Arbitrage Free Risk Loads For Brokers

Christopher M. Steinbach, FCAS, MAAA

ARBITRAGE FREE RISK LOADS FOR BROKERS

CHRISTOPHER M. STEINBACH

Abstract

When actuaries write about an insurance pricing formula, they first identify the formula in question and then discuss the pros and cons of it. This is in stark contrast to capital markets pricing papers, which start with a list of desirable axioms and then proceed by deriving the necessary and sufficient formula that satisfies these axioms. This paper follows the capital markets paradigm, deriving the necessary and sufficient risk load formula from a set of axioms that are uniquely appropriate for insurance. The derivation follows the same basic approach as Black-Scholes, but it differs in that the axioms have been re-selected to be descriptive of how an efficient insurance market operates. The result is a risk load formula commonly known as the Proportional Hazard Transform. This paper also provides an outline of how to implement the Proportional Hazard Transform, making it relevant and accessible to pricing actuaries.

1. MEASURING RISK

In capital market derivations, 'arbitrage free' means that we can not construct a hedging strategy that is guaranteed to make a profit. This assumption is the key to the dynamic hedging strategies that form the basis for many capital market derivations. The simplest version is the Black-Scholes derivative pricing formula. Variants of Black-Scholes have been derived to overcome the overly simplistic assumptions that Black-Scholes relies upon, but they all follow the same basic construction: they all rely on a dynamic hedging strategy to cancel out the risk.

When it comes to insurance, the main problem with the capital market's arbritrage free approach is that it assumes that the financial institution is not the ultimate risk bearer. This is an important observation because the secret to the Black-Scholes derivation is that it is carefully crafted to make the calculation of risk loads somebody else's problem. Since the dynamic hedging strategy transfers all of the risk to commodity speculators, whether these speculators are adequately compensated for this risk becomes their problem. The existence of speculators who are eager to do the hard work for us is all the derivation requires.

Insurance does not have a separate group of speculators; Insurers and reinsurers fill this role themselves. As a result, insurance pricing formulae must explicitly contain risk loads. So, we can not derive these formulae by pretending they are someone else's problem. Nor can we skirt the issue by creating a virtual commodity market, because we would still need formulae that describe the behavior of the virtual commodity speculators. If we want to derive an insurance pricing formula with an adequate risk load in it, we are going to have to take a different approach.

If we want to borrow the capital market's paradigm, then we need to redefine the term 'arbitrage free' so that it is appropriate for insurance. A more appropriate insurance definition would be that every way we carve up a risk produces the same total risk premium. That is, every two insurance programs that offer the exact same coverage will have the exact same price. In this paper, this insurance version of the arbitrage neutral assumption will be represented by the following axioms:

a. Monotonicity: If $F_X(t) \geq F_Y(t)$ for all t, then the price of insuring X is less than or equal to the price of insuring Y.
b. Scale Invariance: If only p% of a risk is proportionately insured, then the proper premium is p% of the full premium.
c. Layer Invariance: If a risk is carved up into a series of layers, then the sum of the premiums for all of the layers equals the full premium.
d. Continuity: Let X_i be a series of risks that converge to X_∞. [That is, the series of CDFs $F_i(x)$ converge to $F_\infty(x)$.] Then the price of X_i converges to the price of X_∞.

The above axioms are useful to those who care about quantifying what price the market will support, such as insurance brokers and regulators. But, they are not appropriate for risk bearers. Nor are they appropriate when the market is not efficient, such as the market for natural catastrophe reinsurance. In these situations, capital constraints dominate and many of the axioms cease being true.

We can now derive the unique pricing formula that satisfies these axioms. The formula is known as the Proportional Hazard Transform (PHT) and it calculates the risk premium for a risk X. This is the amount of money we need to collect for the expected loss plus the profit load.

PHT Theorem: Let S be a collection of all risks with policy limits of L or less, and let S contain all Bernoulli random variables with payoff of L or less. Then the market premium functional H satisfies Axioms (a)-(d) if and only if H can be represented as:

$$H[X;r] = \int_0^L G^r(t)dt$$

where r has an arbitrary value and G(t) is the decumulative distribution of any risk X in S.

Proof: Appendix A for proof.

Therefore, the risk premium for a risk X is equal to H[X;r], where the parameter r is the risk tolerance parameter. The lower r is, the higher the premium is. When r=1, the premium equals the expected loss. When r=0, the premium equals the policy limit. This gives us the following corollary.

PHT Corollary: $r \in [0,1]$ if and only if $H[X;r] \in [E(Loss), Limit]$ where $Limit < \infty$

Proof: Appendix A for proof.

Note that the restriction that all risks must have a policy limit of L or less can be significantly relaxed, and in many instances removed completely. The restriction is only required for proving the axiom of Continuity from the PHT formula – a relationship with no applied value. Weaker assumptions are possible, but they result in significantly more complicated proofs. These enhancements are left for curious readers to pursue.

It must be reiterated that the PHT is not appropriate for risk bearers. In addition to the weaknesses mentioned above, it should be mentioned that the PHT is not a robust methodology for allocating capital. While an implied capital allocation can be derived from the PHT, it is impossible to create a general formula equating the two that permits the underlying parameters to vary. See Appendix B for an example.

Finally, the above axioms are similar to the axioms originally proposed by Wang, Young & Panjer. The key difference is that the axiom of Comonotonic Additivity has been replaced with Scale and Layer Invariance. The change permits actuaries to better assess the applicability of the axioms as well as permitting a more intuitive proof of the PHT Theorem. See Appendix C for the Wang, Young & Panjer axioms.

2. MEASURING DIVERSIFICATION

While the PHT results in an arbitrage neutral risk premium for any one risk, it does not result in an arbitrage neutral risk premium for a diversified pool of risks. Changing the amount of diversification changes the proper value of the risk tolerance parameter r, as the following examples demonstrate.

Example #1: Calculate H[X;r] for X=B(p,1), a Binomial risk of probability p and payout \$1.

$$f[X]=\begin{cases}0 & \text{prob } 1-p\\ 1 & \text{prob } p\end{cases}$$

$$F[X]=\begin{cases}1-p & x<1\\ 1 & x\geq 1\end{cases}$$

$$G[X]=\begin{cases}p & x<1\\ 0 & x\geq 1\end{cases}$$

$$H[X;r] = \int_0^L G^r(t)dt = p^r$$

<u>Example #2</u>: Calculate H[X+Y;r] for X=Y=B(p,1), i.i.d.

$$f[X]=\begin{cases} 0 & \text{prob.}\,(1-p)^2 \\ 1 & \text{prob.}\,2p-2p^2 \\ 2 & \text{prob.}\,p^2 \end{cases}$$

$$F[X]=\begin{cases} (1-p)^2 & x<1 \\ 1-p^2 & 1\le x<2 \\ 1 & x\ge 2 \end{cases}$$

$$G[X]=\begin{cases} 2p-p^2 & x<1 \\ p^2 & 1\le x<2 \\ 0 & x\ge 2 \end{cases}$$

$$H[X;r] = \int_0^L G^r(t)dt = \left(2p-p^2\right)^r + p^{2r}$$

The above examples demonstrate that it is more expensive to insure two risks separately than it is to insure them together, assuming the same risk tolerance parameter is used for both calculations. This can be shown by comparing the price of insuring two identical Bernoulli risks separately, as in example #1, against insuring the risks together, as in example #2. If the two approaches produced identical risk premiums, then we would have:

$$2p^r = (2p-p^2)^r + p^{2r} \quad \text{for all } p$$

which we can rewrite as:

$$(2-p^r) = (2-p)^r \quad \text{for all } p$$

But, this is true only when r equals 0 or 1. That is, this is true only if the premium equals the expected loss or the full limit – the two extremes.

The conclusion is that there are no natural values for the risk tolerance parameter other than the two extremes. Risk tolerance is necessarily a relative measure that reflects the degree of portfolio diversification that a risk benefits from. The better diversified a risk is, the greater the market's risk tolerance for that risk.

For this reason, the PHT does not actually permit actuaries to compute the risk premium for a risk from first principals. Instead, it requires the actuary to first establish the correct risk tolerance parameter for that risk by determining what the market supports, and then it calculates the risk premium from that value. The result is a price that is consistent with the prevailing state of the market. This is the price that the market is expected to be able to bear.

There are several ways to calibrate the risk tolerance parameter r. The simplest approach is to keep a table of risk tolerance values for comparable risks. These are what the market has supported for similar risks. For example:

Example #3: Suppose we have just bound a quota share treaty where the market supported a 60% loss ratio pick and a 65% risk premium. The empirically derived loss distribution is listed below. Then the corresponding r is determined by finding the value that results in the risk premium. The results are as follows:

X	f(X)	F(X)	G(X)	H(X)
0%	0.0%	0.0%	100.0%	100.0%
10%	0.0%	0.0%	100.0%	100.0%
20%	0.0%	0.0%	100.0%	100.0%
30%	0.0%	0.0%	100.0%	100.0%
40%	10.0%	10.0%	90.0%	94.0%
50%	20.0%	30.0%	70.0%	81.1%
60%	40.0%	70.0%	30.0%	49.2%
70%	20.0%	90.0%	10.0%	25.8%
80%	10.0%	100.0%	0.0%	0.0%
90%	0.0%	100.0%	0.0%	0.0%
Sum	100.0%		600.0%	650.0%

Now, since the width of each layer is 10%, the expected loss is 10% of the sum of G(X) and the risk premium is 10% of the sum of H(X). Note that $H(X)=G^r(X)$ in this table. The value that resulted in the risk premium equaling 65% was r=0.589. This value then gets put into a table that lists the name of the account, a description of the business, the date of inception and the value of r that the market supported.

Note that when constructing tables of r's, different reinsurance products for the same line of business can have different r's. That is, r is not always the same from product to product even when the underlying business is the same. There are several reasons for this:

1. Different reinsurance products are supported by different sectors of the reinsurance market. Each sector has a different amount of capacity available to it, resulting in differences in supply by product.
2. Different reinsurance products have different regulatory and statutory consequences. Some are treated more favorably than others, resulting in differences in demand by product.
3. Large and small losses can come from different causes of loss, each with a different parameter variance. Thus, parameter variance can vary by product.

Finally, when a risk is carved up, we are faced with the problem that the sum of the parts may not equal the whole. Equality holds for programs carved up vertically and horizontally, but it does not hold for other structures. Notably, when we refer to "Layer Invariance," we are referring to aggregate stop loss layers only. It does not hold for excess reinsurance because Layer Invariance slices up the loss distribution after frequency has been reflected. The result is that excess reinsurance structured as two narrow layers costs slightly more than one wide layer, as the following example demonstrates.

> Example #4: Suppose we are analyzing a line of business which has a frequency that is Poisson with a mean of 10 and a severity that is Exponential with a mean of \$1M. We are interested in pricing the \$1M x \$1M, \$3M x \$2M and \$4M x \$1M excess layers. If r=0.589 then the risk premium for the \$1M x \$1M, \$3M x \$2M and \$4M x \$1M layers are \$2,842,000, \$2,049,000 and \$4,740,000 respectively. In other words, reinsuring both layers together is 3.1% cheaper than reinsuring them separately. (See Appendix D for the calculation details.)

There are times when this inequality should be compensated for, such as when reinsurers all take the same shares on two consecutive excess layers. For example, if the reinsurers in Example #4 behave this way, then the pricing of the individual layers should be multiplied by a compensation factor of 0.969 = \$4,740,000 / (\$2,842,000 + \$2,049,000) to force the sum of the parts to equal the whole.

When pricing any structure, the actuary should always check the final results to see if diversification is being properly applied. In many situations, diversification is not being properly applied and a compensation factor is required. Also, when borrowing r's from comparable reinsurance programs, be careful what you consider a comparable program to be. Not only must the underlying risk be comparable, the market's acceptance of the product must also be comparable.

3. MEASURING DURATION

A practical risk load formula has three components to it: a measure of risk, a measure of diversification, and a measure of duration. The PHT formula includes a measure of risk and the compensation factor corrects for diversification, but we have yet to reflect duration. The main complication we must address is that risk can run off very differently from expected loss. For example, the fact that 50% of the loss has incurred does not mean that 50% of the risk has occurred.

We can model the run-off of risk by borrowing ideas from IBNR theory. In IBNR theory, the Paid Loss Lag at time t is defined to be the percentage of the total expected loss that we expect to have been paid by time t. Similarly, the Incurred Loss Lag at time t is defined to be the percentage of the total expected loss that that we expect to have incurred by time t. We can define the Risk Lag at time t as the percentage of the total expected risk that we expect to have occurred by time t.

The concept of a Risk Lag permits us to use the Paid Lags and Incurred Lags as benchmarks for estimating Risk Lags. First notice that no risk remains after a loss has been fully paid. Therefore, the Paid Lags can serve as an approximate upper bound for the Risk Lags. Also notice that all of the risk remains until the claim has been first reported. Therefore, the Incurred Lags can serve as an approximate lower bound on the

Risk Lags. This permits us to estimate Risk Lags as weighted averages of Paid and Incurred Lags.

Where exactly the Risk Lags lie between the Paid Lags and Incurred Lags would depend on how claims are handled. If case reserves are always established with 100% accuracy, then the only risk occurs when the loss occurs, so the Risk Lag should equal the Incurred Lag. If case reserves have zero credibility until the loss is paid, then risk does not diminish until the loss is paid and the Risk Lag should equal the Paid Lag. An intermediate reserving approach would result in an intermediate Risk Lag.

While this model of duration is not exceptionally precise, it does produce reasonable and practical estimates. Its strength is that it permits actuaries to make intelligent estimates of how risk runs-off and it permits relatively sophisticated calculations of the cost of capital. More complicated structures can be created to model the time value of risk more accurately, but these are left for ambitious readers to pursue.

4. CONCLUSION

The PHT calculates risk premiums that are consistent with what an insurance market bears, reflecting the market's tolerance for risk. This makes the PHT especially useful to brokers who are interested in predicting the price that the market should support. But, the PHT assumes that an efficient insurance market exists. To the extent that an efficient market does not exist, the result of the PHT will not be accurate. None-the-less, the PHT derived price should be a lower bound on the market price, permitting benchmarking even when the market is not efficient.

The PHT is significantly less useful to risk bearers. This can be seen in the way the axioms do not reflect the risk bearer's capital structure. Risk bearers can use the PHT formula to estimate what the market should bear for a risk, but they should not use the formula to establish if this risk is an effective use of their capital. Other formulae, such as Value at Risk, are more appropriate for this purpose.

Interestingly, while the PHT does not reflect the capital structure of an individual company, it does make assumptions about the capital utilization of the industry as a whole. This is evident in the fact that the resulting risk loads are not zero, indicating that the industry does have capital at risk. But, instead of modeling capital as a fixed quantity the way individual insurers do, the PHT assumes that market forces will require insurers to actively manage their capital to match the risk they assume. In other words, the companies who use their capital efficiently will determine the market and everyone else must struggle to keep up.

Considerably more work needs to be done on pricing & capital allocation. We have historically assumed that the Law of Large Numbers makes our industry different and our pricing formulae unique. But, it is increasingly obvious that insurance is a form of finance and that the financial markets paradigms should be applied to insurance. Hopefully, this paper contributes to the change in the direction of actuarial discourse, closing some of the divides that currently exist between insurance and the capital markets.

REFERENCES

[1] Feldblum, Sholom, "Risk Loads for Insurers," PCAS LXXVII, pp.160-195

[2] Rudin, Walter, "Real & Complex Analysis," McGraw-Hill Book Company, 1987

[3] Wang, Shaun, "Insurance Pricing and Increased Limits Ratemaking by Proportional Hazard Transforms," Insurance: Mathematics and Economics 17, 1995, pp. 43-54

[4] Wang, Shaun, "Premium Pricing by Transforming the Layer Premium Density," ASTIN 26, 1996, pp. 71-92

[5] Wang, Shaun, Virginia Young and Harry H. Panjer, "Axiomatic Characterization of Insurance Price," Insurance: Mathematics and Economics 21, 1997, pp.173-183

[6] Wheeden, Richard L., and Zygmund, Antoni, "Measure and Integral, An Introduction to Real Analysis," Marcel Dekker, Inc., 1997

APPENDIX A

PROOF OF THE PHT THEOREM & COROLLARY

Lemma 1: Let B(q,k) represent a Bernoulli risk with probability q and severity k. Let H(q,k) represent the premium we would charge for insuring this risk. The unique formula for H(q,k) that satisfies axioms (a), (b) & (e) is the following:

$$H(q,k) = q^r k \qquad [1]$$

where r is an arbitrary risk tolerance parameter.

Proof: Let us model B(q,k) as a biased coin toss, where the coin has a probability p of heads (p is not necessarily equal to q in [1]). Consider the following process: Flip a biased coin. If heads, flip it again. If heads again, then collect $k. This process can be insured in two ways. First, it can be considered a Bernoulli risk with p^2 probability of heads and a pay-out of k. Second, it can be considered as a Bernoulli risk with probability p and a pay-out that buys another insurance policy insuring against the second coin toss. Since these two insurance structures provide the exact same protection, the axiom of Monotonicity says that they must have the exact same price. Thus, we can equate the risk premiums for these two different insurance structures:

$$H(p^2,k) = H(p,H(p,k)) \qquad [2]$$

Furthermore, the axiom of Scale Invariance implies that the premiums are linearly proportional to the payoff:

$$H(p,X) = X\, H(p,1) \qquad [3]$$

Applying [3] to [2] we get:

$$H(p^2,1) = H^2(p,1) \qquad [4]$$

For the sake of notational simplicity, the k's have been cancelled out. We will reintroduce them at the end. Applying the above argument repeatedly gives us:

$$H(p^m,1) = H^m(p,1) \text{ for all positive integers } m \qquad [5]$$

Which we can rewrite as:

$$H^{1/m}(p^m,1) = H(p,1) \text{ for all positive integers } m \qquad [6]$$

Letting n be another integer (not necessarily equal to m), we can rewrite [6] with n replacing m. Equating the two versions of [6] gives us:

$$H^{1/m}(p^m,1) = H^{1/n}(p^n,1) \text{ for all positive integers } m,n \qquad [7]$$

or

$$H(p^m,1) = H^{m/n}(p^n,1) \text{ for all positive integers } m,n \qquad [8]$$

Now, let $q=p^n$ and substitute this into [8]:

$$H(q^{m/n},1) = H^{m/n}(q,1) \text{ for all positive integers } m,n \qquad [9]$$

Applying the Continuity assumption gives us:

$$H(q^v,1) = H^v(q,1) \text{ for all positive real numbers } v \qquad [10]$$

Which we can write as:

$$H^{1/v}(q^v,1) = H(q,1) \text{ for all positive real numbers } v \qquad [11]$$

Now, since v can be anything, let us select v such that $q^v = 0.5$. In other words, v=ln(0.5)/ln(q). Substituting this into [11]:

$$H(q,1) = H^{\ln(q)/\ln(0.5)}(0.5,1) \qquad [12]$$

Which we can rewrite as:

$$H(q,1) = q^{\ln(H(0.5,1))/\ln(0.5)} \qquad [13]$$

Letting r = ln(H(0.5,1)) / ln(0.5), reintroducing k and applying [3] again, we get:

$$H(q,k) = q^r k \qquad [14]$$

which is the PH-Transform for B(q,k).

Lemma 2: Let H[Z] represent the premium charged for an arbitrary risk Z. Let G(z) be the decumulative distribution of Z. The unique H[Z] that satisfies axioms (a) - (e) is the following:

$$H[Z] = \int_0^\infty G^r(t)\,dt \qquad [15]$$

Proof: By the Layer Invariance assumption, the price for risk Z is the sum of the prices for each of its layers. Carve the risk Z into a series of layers, each of width i. Let j enumerate the layers with j=0 being the first layer. Now we must evaluate the prices of the layers.

First, by the axiom of Monotonicity, the price of excess layer number j is bounded from above by a Bernoulli risk of probability G(ji) and severity of i. [Probability of entering the layer & severity of one layer width. This prices every partial loss in the layer as if it were a total loss, resulting in a price that is too high.] So we can write:

$$H[Z] \le \sum_{j=0}^{\infty} H[G(ji), i] \qquad [16]$$

The Continuity assumption permits us to take the limit as i approaches zero giving us:

$$H[Z] \le \lim_{i\to 0} \sum_{j=0}^{\infty} H[G(ji), i] \qquad [17]$$

Applying Lemma 1 and evaluating:

$$H[Z] \le \int_0^{\infty} G^r(t)\,dt \qquad [18]$$

Second, by the axiom of Monotonicity, the price of excess layer j is bounded from below by a Bernoulli risk of probability G((j+1)i) and severity i. [Probability of exhausting the layer & severity of one layer width. This ignores partial losses entirely, resulting in a price that is too low.] So, we can write:

$$H[Z] \ge \sum_{j=0}^{\infty} H[G((j+1)i), i] \qquad [19]$$

The Continuity assumption permits us to take the limit as i approaches zero giving us:

$$H[Z] \ge \lim_{i\to 0} \sum_{j=0}^{\infty} H[G((j+1)i), i] \qquad [20]$$

Applying Lemma 1 and evaluating:

$$H[Z] \ge \int_0^{\infty} G^r(t)\,dt \qquad [21]$$

Equations [18] and [21] together produce [15].

Lemma 3: Let $\{G_k^r\}$ be a sequence of measurable functions on E=[0,L] such that $G_k^r \to G^r$ a.e. in E. Since $|E|<+\infty$ and $|G_k^r|\le 1$ a.e. in E, then $\int_E G_k^r \to \int_E G^r$.

Proof: This is a specific case of the Bounded Convergence Theorem. See any Real Analysis textbook (such as Wheeden & Zygmund) for the proof.

PHT Theorem: Let S be a collection of all risks with policy limits of L or less and let S contain all Bernoulli random variables with payoff of L or less. Then the market premium functional H satisfies Axioms (a) - (e) if and only if H can be represented as

$$H[X;r] = \int_0^L G^r(t)dt \qquad [18]$$

where r has an arbitrary value and G(t) is the decumulative distribution of any risk X in S.

Proof:

⇒ Apply Lemma 2

⇐ The PHT has the following properties:

a) If $F_X(t) \geq F_Y(t)$ for all t, then $G_X(t) \leq G_Y(t)$ for all t and the result immediately follows.

b) The price of a risk X is:

$$H[X;r] = \int_0^\infty G^r(t)dt \qquad [19]$$

Let v=p·t and dv=p·dt, then

$$p \cdot H[X;r] = \int_0^\infty G^r\left(\frac{v}{p}\right)dv \qquad [20]$$

And since the integral is H[pX;r], this proves Scale Invariance.

c) The price for the layer that starts with A and ends with B equals:

$$H[X;r] = \int_A^B G^r(t)dt \qquad [21]$$

Layer Invariance immediately follows.

d) Apply Lemma 3.

PHT Corollary: r ε [0,1] if and only if H ε [E(Loss),Limit] where Limit<∞

Proof:

⇒ If r=0, then H=Limit. If r=1, then H=E(loss). All that is left is to prove H is strictly monotonic. We know that $0 \leq G(t) \leq 1$ for all t. Therefore, if r<s, then $G^r(t) > G^s(t)$ and H[X;r]>H[X;s] for all H≠∞. In our situation, H is bounded above by Limit, so H is necessarily finite.

⇐ Since all finite H are strictly monotonic with respect to r, there is only one r that results in H=Limit and only one r that results in H=E(Loss). We know from above that these are r=0 and r=1 respectively. Application of strict monotonicity completes the proof.

APPENDIX B

IMPLIED CAPITAL ALLOCATION

While the PHT can be used for calculating an implied capital allocation, it is not a robust approach. For example, a Bernoulli risk with a probability of p and a payout of $1 has an expected loss of p and a risk premium of p^r. This means that the load for risk is p^{r-1}. If the market supports an ROE of R, then the implied capital allocation is p^{r-1}/R.

The problem occurs when the insurance market's pricing changes (either getting harder or softer). Continuing the above example, if we assume that the ROE changes to R' while the capital allocation stays the same, then the new price H' for the risk is:

$$H' = p + p^{r-1} \times R' / R = p^{r'}.$$

Notice that the capital allocation is a function of r & R and the new risk tolerance parameter r' is a function of p. This is the problem. If the PHT is equivalent to a capital allocation methodology, then we should have a capital allocation that is a function of only p and the cost of capital should be invariant with respect to p. Instead, the PHT results in the two being commingled. It appears to be impossible to disaggregate the PHT's risk load into an amount of capital formula times a cost of capital formula.

For example, suppose we have three $1 Bernoulli risks with p equal to 25%, 50% and 75% respectively. If r=0.85, then the risk pure premiums for the three risks are 0.308, 0.555 and 0.783 respectively. If R=10%, then the implied capital allocations are 0.578, 0.548 and 0.331 respectively. If the ROE now changes to 15%, then the new risk pure premiums are now 0.337, 0.582 and 0.800 respectively. But, this results in the new risk tolerance parameter varying by risk. The risk tolerance parameters for the three risks are now 0.785, 0.780 and 0.777 respectively. Notice that the only way to make the new risk tolerance parameter constant is to either vary the new ROE or to change the capital allocations. The risk tolerance parameter, the ROE and the capital allocation can not all be invariant.

Therefore, the PHT approach to pricing and the capital allocation approach to pricing are incompatible. As the insurance market gets harder or softer, one of the following must be true:

1. The PHT's risk tolerance parameter r varies by risk
2. The market's ROE varies by risk
3. The capital allocations change in addition to the cost of capital changing

In other words, the assumptions underlying the two pricing methodologies can not all be true. Further research must be performed to determine which set of assumptions better explains the behavior of the insurance market.

APPENDIX C

THE WANG-YOUNG-PANJER AXIOMS

In 1997, Wang, Young and Panjer published the paper "Axiomatic Characterization of Insurance Price" in which they identified and proved a set of necessary and sufficient conditions for the PHT. The axioms Wang, Young and Panjer identified were as follows:

1. Conditional State Independence: For a given market condition, the price of an insurance risk X depends only on its distribution.
2. Monotonicity: For two risks X and Y in $\mathcal{X}$, if $X(\omega) \leq Y(\omega)$, for all $\omega \in \Omega$, a.s., then $H[X] \leq H[Y]$.
3. Comonotonic Additive: if X and Y in $\mathcal{X}$ are comonotonic, then $H[X+Y] = H[X]+H[Y]$.
4. Continuity: For $X \in \mathcal{X}$ and $d \geq 0$, the functional H satisfies

   $$\lim_{d \to 0+} H[(X - d)_+] = H[X] \quad \text{and} \quad \lim_{d \to \infty} H[\min(X, d)] = H[X]$$

   in which $(X - d)_+ = \max(X - d, 0)$.
5. Reduction of compound Bernoulli risks: Let $X = IY$ be a compound Bernoulli risk, where the Bernoulli frequency random variable I is independent of the loss severity random variable $Y = X \mid X>0$. Then the market prices for risks $X = IY$ and $IH[Y]$ are equal.

From these axioms, Wang, Young and Panjer prove the following:

W.Y.P's PHT Theorem: Assume that a collection of risks $\mathcal{X}$ contains all of the Bernoulli random variables. Then the market premium functional H satisfies Axioms 1-5 if and only if H can be represented as

$$H[X; r] = \int_0^\infty G^r(t)dt$$

where r is some unique positive constant and G(t) is the decumulative distribution of any risk in $\mathcal{X}$.

APPENDIX D

EXCESS REINSURANCE EXAMPLE

Assume that frequency is Poisson with a mean of 10, severity is Exponential with a mean of $1M, and r=0.589. We want to price the excess reinsurance layers $1M x $1M, $3M x $2M and $4M x $1M. First we determine the decumulative distribution G(X) by simulating 10,000 years of experience. Then we calculate H(X):

$1M x $1M Excess Layer

Discretization Point X	Midpoint	G(X)	H(X)
-	500,000	89.1%	93.4%
1,000,000	1,500,000	62.0%	75.5%
2,000,000	2,500,000	33.3%	52.3%
3,000,000	3,500,000	14.4%	31.9%
4,000,000	4,500,000	4.9%	16.9%
5,000,000	5,500,000	1.3%	7.7%
6,000,000	6,500,000	0.4%	3.9%
7,000,000	7,500,000	0.2%	2.6%
8,000,000	8,500,000	0.0%	0.0%
Sum		205.6%	284.2%
Layer Width		1,000,000	1,000,000
Expected Loss		2,056,000	
Risk Premium			2,842,000

$3M x $2M Excess Layer

Discretization Point X	Midpoint	G(X)	H(X)
-	500,000	53.8%	69.4%
1,000,000	1,500,000	30.4%	49.6%
2,000,000	2,500,000	16.6%	34.7%
3,000,000	3,500,000	6.9%	20.7%
4,000,000	4,500,000	3.2%	13.2%
5,000,000	5,500,000	1.4%	8.1%
6,000,000	6,500,000	0.6%	4.9%
7,000,000	7,500,000	0.2%	2.6%
8,000,000	8,500,000	0.1%	1.7%
Sum		113.2%	204.9%
Layer Width		1,000,000	1,000,000
Expected Loss		1,132,000	
Risk Premium			2,049,000

$4M x $1M Excess Layer

Discretization Point X	Midpoint	G(X)	H(X)
	500,000	89.1%	93.4%
1,000,000	1,500,000	71.7%	82.2%
2,000,000	2,500,000	54.1%	69.6%
3,000,000	3,500,000	38.7%	57.2%
4,000,000	4,500,000	26.2%	45.4%
5,000,000	5,500,000	16.9%	35.1%
6,000,000	6,500,000	10.1%	25.9%
7,000,000	7,500,000	6.2%	19.4%
8,000,000	8,500,000	3.4%	13.6%
9,000,000	9,500,000	1.9%	9.7%
10,000,000	10,500,000	1.0%	6.6%
11,000,000	11,500,000	0.6%	4.9%
12,000,000	12,500,000	0.3%	3.3%
13,000,000	13,500,000	0.2%	2.6%
14,000,000	14,500,000	0.1%	1.7%
15,000,000	15,500,000	0.1%	1.7%
16,000,000	16,500,000	0.1%	1.7%
17,000,000	17,500,000	0.0%	0.0%
Sum		320.7%	474.0%
Layer Width		1,000,000	1,000,000
Expected Loss		3,207,000	
Risk Premium			4,740,000

Note that, in the above calculations, the discretization includes a point at the origin. The probability assigned to a discretization point equals the cumulative probability from the midpoint directly below the discretization point to the midpoint directly above the descretization point. For example, the probability assigned to the $2.0M discretization point is the cumulative probability that the loss will be between $1.5M and $2.5M. This discretization process results in an expected loss that is slightly lower than the simulated expected loss. For example, the $4M x $1M layer expected loss is calculated above to be $3,207,000 while it the simulation really resulted in an expected loss of $3,211,307.

Also note that the values of G(X) and H(X) are both rounded to the first decimal point, as shown above. This was done solely to make the calculations easier for readers to follow.