funds could also be borrowed from banks or other commercial lenders at a higher interest rate. Since the self-insurer would want to prevent its debts from accumulating, each loan should be repaid by the time of the event's next expected occurrence in $1/p_i$ years. By viewing the policyholder as the insurer, this makes it possible to use equation (1) to determine the self-insurance price for the exposure.

Even if insurance is more expensive than a policyholder's self-insurance price, there may be reasons other than price for the policyholder to continue to purchase insurance. One of the benefits to self-insurance is that it avoids some the insurer's expense loadings, such as commissions and premium tax. An advantage to insurance is that it provides financing for the potential damages at a predetermined cost. For self-insurance, financing would be obtained after the damages occur. A self-insurer takes the risk that interest rates may be excessive and that financing may not be available at any price. If this prevents the damaged property from being repaired, it can have the effect of converting a partial loss into a total loss. A decision to self-insure would need to consider whether the benefits of self-insurance offset its disadvantages.

Risk Diversification and Price

Risk diversification has two effects on an insurer's price. As discussed in "Pricing for Systematic Risk," diversification reduces the price an insurer requires for a portfolio of exposures. In addition, it affects the insurer's price for every exposure within that portfolio. The paper also noted that a portfolio can be defined as a single market segment or as all market segments combined.

One way to provide a solution to the problem of insurance pricing is to assume that each insurer diversifies its risk over all market segments. If $W_1, \ldots, W_n$ represent the cumulative exposures for all n market segments, and $W_T = \sum W_i$ is the total risk exposure over all market segments, then the systematic risk pricing model can be used to determine the risk margin for each market segment. Under the assumption that the damages in distinct market segments are independent, the risk margin M_i for market segment W_i is $k\sigma_i^2$, where $k = M_T / \sigma_T^2$. Unfortunately, this solution has several shortcomings. First, insurance exposures are not actively traded in a secondary market, undermining the entire rationale for this approach. Second, this method does not resolve the problem of determining the appropriate reward associated with taking risk. Instead, it simply shifts the problem from the individual market segments W_i to the insurance market W_T as a whole. Third, market competition may permit insurers to price each market segment for its own risk. Due to the absence of a secondary market, the market pressures that would cause insurance prices to decrease in response to risk diversification across market segments may not exist.

There are several reasons that can be provided to support the view that insurance prices should reflect risk diversification within each market segment but not across market segments. For instance, in other industries, companies use diversification across market segments to reduce their risk rather than to reduce the prices they charge their customers. Diversification across markets would be an ineffective business strategy if all of the benefits accrued to the customers through lower prices rather than to the company. Similarly, if expansion across market segments reduced insurer's profit margins without improving their risk vs. return tradeoffs, insurers would have little incentive to expand into new markets. A second point to consider is the overall profitability of the insurance industry. According to an ISO study of industry results, 2001 was the first year in history that the property/casualty insurance industry experienced a net loss after taxes. The assumption that insurers retain the benefits of risk diversification across market segments provides a reason why the industry is so consistently profitable, especially in comparison to the airline, steel, and automotive industries, all of which are frequently lose money. Third, insurers are obligated by the long-term nature of their liabilities to have sufficient capital to settle all claims. If an insurer lost a considerable portion of its capital, it might need to obtain additional funds in the capital markets. However, investors would generally prefer to use their money to fund investments that earn a future profit rather than use it to settle an insurer's debts. Since it may be difficult for an insurer to raise additional capital, capital preservation should be an important consideration in its business operation. One approach the insurer can use to improve its ability to preserve capital is to retain the benefits of risk diversification across market segments.

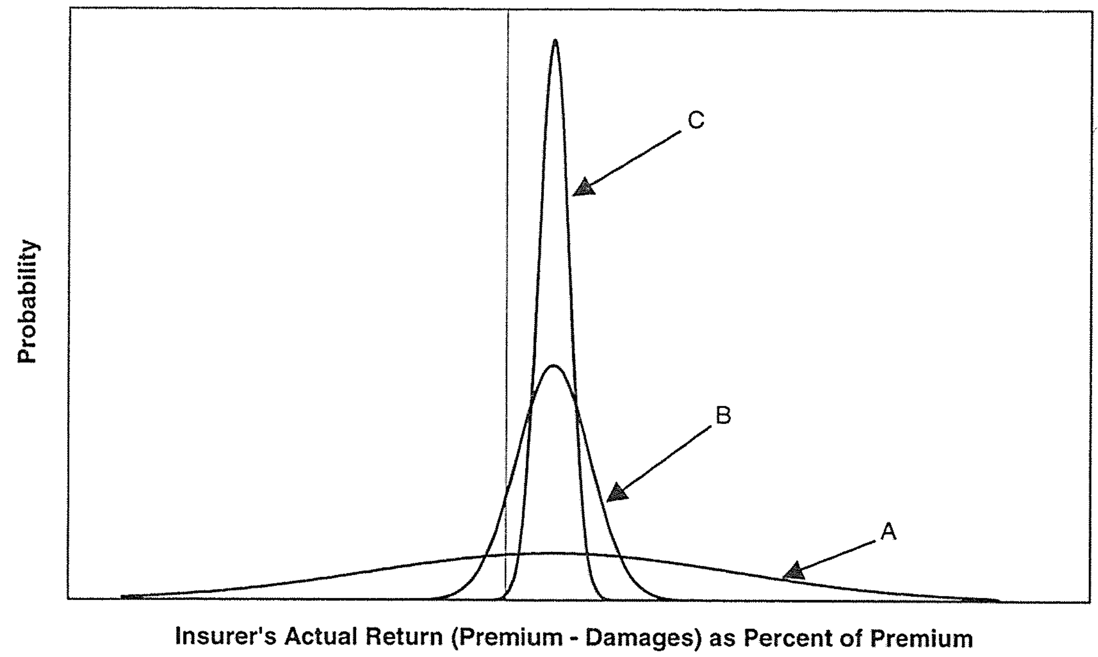
Based on the reasons provided above, this paper will adopt the assumption that an insurer's risk diversification within a market segment benefits its policyholders through reduced prices while diversification across market segments has little or no effect on the insurer's price. For the purpose of this discussion, a market segment will be considered to represent a group of exposures that are priced as a single entity, such as an insurer's Personal Automobile Liability book of business in a single state. For each market segment, the risk pricing model in (4) indicates that the insurer expects to be profitable but that it may experience a loss in any given year. Since the insurer would retain the benefits of risk diversification across market segments, this would reduce the insurer's probability of experiencing a loss over its entire portfolio. If the insurer is sufficiently well diversified and charges the prices indicated by the risk pricing model, it may be possible for it to be profitable over its entire portfolio virtually every year. Whether this is a reasonable characterization of the experience for large, well-diversified insurers will be left to others to address.

Figure 2 illustrates the effect of risk diversification across market segments on the insurer's financial results. The term "exposure" now refers to the experience of a single market segment. Each market segment is assumed to be priced for its own risk based on the risk pricing model in (4). For a single market segment, indicated by distribution A, the insurer has a significant probability of experiencing a loss. If it participates in five similar market segments, as indicated by distribution B, the insurer has the same expected profit as a percent of premium as for distribution A but with a much lower probability of experiencing a loss. If the insurer participates in 12 market segments, as indicated by distribution C, the insurer's expected profit as a percent of premium is still unchanged, but its probability of a loss becomes minimal. If the assumption that the insurer retains the benefit of risk diversification across market segments is valid, then the distribution becomes narrower, the profit becomes more certain, and the insurer's probability of a loss on its entire portfolio goes to zero as the number of market segments increases

One point that should be made about the results described above and shown in Figure 2 is that the description of the effect of risk diversification across a portfolio is more relevant to a large, well-diversified, primary insurer than it is to a reinsurer. Primary insurers tend to write relatively small independent exposures. Due to its small size, the outcome for any particular exposure would have little effect on the insurer's expected aggregate profit. In comparison, a reinsurer's portfolio may consist of exposures that are relatively large relative to the expected aggregate profit. Another issue is that reinsurance exposures may be more likely to be correlated with one another. Since this limits the reinsurer's ability to diversify risk across its portfolio, the reinsurer would be at greater risk of experiencing a loss on its entire portfolio. This suggests that a reinsurer's portfolio may be more consistent with distribution B rather than distribution C in Figure 2. Reinsurance and primary insurance also differ in that reinsurers tend to have long-term relationships with the primary carrier. Rather than attempting to earn a profit each year, a reinsurer may instead focus on the long-term profitability of the relationship. In this situation, the reinsurer could choose to recover its capital contributions retroactively in the years following a loss using the approach described in the discussion of Tables 1 and 2. Since a retroactive pricing strategy is inconsistent with the risk pricing model in (4), the insurer's aggregate profit and loss distribution would differ from those shown in Figure 2. Retroactive pricing may be a realistic approach for reinsurers to use since the damage distributions for their exposures may be uncertain. However, since the focus of this paper is on risk margins for prospective pricing, retroactive pricing in the reinsurance industry will not be given further consideration.

Before returning to the subject of risk diversification, a comparison between the reinsurance pricing approach described by Mango (2003) to the method described here may be worthwhile. In Mango's approach, "uncertainty is reflected between scenarios, not within them." His approach evaluates each scenario individually in order to determine the capital consumed for each outcome. His approach of evaluating risk by treating each outcome individually is consistent with the approach used in this paper. In addition, Mango's definition of the capital consumed for each outcome corresponds to the "deficit" as defined in Tables 1-4. Another similarity is that Mango determines a risk loading for each business segment rather than for the company as a whole. This is consistent with the earlier discussion of pricing each market segment for its own risk. One difference between the two methods is that Mango's surcharge on the capital consumed is based on the quantity consumed, while the pricing model in (1) bases the surcharge on the term of the loan. The risk pricing model in (4) uses a uniform surcharge factor in order to eliminate the need to identify the source of money used to fund the deficit. A second difference is that the risk pricing model in (4) considers the cost of the capital consumed to be the risk margin in the premium. Mango appears to develop the surcharge for capital consumption as a separate charge in addition to the risk margin in the premium. However, a careful examination of his approach suggests that this is not actually the case. The third and most important difference between Mango's approach and the pricing method described in this paper is that Mango focuses on the special problems of reinsurers. His approach develops the surcharge for capital consumption for each business segment individually, without examining the offset of profits and losses over the reinsurer's entire portfolio. The ability of an insurer to use diversification to reduce its risk and its capital consumption over their portfolio is the basic subject of this paper. In particular, this paper focuses on well-diversified insurers that are capable of successfully using diversification across market segments to eliminate almost all of their insurance risk, as indicated by distribution C in Figure 2.

Figure 2
Distribution of the Insurer's Returns



Risk diversification across market segments has a direct implication on the insurer's cost of providing conditional risk financing. In particular, it makes it possible to determine the value for α in equation (4). Consider an insurer that has successfully diversified its exposure over a large number of market segments. To illustrate, assume that the distribution of the insurer's outcomes for each market segment are similar to distribution A in Figure 2 but that its average results over its entire portfolio are similar to distribution C. Recall that the insurer uses its own capital only if the damages exceed the premium. Since the insurer is almost certainly profitable over its entire portfolio, the funds needed to pay for a deficit in the unprofitable market segments can be obtained from the gains in the profitable segments. The insurer would not be required to use its own capital to support its operation since it can obtain the required funds from the profits earned in the remaining market segments. In this sense, the policyholders would provide the funds needed to support the insurance operation without any use of the insurer's capital. Since the policyholder would no longer need to borrow the insurer's capital, the surcharge parameter α can be reduced provided that the insurer's portfolio distribution is still profitable. The insurer could continue to enter additional market segments until it reached the minimum value for α of 1. For a well-diversified insurer, this implies that the surcharge parameter α in equation (4) should be equal to 1.

The requirement that α must be equal to or greater than 1 can be related to existing research on risk pricing. A value for α of 1 implies that a well-diversified insurer should price each market segment so that the expected gain is twice its expected loss. This can be obtained by restating the risk pricing model in (4) for $\alpha = 1$ as:

$$(5) \qquad \int_{x<P} (P - x)\, dF(x) = 2\int_{x>P} (x - P)\, dF(x)$$

This result, which can be interpreted as *Expected Gain = 2*Expected Loss*, is consistent with Starmer (2000, p. 365), who states: "Benartzi and Thaler show that, assuming people are roughly twice as sensitive to small losses as to corresponding gains (which is broadly in line with experimental data relating to loss aversion), the observed equity premium in consistent with the hypothesis that investments are evaluated annually."

The conclusion that the surcharge factor α for a well-diversified insurer should be 1 is based on the assumption that the insurer has essentially eliminated its risk of experiencing a loss by diversifying over a large number of market segments. Other insurers may not be able to achieve this degree of financial security due to the limited size of their markets, statistical correlation between exposures, high expenses, or other reasons. In this situation, a value for α in excess of 1 may be needed in order for these insurers to achieve the same degree of financial security as a well-diversified insurer. For instance, suppose that the insurer's countrywide profit distribution based on a surcharge factor of $\alpha = 1$ is similar to distribution B in Figure 2. Since the insurer would have a significant probability of experiencing a loss, it would occasionally need to contribute its own capital to settle all of its claims. To correct this, the insurer can select a surcharge factor α in excess of 1 that shifts its aggregate profit distribution far enough to the right to ensure that it consistently earns a profit, but not so large that it becomes uncompetitive. In this sense, the value of α represents the insurer's degree of success in reducing its risk through diversification across market segments.

One limitation of the procedure described above is that since the tails of the distribution may extend to infinity, it may not always be possible to shift the distribution far enough to the right to ensure that the insurer would never need to use its own capital. One way to address this issue is for the insurer to purchase stop-loss reinsurance. The price for the reinsurance could be determined by applying the risk pricing model in (4) to the adverse outcomes in the insurer's aggregate profit distribution. Whether the net cost of the reinsurance is passed on to the policyholders would be a matter for the insurer to decide.

Comparison to Expected Utility Theory

The risk pricing model in (4) has been developed based on the role of capital in support of an insurance transaction. This formula can also be obtained more directly through the use of expected utility theory (EUT). One concern with the use of expected utility theory for the pricing of individual exposures is that it gives no recognition to the insurer's ability to reduce its risk through diversification within a market segment, as discussed in "Pricing for Systematic Risk." To avoid this concern, the EUT model will be used only to determine the insurer's price for a market segment as a whole. Given the insurer's price for the market segment, the prices for individual exposures within the market segment can be determined through the use of the systematic risk pricing model or any other method that provides an equitable allocation of the risk margin for the market segment to the individual policies that compose it.

Expected utility theory is a general decision making technique used by economists to evaluate an individual's choices under uncertainty. A detailed discussion of this subject can be found in Borch (1990) and Robison and Barry (1987). Expected utility theory is based on a series of axioms that describe an individual's preferences among simple and compound lotteries, as specified in Appendix A. The axioms are used to demonstrate the existence of a utility function that can be used to rank an individual's preferences. Since utility depends on preferences, each person may have a different utility function. An optimal decision is the one that maximizes an individual's expected utility.

The expected utility model is generally described in terms of a utility function $U(w)$, which is a continuous and increasing function of wealth, w. Utility is defined in terms of wealth rather than income in order to incorporate the individual's capital constraint into the evaluation of the optimal decision. Since the insurer is required to be risk averse, the utility function U is concave downward. The expected utility for an uncertain prospect (that is, a random variable) is determined from the utility for each outcome in combination with the probability distribution of the potential outcomes, i.e., $U(X) = \sum U(x_i)p_i$, where p_i is the probability corresponding to outcome x_i.

Traditionally, the insurer's certainty equivalent price for accepting a transaction for the uncertain damages X is defined to be the unique value $P(X)$ such that:

$$EU(w - X + P(X)) = U(w) \qquad (6)$$

The interpretation of this formula is that $P(X)$ is the price that makes the insurer indifferent between the two options of accepting the exposure X for the premium $P(X)$ and not accepting the exposure. This formula compares the expected utility of the insurer's final but uncertain wealth of $w - X + P(X)$ to the utility of its initial wealth of w. The utility of initial wealth, $U(w)$, can be arbitrarily selected to be 0.

Expected utility theory pricing has the following basic properties:

$$P(c) = c \qquad (7)$$

$$P(X + c) = P(X) + c \qquad (8)$$

$$E(X) <= P(X) <= max(X) \qquad (9)$$

$$\text{If } X(\omega) <= Y(\omega) \text{ for all outcomes } \omega, \text{ then } P(X) <= P(Y). \qquad (10)$$

One important feature of the expected utility model in equation (6) is its dependence on the insurer's initial and final wealth. However, this relationship between utility and wealth is not required by the axioms of expected utility theory. By eliminating the reference to the insurer's wealth in equation (6), the expected utility theory model can be restated in terms of the effect of the insurance transaction on the insurer's income:

$$EU(P(X) - X) = 0 \qquad (11)$$

The interpretation of this formula is that $P(X)$ is the price that makes the insurer indifferent between accepting the transaction for an uncertain gain or loss of $P(X) - X$ and not accepting the transaction.

At this point, assume that the insurer's price for a pro-rata portion of an exposure is the pro-rata price:

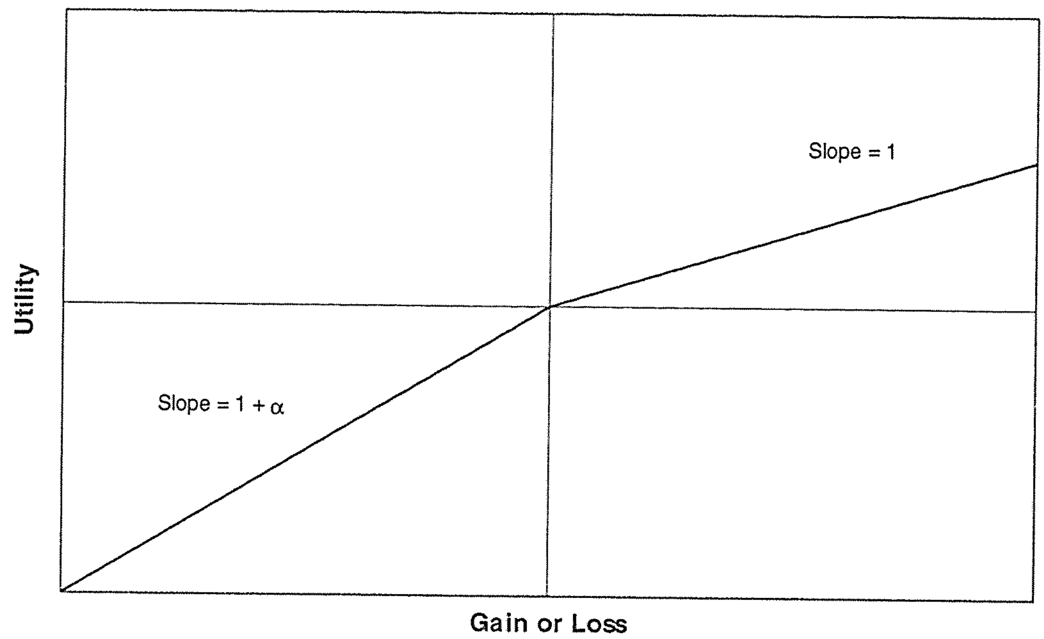
$$(12) \qquad P(aX) = aP(X) \text{ for } a >= 0$$

This property is essentially a statement about the effect of risk diversification on price. In "The Diversification Property," the pricing formula in equation (12) was demonstrated to be equivalent to the diversification property. This property states that $P(X + Y) <= P(X) + P(Y)$, with equality holding if and only if the two exposures are perfectly correlated with one another. The basis for the diversification property is that since diversification over imperfectly correlated exposures reduces the standard deviation, i.e., $\sigma_{X+Y} < \sigma_X + \sigma_Y$, it should also reduce the insurer's price. On the other hand, if the two exposures are perfectly correlated with one another, then no risk diversification occurs. Since this results in no reduction to the standard deviation, $\sigma_{X+Y} = \sigma_X + \sigma_Y$, there is no reduction in price.

It should be noted that the diversification property is not true for expected utility theory in general. For instance, if X is a Bernoulli random variable and the insurer's utility function $U(x)$ is $1 - e^{-0.2x}$, then it can be shown that $P(2X) > 2P(X)$. In addition, this particular utility function has the property that $P(X + Y) = P(X) + P(Y)$ whenever X and Y are independent. While this may appear to be an appealing characteristic, it implies that the insurer's financial condition (i.e., its utility) is not improved by insuring additional independent exposures. Moreover, the insurer's financial condition would become worse whenever the insurer's exposures are not all independent of one another. Since an insurer with this utility function obtains no benefit from risk diversification and may actually be in a worse financial condition after insuring a portfolio of exposures, the insurer has no incentive to provide coverage and may choose to withdraw from the insurance business instead.

The scalability property in (12) makes it possible to determine the general form of the insurer's utility function. Appendix B demonstrates that the utility function consists of two rays meeting at 0, as illustrated in Figure 3. The slope for the outcomes corresponding to gains can be arbitrarily selected to be 1, while the slope for the adverse outcomes is greater than or equal to 1. In this context, α represents the insurer's risk aversion parameter rather than its surcharge factor for the use of its own capital. Since the specific form of the utility function is known, it can be substituted into equation (11) in order to determine the insurer's certainty equivalent price. The resulting price function is identical to the risk pricing model in (4).

Figure 3
Shape of the Utility Function



The Standard Deviation Pricing Formula

The properties of expected utility theory in equations (7)-(10) and the scalability property in (12) make it possible to obtain a rough estimate of the required risk margin for each market segment. If the exposures in a market segment are independent, the central limit theorem states that the distribution of the average damages should be approximately normal. To make use of this result, consider a normally distributed exposure X with mean μ and standard deviation σ. Since X can be expressed as $\mu + \sigma Z$ where Z is a standard normal distribution, this gives $P(X) = \mu + \lambda\sigma$, where λ is defined as $P(Z)$. For an insurer with a risk aversion parameter of $\alpha = 1$, the value of λ as determined from equation (4) is approximately 0.3. Consequently, the risk margin for each market segment should be approximately 0.3σ.

As an application of this result, suppose that an insurer wants to determine the indicated rate change for a market segment. Assume that the trended current rate level loss ratios for the past five years are 0.70, 0.90, 0.80, 0.90, and 0.70, with a mean value of 0.80 and standard deviation of 0.10. Suppose that expenses are 35% of the premium and that the insurer expects to earn no investment income on the exposure. The total damages and expenses for the exposure can be expressed as $Y = X + e(Y)$, consisting of damages of X and expenses of $eP(Y)$. Applying (12), the premium for Y is $P(Y) = P(X)/(1-e) \cong (E(X) + 0.3\sigma)/(1-e)$. Using this result, the insurer's indicated rate change is (0.80 + (0.3)(.10))/(1-.35) – 1 = 27.7%. After applying the rate change, the indicated loss ratio and standard deviation become 0.627 and 0.078, respectively. Since the indicated risk margin at the revised rate level is 0.023 = (0.3)(.078) and the expense ratio is 0.35, the sum of the loss ratio, risk margin, and expense ratio is equal to 100% of the premium, as intended.

Prior to applying this result, it should be recognized that λ is approximately 0.3 when the insurer's risk aversion parameter α is 1. For other values of α, the value for λ would need to be recalculated. A second application of the standard deviation pricing formula is in testing the consistency of the insurer's risk margins across all of its market segments. If P, μ, and σ are for each market segment are known, and each market segment is approximately normally distributed, then the risk margins are consistent provided that $\lambda = (P - \mu)/\sigma$ is consistent across market segments.

Methods for Evaluating the Premium

In order to evaluate the certainty equivalent price for the uncertain damages, the risk pricing model in (4) must be solved for $P(X)$. This section reviews several techniques that can be used to determine the premium.

In simple cases, it may be possible to evaluate the integral directly. For example, suppose that the insurer's risk aversion parameter α is 1 and that X has the following distribution:

$$X = 1000 \qquad \text{where } Prob(X = 1000) = 0.50$$
$$X = 2000 \qquad \text{where } Prob(X = 2000) = 0.50$$

Since \$1500 = $E(X) <= P(X) <=$ \$2000, the expected risk component in (4) can be evaluated as $(\$2000 - P(X)) * 0.50$ while the expected return is $P(X)$ – \$1500. Equating the two results and solving gives $P(X)$ = \$1666.67. In this example, the premium consists of expected damages of \$1500 plus a risk margin of \$166.67.

The premium can also be obtained through a recursion process similar to that used in preparing Tables 1-3. Starting with an initial estimate P_1 of $P(X)$, a second estimate P_2 can be obtained by:

$$(13) \qquad \int_{x>P_1} (x - P_1)\, dF(x) = P_2 - E(X)$$

By adding P_1 to both sides of the equation, this can also be expressed as $P_2 = P_1 +$ *Expected Risk at* P_1 – *Expected Return at* P_1. For example, using the two outcome exposure introduced above, select P_1 as $E(X)$ = \$1500. Since the integral in formula (13) can evaluated as (\$2000 – \$1500) * 0.50 = 250, this gives a second estimate of the premium of P_2 = \$1750. After several iterations, the premium converges to \$1666.67.

A third method for evaluating the premium is to graph the expected risk and the expected return functions in equation (4), as shown in Figure 1. The intersection of the two curves determines the certainty equivalent price.

The fourth technique for determining the certainty equivalent price relies on the concept of synthetic probabilities. Synthetic probabilities represent a distortion of the actual probabilities to correspond to the risk of the exposure. Given the correct set of synthetic probabilities, the insurer's certainty equivalent price $P(X)$ can be evaluated as $E^*(X)$, where the expectation operator E^* is evaluated based on the use of the synthetic probabilities.

If the certainty equivalent premium $P(X)$ is known, the synthetic probabilities can be obtained by multiplying the actual probabilities by $(1 + \alpha)$ for those outcomes $x > P(X)$ and rescaling the entire set of probabilities to sum to 1. If $f(x_i)$ represents the density function or probability for outcome x_i, then the corresponding density function for the synthetic probability is $f^*(x_i)$, where:

(14)
$$f^*(x_i) = k(1 + \alpha)f(x_i) \quad \text{for } x_i > P(X)$$
$$f^*(x_i) = kf(x_i) \quad \text{for } x_i <= P(X)$$

The value of k is selected to ensure that the total synthetic probability is 1. A detailed discussion of synthetic probabilities is provided in the discussion of the arbitrage theorem in Appendix C.

Synthetic probabilities can be especially effective in evaluating the premium for an exposure with a finite set of outcomes. The initial step in the evaluation is to arrange the damage outcomes in increasing order. The second step is to assume that the true premium that falls between a selected pair of outcomes x_i and x_{i+1}. Select a provisional premium P_1 between x_i and x_{i+1} and use the synthetic probabilities constructed in (14) to calculate $E^*(X)$. If $E^*(X)$ falls between x_i and x_{i+1}, then the assumption is true and the price $P(X)$ for the exposure is $E^*(X)$. If not, another pair of outcomes x_j and x_{j+1} can be tested. The new pair of outcomes should selected so that the new provision premium P_2 falls between the values for P_1 and $E^*(X)$ from the previous step. This process can be continued iteratively until the premium is determined.

As an application of this procedure, consider the exposure specified in Table 1. Since the expected damages for the exposure are \$1000, the true premium must fall in the interval between outcomes B and C. For $\alpha = 1$, the synthetic probability procedure doubles the probability for outcome C to .500. After rescaling, the synthetic probabilities for outcomes A, B, and C become .200, .400, and .400, respectively. The expected damages using the synthetic probability distribution are \$1400, which agrees with the premium determined in Table 3.

The Effect of Expenses on Price

The risk pricing procedure developed above can be easily modified to incorporate the insurer's expenses. Let X represents the insurer's experience for a single market segment. Suppose that in accepting the uncertain damages X in exchange for a premium of P, the insurer also incurs various expenses. These expenses represent the transaction costs arising out of issuing and providing service on the policies in the market segment. Any overhead expenses that cannot be directly attributed to the market segment will not be considered in this analysis. Let Y represent the uncertain loss adjustment expenses. The fixed expenses, i.e., those that are fixed dollar amounts independent of the premium, will be designated by f. Let the variable expenses such as commissions and premium taxes be expressed as a percentage g of the premium so that the amount in dollars is gP. Any federal income taxes owed will depend on the insurer's pre-tax profit or loss of $P - (x + y + f + gP)$. Assume that federal income taxes are based on a flat rate of t so that the federal income taxes owed are $(P - (x + y + f + gP))t$. Combining these results, the insurer accepts the uncertain outcome Z in exchange for a premium of P_Z where Z represents the total of the insurer's costs:

(15)
$$Z = X + Y + f + gP_Z + (P_Z - (X + Y + f + gP_Z))t$$

Consider the simplest case, in which all expenses are fixed and taxes are zero. Setting Y, g, and t equal to 0 results in $Z = X + f$. Applying properties (7)-(10) to Z results in a premium of $P_Z = P(X) + f$. This indicates that the insurer's premium P_Z after taking expenses into consideration is equal to sum of the premium $P(X)$ based on no expenses plus the fixed expenses f. Following a similar approach, the premium for the exposure in equation (15) can be evaluated as:

$$P_Z = (P(X + Y) + f)/(1 - g) \qquad (16)$$

Notice that the federal income tax rate of t does not appear in the premium. This differs from the familiar ratemaking procedures described in Feldblum (1992) and Myers and Cohn (1987) that directly include federal taxes into the calculation of the required premium. The reason for this difference is that the Feldblum and Myers and Cohn models focus on obtaining a fair return for the investor, while the pricing model developed above looks at risk from the insurer's perspective. The interpretation of the result in (16) is that the taxation of underwriting gain and loss places the federal government in the role of a pro-rata reinsurer. The government can be considered to assume a portion, t, of the premiums, damages, and expense components of the insurance transaction, while the insurer retains the residual portion, $1 - t$, of the various components of the transaction. The insurer's risk and return components for the transaction are both reduced by the same factor, $1 - t$, resulting in no change to the insurer's certainty equivalent price.

The primary difference between income taxes and expenses is that expenses always reduce the insurer's income. Income taxes may be positive, negative, or zero, depending on the insurer's income. High expenses can make an otherwise profitable insurer unprofitable. Income taxes would reduce the insurer's income but would not make it unprofitable.

Before leaving this subject, two further points need to be clarified. First, the conclusion that federal taxes have no effect on the insurer's price applies only to a flat tax rate. Since the current federal income tax provisions differ from a true flat tax, a loading for income taxes may need to be included in the premium. However, this loading may be small enough to be ignored.

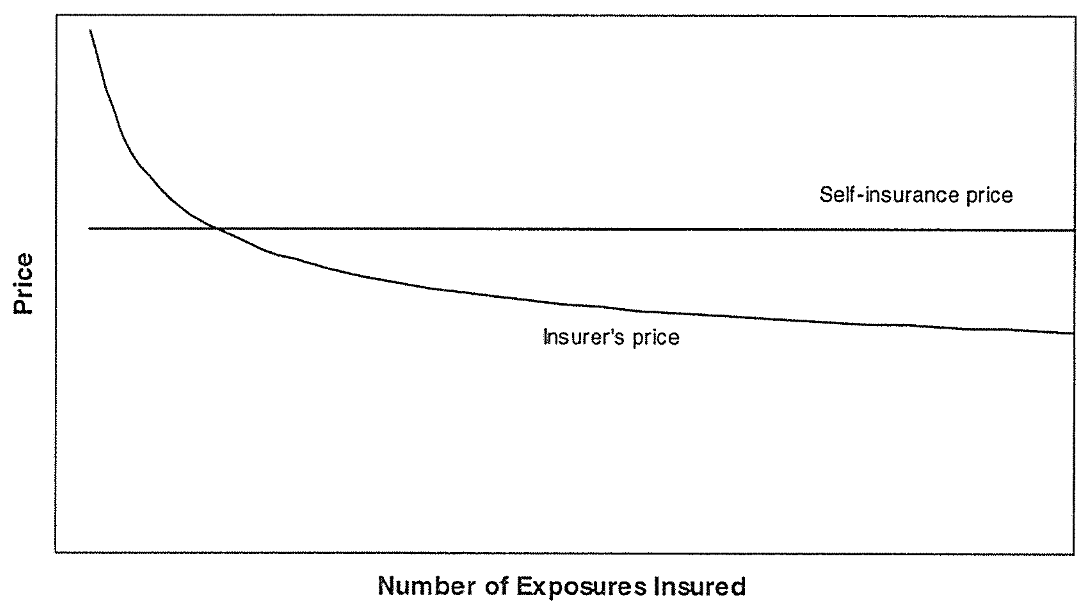
Second, even though federal income taxes have no effect on the insurer's price in a single market segment, they have a negative financial impact over the insurer entire operation. For example, suppose that distribution A in Figure 2 represents the insurer's experience for a single market segment. Since federal income taxes have no impact on the price the insurer charges, their primary effect is to narrow the width of the gain/loss distribution. This reduces the insurer's potential gain but also reduces its potential loss. Even though taxes have a neutral effect for a single market segment, this is not true over the insurer's entire operation. To illustrate this point, suppose that distribution C in Figure 2 represents the insurer's experience across its entire portfolio. Since the insurer has essentially no risk of experiencing a loss at this level of aggregation, federal income taxes provide no measurable benefit to the insurer in terms of reducing its risk of loss. Instead, taxes simply reduce the insurer's profits without providing an offsetting benefit.

The Mutually Acceptable Price

The following discussion returns to the subject of risk diversification within a market segment and its effect on the price for an exposure. In "Pricing for Systematic Risk," competitive pressures were described as compelling the insurer to reduce its prices to reflect its reduction in risk over the market segment. Even in the absence of competition, an insurer's price needs to be competitive with self-insurance. In order for a risk transfer to be acceptable to both parties to the transaction, the price offered by the insurer needs to be less than the policyholder's self-insurance price. Both prices can be determined using the risk pricing model in (4). However, the insurer's price includes its risk margin as well as an expense loading. If the expense loading is too large, the insurer's premium may exceed the policyholder's self-insurance price. In order to offer a competitive price, the insurer would need to take into account the risk diversification it achieves within the market segment in order to offer an acceptable price.

Suppose that the insurer has n independent and identically distributed exposures in market segment $Y = \Sigma X_i$, and assume that each policy has fixed transaction expenses of e. Using the approximation based on the standard deviation pricing formula, the insurer's required premium for the market segment as a whole is $P(Y + ne) = E(Y) + \lambda\sigma_Y + ne$, so that its price for an individual exposure X_i would be $P(X_i + e) = \mu + e + \lambda\sigma_X/\sqrt{n}$. As the number of exposures n increases, the insurer's price for each exposure would decrease. Provided that the transaction expenses e are not excessive (i.e., $\mu + e$ is less than the self-insurance price) and n is sufficiently large, the insurer's price would be less than the self-insurance premium as illustrated in Figure 4. In this situation, any premium between the insurer's price and the self-insurance price would be a mutually acceptable price for both participants in the risk transfer. However, if the insurer's expense loading e is excessive or the exposures are perfectly correlated with one another, a mutually acceptable price for the transaction may not exist.

Figure 4
Comparison of Certainty Equivalent Prices



The Effect of Deductibles on the Mutually Acceptable Price

The previous discussion found that a mutually acceptable price for a transaction exists provided that the insurer's expense loading is not excessive and that the insurer is able to diversify its risk over the market segment. Since the risk margin diminishes as the number of exposures increases, the existence of a mutually acceptable price hinges on the insurer's expense loading. The most convenient measure of an insurer's expense loading is its loss cost multiplier. For example, the exposure in Table 3 has a loss cost multiplier of 1.40 ($1400/$1000). Since this is large enough to accommodate a reasonable expense loading, a mutually acceptable price for this exposure may exist. For another exposure, the loss cost multiplier may be so small that the insurer would not be able to recover its expenses. For example, if the self-insurer's loss cost multiplier for another exposure is only 1.20 and the insurer's loss cost multiplier (representing the expense loading only) is 1.30, a mutually acceptable price for the risk transfer will not exist regardless of the number of exposures insured.

One technique an insurer can use to ensure the existence of a mutually acceptable price for an exposure is to offer a policy with a large deductible. For example, if the insurer offered a deductible of $500 on the exposure in Table 3, the loss cost multiplier for the excess coverage would rise to 1.60. Presumably, the self-insurer's loss cost multiplier for the excess coverage would be even higher. If the insurer's expense loading is 0.40 of the expected damages and its risk margin can be reduced to 0.20 through risk diversification, the insurer's actual premium would be based on a loss cost multiplier of 1.60, ensuring that a mutually acceptable price exists.

In determining the loss cost multiplier of 1.60 for the excess coverage, the insurer made the assumption that the policyholder evaluates the acceptability of the excess coverage independently of the decision to self-insure the damages below the deductible. More properly, the insurer also needs to consider whether the policyholder would prefer to self-insure both portions of the exposure. Based on the diversification property, $P(X + Y) <= P(X) + P(Y)$ for any two exposures X and Y. If X represents the retained exposure and Y the excess exposure, then the policyholder's self-insurance premium $P_S(X + Y)$ for the total exposure is less than the sum of its prices for the individual exposures, $P_S(X) + P_S(Y)$. In order for the policyholder to prefer to purchase the excess coverage for a premium of $P_I(Y)$ over self-insuring the total exposure, the combined premium $P_S(X) + P_I(Y)$ should be less than $P_S(X + Y)$ so that $P_I(Y) < P_S(X + Y) - P_S(X)$. Since the self-insurance premium $P_S(X)$ for the retained portion of the exposure in Table 3 is $428.57, the insurer's premium $P_I(Y)$ for the excess coverage must be no greater than $971.43. This limits the insurer's loss cost multiplier for the excess coverage to 1.55 rather than the 1.60 value determined above.

The Competitive Market Price

The policyholder's decision to purchase insurance or to self-insure an exposure is based on finding a mutually acceptable price for the transaction. By applying this analysis to every exposure, the supply and demand functions for the market segment can be determined.

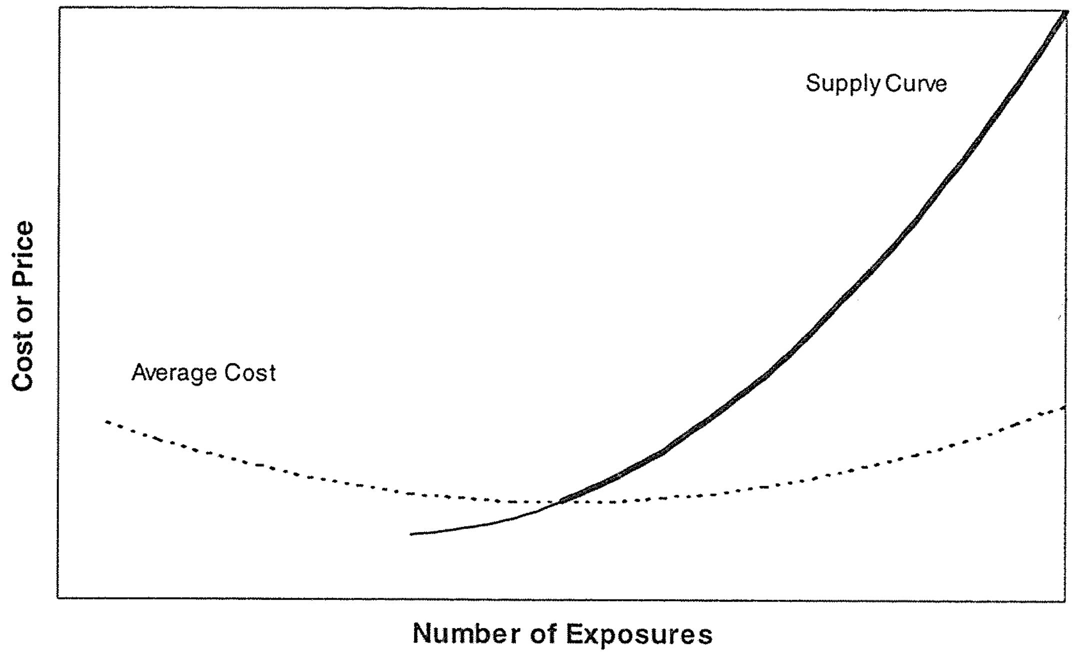
The first step in the analysis is to construct a demand curve for insurance. The demand curve represents the collective interest in purchasing insurance at a given price P. For any individual, the demand for insurance can be represented by a function whose value is 1 for any market price

P below the self-insurance price and 0 for any market price above that point. The market demand curve is the cumulative demand for insurance at each price, as measured by the total number of policies that would be purchased at a given price. As such, the market demand curve is simply the sum of the demand curves for each exposure.

The next step is to determine the supply curve for each insurer under conditions of perfect competition. These conditions require that each insurer have no influence over the market price and that exposures having identical risk characteristics are charged the same price. According to Thompson and Formby (1993), the supply curve for a particular company is based on its cost function. For an insurer, the total cost for a given number of exposures consists of the total expected damages, the insurer's fixed overhead expenses, the transaction expenses, and the insurer's cost of risk, where the cost of risk is the risk margin required by the risk pricing model in (4). The total cost can also be expressed as the sum of the insurer's certainty equivalent price for the market segment and the insurer's fixed overhead expenses. The insurer's total cost exceeds the insurer's breakeven cost for the market segment by the amount of the insurer's profit margin.

Once the insurer's total cost for the market segment is known, its average cost curve can be determined. This analysis will be limited to a discussion of the insurer's costs over the short term so that the insurer's staffing and other overhead expenses can be considered to be fixed costs. Due to risk diversification within the market segment and to a portion of the costs being fixed, the insurer's average cost curve should decrease as the number of exposures increases, as indicated in Figure 4. However, this overlooks the possibility that exposures differ in their risk characteristics. Due to imperfect information, rapid expansion by an insurer is likely to result in a lower quality book of business than its existing portfolio, thereby increasing the insurer's loss costs. As the number of exposures increases, the insurer will eventually reach a point of diminishing returns, i.e., a point where its average costs begin to increase. Consequently, the insurer's short-run average cost curve will be assumed to be "U" shaped as shown in Figure 5.

Figure 5
Insurer's Supply Curve



Differentiating the formula *Total Profit = Total Revenue – Total Cost* shows that the insurer maximizes its profit at the point where its marginal cost is equal to its marginal revenue. Since the market is assumed to be perfectly competitive, the insurer's marginal revenue is identical to the market price, where the market price can be any value along the vertical axis in Figure 5. If the insurer's average cost curve is "U" shaped as in Figure 5, the insurer's marginal cost curve will intersects the average cost curve at its minimum and remain above the average cost curve for all greater quantities sold. Since the insurer maximizes its profits by insuring the number of exposures indicated by its marginal cost curve, this means that the insurer's supply curve coincides with its marginal cost curve. The supply curve extends below the average cost curve until it reaches the insurer's average variable cost curve. At the intersection of the supply curve and the average cost curve, the insurer recovers all of its costs and earns a normal profit. For any market price above this point, the insurer would earn profits in excess of the normal profit, as measured by the vertical distance between the supply curve and the average cost curve. If the market price is below the intersection of the supply curve and the average cost curve, the insurer would earn less than its normal profit or may be unprofitable. If inadequate profits or losses persist over the long term, the insurer can withdraw from this market segment in order to pursue other market segments in which it can obtain a more adequate return.

The final step in the analysis of the competitive market price is to compare supply and demand for the market as a whole. Given the supply curve for each insurer, the market supply curve is the sum of the number of exposures that each insurer is willing to insure at each price. In a perfectly competitive market, the market price is the price at which the market supply and market demand curves intersect. At this price, the return earned by each insurer would differ due to the differences in their average costs.

The Time Value of Money

Up to this point, the discussion has been limited to exposures with uncertain damages whose actual outcome is discovered immediately after the risk transfer has been completed. The model will now be extended to damages in which the outcome is realized at a future time.

Suppose that all of the potential outcomes for X are realized at a future time t instead of time 0. The certainty equivalent price $P_t(X)$ for the risk transfer can be evaluated at time t using the risk pricing model in equation (4). Since this is a fixed amount, it can be discounted to time 0 at the appropriate risk-free rate, v_t, so that the price at time 0 to transfer the uncertain damages is:

(17) $$P_t(X)v_t$$

The certainty equivalent price can also be evaluated using a second approach, by discounting each outcome for X to time 0 at the risk-free rate. Consider the random variable Y, defined as the value of the uncertain outcome X discounted to time 0 at the risk-free rate:

(18) $$Y = Xv_t$$

Under this approach, the outcomes for X are now considered to occur at time 0. The certainty equivalent price for Y at time 0 is:

(19) $$P_0(Y) = P_0(Xv_t) = P_0(X)v_t$$

The prices in (17) and (19) are identical provided that $P_t(X) = P_0(X)$, i.e., that the insurer's prices are consistent over time. For this to be true, the insurer's risk aversion parameter α must be stable over time. Since the risk aversion parameter α for a well-diversified insurer is 1, either of the two methods for determining the present value for an exposure with uncertain future outcomes will produce the same price.

In order to extend the result developed above to exposures with multiple payments made over time, let each x_i for $i = 1$ to m represent a single outcome for the uncertain damages X. Each outcome x_i represents a stream of cash flows $x_{i,j}$ with $j = 1$ to n, where j indicates the point in time at which the payment is made, so that:

(20) $$x_i = (x_{i,1}, x_{i,2}, x_{i,3}, \ldots, x_{i,n}) \text{ for } i = 1 \text{ to } m$$

For each outcome x_i, the individual payments $x_{i,j}$ can be discounted to present value at the appropriate risk-free rate v_j. Replacing each outcome x_i with its present value $y_i = \Sigma x_{i,j}v_j$, the certainty equivalent value for a stream of payments X will be defined as $P(Y)$. This method meets two basic requirements. First, it produces the correct price for an exposure whose cash flows are certain, and second, the method is consistent with the pricing requirement in (19) for an exposure whose uncertain outcomes are paid at time t. Appendix C provides another perspective on this result by demonstrating the consistency of the risk pricing model with the arbitrage theorem.

Conclusion

The procedures commonly employed by actuaries to determine the appropriate risk margin in the insurer's rate are derived from financial analysis methods that evaluate risk from the perspective of the investor. This paper describes an alternate approach that determines risk margins based on the risk of the exposure, taking into account the insurer's ability to diversify its risk within and across market segments.

The pricing procedure is based on two concepts. The first is the role of capital in an insurance transaction. Capital is needed only when the damages for an exposure exceed the premium. The risk pricing model in (4) determines the prospective price for an exposure based on the requirement that the policyholder repay the insurer's capital contribution as if it were a loan. The pricing formula is consistent with an expected utility theory model based on a utility function consisting of two rays meeting at the origin. The price for a transaction is determined by finding the point at which the risk and return are in balance. Based on this approach, federal income taxes should have only a very limited or no effect on the insurer's price. The model also provides a means for determining the price for an exposure with future cash flows by discounting each cash flow to present value at the appropriate risk-free rate.

The second concept is that risk diversification affects price. The insurer is able to diversify its risk both within and across market segments. The paper assumed that risk diversification within market segments benefits the policyholders by reducing the insurer's price. Since insurers need to conserve capital, risk diversification across market segments has been assumed to have no effect on the insurer's price. For a sufficiently well diversified insurer, any losses in one market segment are covered by the gains in other market segments so that the insurer would rarely need to use its own capital to support its insurance operation. A less well-diversified insurer can improve its ability to conserve capital by increasing its risk aversion parameter α. The ability of policyholders to self-insure places a limitation on the insurer's ability to α to increase its risk margin. By explicitly recognizing the effect of risk diversification on price within a market segment, it was shown that a mutually acceptable price exists provided that the insurer's expenses are not excessive and that the insurer's risk margin can be sufficiently reduced by diversification within the market segment. This result can be applied to each exposure in the market segment in order to develop supply and demand curves and determine the competitive market price.

Author's note: This paper is based on material presented at the 11th AFIR Colloquium.

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Appendix A: The Axioms of Expected Utility Theory

Starmer (2000, p. 334) notes that expected utility theory (EUT) can be derived from three axioms: the ordering of preferences, continuity, and independence. Expected utility theory does not require that the outcomes be expressed in monetary terms or even numerically. The only requirement is that the individual must be able to identify a preference between any two uncertain prospects. Taking a more practical viewpoint, it can be assumed that all outcomes are expressed in monetary terms. For the two uncertain random variables X and Y, the notation $X \gg Y$ will represent the individual's assessment that X is a more desirable prospect than Y (i.e., that X is preferred to Y).

The description of preferences and the associated notation will be used here in the normal manner, that is, to represent investments rather than insurance exposures. For insurance exposures, the notation $X \gg Y$ might instead be understood to mean that X is a worse prospect than Y. For instance, if every damage outcome for X exceeds the corresponding damage outcome for Y, then X is a worse prospect than Y. This might be stated as $X \gg Y$ *if* $X(\omega) > Y(\omega)$ for all outcomes ω, provided that all damages are paid at time 0. In order to keep the terminology consistent, insurance damages can be considered to be negative amounts rather than positive amounts. On this basis, the normal terminology for investment decisions would apply as well to insurance transactions.

The ordering axiom requires completeness, i.e., that either $X \gg Y$ or $Y \gg X$, or both. The third possibility is denoted as $X \sim Y$. Ordering also requires transitivity, so that if $X \gg Y$ and $Y \gg Z$, then $X \gg Z$.

The continuity axiom requires that if $X \gg Y$ and $Y \gg Z$, then there exists a probability p such that the compound lottery $(X,p,Z,1-p)$ is equally preferred to Y. The notation $(X,p,Z,1-p)$ represents uncertain damages which are equal to X with probability p or equal to Z with probability $(1-p)$. The two axioms of ordering and continuity imply the existence of a preference function U that assigns a numerical value to each random variable with $U(X) >= U(Y)$ if and only if $X \gg Y$.

The third axiom, that of independence, requires that for all X, Y, and Z, whenever $X \gg Y$ then $(X,p,Z,1-p) \gg (Y,p,Z,1-p)$ for all probabilities p. Starmer (2000, p. 335) observes that "the independence axiom of EUT places quite strong restrictions on the precise form of preferences: it is this axiom which gives the standard theory most of its empirical content (and it is the axiom which most alternatives to EUT will relax)." Further discussion of the independence axiom and the alternatives to expected utility theory can be found in Machina (1987) and Starmer (2000). Given the three axioms, it can be shown that there exists a utility function u such that for all random variables X, the preference function U can be expressed as an expected utility:

$$(21) \qquad U(X) = \sum p(x_i)u(x_i)$$

where $p(x_i)$ represents the probability of outcome x_i.

Appendix B: The Form of the Utility Function

This section will demonstrate that the utility function consisting of two rays, as shown in Figure 3, satisfies equation (12). It will also be shown that the utility function must be of this form.

Assume that the utility function has the form shown in Figure 3. For any damage exposure X, $P(X)$ is the unique value that satisfies the expected utility formula:

$$EU(w - X + P(X)) = 0 \quad (22)$$

Using the definition of U, this can also be expressed as:

$$-e \int_{x>P(X)} (x - P(X))\, dF(x) + c \int_{x<=P(X)} (P(X) - x)\, dF(x) = 0 \quad (23)$$

Consider the random variable $Y = aX$. The expected utility formula for Y is:

$$EU(w - aX + P(Y)) = -e \int_{ax>P(Y)} (ax - P(Y))\, dF(x) + c \int_{ax<=P(Y)} (P(Y) - ax)\, dF(x) \quad (24)$$

$P(Y)$ is the unique value that makes this result equal to zero. Consider the possibility that $P(Y) = aP(X)$. Substituting this on the right hand side of the formula permits a to be factored out so that:

$$EU(w - aX + P(Y)) = aEU(w - X + P(X)) \quad (25)$$

Since the expected utility formula on the right side of this equation is 0 by definition, this shows that $P(Y) = aP(X)$ is the certainty equivalent price for $Y = aX$, so that $P(aX) = aP(X)$.

The next objective is to demonstrate that U must be of the form shown in Figure 3. Assume that equation (12) is true and consider a random variable X with two outcomes, 1 and x, with $x < 0$. According to the continuity axiom of expected utility theory, there exists a probability p for which $EU(X) = 0$. Hence:

$$U(x)p + U(1)q = 0 \quad (26)$$

so that:

$$U(x) = -U(1)q/p \quad (27)$$

Next, consider the random variable $Y = 2X$. Since $P(X)$ is the unique solution to $EU(X - P) = 0$, and $EU(X) = 0$, this shows that $P(X) = 0$. Consequently, $P(Y) = 2P(X) = 0$, so that $EU(Y - P(Y)) = EU(Y)$. Since $EU(Y - P(Y)) = 0$ by definition, this means that $EU(Y) = 0$. However:

$$EU(Y) = U(2x)p + U(2)q = 0 \quad (28)$$

or

$$U(2x) = -U(2)q/p \quad (29)$$

Define $k = U(2) / U(1)$. Since U is concave and increasing, this requires that $1 < k <= 2$. Also, the value for $U(2x)$ is related to the value for $U(x)$ as follows:

(30) $$U(2x) = kU(x)$$

Consider the specific value of $x = -1$, so that $U(-2) = kU(-1)$. Since $U(-1) < 0$ and $k <= 2$, this implies that:

(31) $$U(-2) = kU(-1) >= 2U(-1)$$

However, if $U(-2) > 2U(-1)$, then U is not concave. Consequently, k must be equal to 2, so that for any $x < 0$:

(32) $$U(2x) = 2U(x)$$

A similar result can be obtained for any other random variable $Y = tX$ for $t > 0$, so that:

(33) $$U(tx) = tU(x)$$

The solution to this formula is a straight line ending at the origin:

(34) $$U(x) = ex \quad \text{for } x <= 0$$

A similar argument can be used to show that for $x >= 0$,

(35) $$U(x) = cx \quad \text{for } x >= 0$$

In order for U to be concave, e must be greater than c. U can be converted into the utility function shown in Figure 3 by dividing equations (34) and (35) by c. This shows that the requirement that $P(aX) = aP(X)$ for all $a > 0$ implies that the utility function must be of the form shown in Figure 3.

Appendix C: The Arbitrage Theorem

The arbitrage theorem from the field of financial theory is the basis for developing what is known as arbitrage-free pricing. Arbitrage represents an opportunity to make a risk-free return greater than that of risk-free Treasury bills by taking positions in different assets. If these opportunities do not exist, prices are described as being arbitrage-free. Neftci (1996) describes the arbitrage theorem as providing a connection between risk and the time value of money.

The arbitrage theorem can be described in terms of the following example. Consider three assets, each with a term of one year. The first asset is a bond with a current price of \$1 and having a risk-free yield of r_f. The second asset has a current price of S and pays either T - *\$1,000* with probability p or T - *\$2,000* with probability q, where $p + q = 1$. Both S and T are known values. The third asset is an insurance policy that has a current price of P and pays an indemnity of either \$1,000 or \$2,000. The Arbitrage Theorem states that an arbitrage-free price P exists for the insurance if and only if there exist positive constants u and v such that:

$$\begin{bmatrix} 1 \\ S \\ P \end{bmatrix} = \begin{bmatrix} 1 + r_f & 1 + r_f \\ T - 1000 & T - 2000 \\ 1000 & 2000 \end{bmatrix} * \begin{bmatrix} u \\ v \end{bmatrix}$$

The matrix equation relates the present value of the three assets to their values one year in the future. The first row represents the ability of the individual to borrow money at the risk-free rate, and can be written as:

(36) $$1 = (1 + r_f) * (u + v)$$

The values u and v are known as synthetic probabilities. If $y = (1 + r_f) * u$ and $z = (1 + r_f) * v$, it can be seen that y and z are positive values that resemble probabilities. That is:

(37) $$1 = y + z$$

The second row of the matrix relates the current purchase price for an asset to the value of the uncertain future outcomes. The third row represents the individual's ability to purchase insurance at a premium P to offset the uncertainty of the outcomes for the second asset. The uncertainty can be eliminated since the outcomes for the third asset are negatively correlated with those of the second asset.

Consider the equation describing the second asset:

(38) $$S = (T - 1000) * u + (T - 2000) * v$$

Multiplying by $(1 + r_f)$, this becomes:

(39) $$S * (1 + r_f) = (T - 1000) * y + (T - 2000) * z$$

The right hand side of the equation resembles an expected value calculation with the synthetic probabilities y and z used in place of the true probabilities p and q. The transformation of the true probability distribution into a synthetic probability distribution can be understood as a consequence of the risk aversion of the individual.

The following discussion will provide the construction of the synthetic probabilities corresponding to the risk pricing model in equation (4). The initial step is to discount the uncertain outcomes to their present values X using the risk-free rate. The certainty equivalent price P for the exposure X can be expressed as:

(40) $$EU(P - X) = 0$$

or:

(41) $$\int U(P - x)f(x)dx = 0$$

where $f(x)$ is the probability density function of X.

The function $U(P - x)$ is:

(42) $$\begin{aligned} &(P - x) \quad for\ P - x \geq 0 \\ &(1 + \alpha)(P - x) \quad for\ P - x < 0 \end{aligned}$$

Define $g(P) = f(P)$. For all other values, define g(x) as:

(43) $$g(x) = U(P - x)f(x) / (P - x)$$

Since $g(x)$ is non-negative and $\int g(x)dx$ is finite, a synthetic probability density function $h(x)$ can be defined as:

(44) $$h(x) = g(x) / \int g(x)dx$$

Based on these definitions, the expected utility can be restated as:

(45) $$\int U(P - x)f(x)dx = \int (P - x)g(x)dx$$

Since P is the certainty equivalent price, the left side of this equation is 0 so that:

(46) $$\int xg(x)dx = P \int g(x)dx$$

Dividing both sides by $\int g(x)dx$ gives:

(47) $$\int xh(x)dx = P$$

or:

(48) $$E^*(X) = P$$

This result shows that the certainty equivalent price P can be determined as the expected value of the outcomes using the synthetic probability distribution h.