A Cash Flow Model for Forecasting Underwriting Investment Income

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Abstract

The investment income received by a property-casualty company can be a prime component in its pricing and decision to write some lines of business that generate underwriting losses. In times of high interest rates it can enable the insurer to write during soft markets and to gain market share by taking on previously uninsurable risks.

Without the dynamic aspect of doing business the problem of investment income would reduce to watching a static amount of money, the surplus, accumulate in a savings or investment fund. The timing of the acquisition of new revenue, the uncertainty of the reserves and the payment of losses complicate the estimation of future investment income even if the amount of future written premium were known for a certainty. However a model that allows for a scenario testing of the random elements would be a useful tool in forecasting investment income.

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1. Introduction

Constructing a cash flow model involves the inevitable use of assumptions. In the following a deterministic model is created that uses certain simplifications. Among them is that the investment rate of return is known and does not vary from year to year. Another is that this rate is the same rate used to discount the reserves and a third is that the ultimate loss ratio has been accurately forecast. Since the principle impetus for the research done here is to construct a picture of insurance company financial position at future times under different operating conditions, these assumptions are simply consistent with the omniscience implied by a scenario analysis. To avoid too much simplicity the model is re-worked by allowing the rate of return to vary from year to year, although the rate used to discount the reserves is assumed to remain constant.

Some of the tricks for handling multiple sums and solving recurrence relations that are employed in the Appendices can be found in [KN].

2. The Cash Flow Algorithms

The first step in estimating investment income for an insurance company is to identify the cash that there is available to invest. Setting aside for the moment the cash in surplus and ongoing cash flow from current investments the source of new available cash has to be in the written business. We will assume that if the company stopped doing business immediately there would be enough cash and invested assets to pay the liabilities. Surplus would not be impacted and future investment income would come only from the surplus. If it does not stop business the increase in investment income will come from the operating cash flows from business written and not from a more clever investing strategy.

In line with these remarks we will start with a single policy written for at the beginning of the year net of re-insurance for an amount P. The cash invested from P will be P less the underwriting expense plus ceded commissions received. We will assume for the moment that premium is received, expenses are paid and, initially, reinsurance cessions are consummated and paid at the time the policy is first written. The errors induced by these timing assumptions can be refined later.

Let C_j be the loss fund at time $t=j$. If E is the underwriting expense and profit percentage, R is the ceding commission, W is the original written premium and c is the percentage ceded, then C_0, the initial cash fund , is $(1-c)W - EW + cWR$. In order for this fund to be positive the inequality $R \geq 1 - (1-E)/c$ should be true. Companies are shrewd enough negotiators to guarantee this will be a fact. In the exposition that follows the time j will refer to j years although it could be any time interval if the claim settlement pattern conforms to it. P=(1-c)W will be the retained premium that enters all of the formulas. [1]

There won't be a separate return on loss reserves and unearned premium. The entire reserve at the inception of the policy is LP and this will be reduced as payments are made. The return on year *n* will follow the recurrence:

$$I_n = iC_{n-1} - p_n LPr \qquad (2.1)$$

Here i is the rate of return, p_j is the percentage of ultimate paid in year j, L is the projected ultimate loss ratio and $r = (1+i)^{1/2}-1$ under the assumption that the amount paid in year j averages to the middle of the year. This simply states that the return during the year is from the cash fund at the beginning of the year less the interest lost from a payment made during the middle of the year.

The next task is to estimate the size of the fund that supports this policy at a future point in time, say *n*. The completion of this task involves first measuring the impact of taxes on the investments earned so far. We will define a tax variable, T_n at time n as the change in the discounted reserve less the amount paid during the year.

$$T_n = d_{n-1}R_{n-1} - d_n R_n - p_n LP \qquad (2.2)$$

Here d_n refers to a discount factor defined as

$$d_n = \frac{\sum_{k=n+1}^{N} p_k v^{k-n-1/2}}{\left(1-\sum_{k=1}^{n} p_k\right)} = \frac{LP\sum_{k=n+1}^{N} p_k v^{k-n-1/2}}{R_n} \qquad (2.3)$$

where N is the year of the last payment and $v = 1/(1+i)$

If S_j is the contribution made to the loss fund for some year $j \leq n$, then the loss fund at time n is

$$C_n = C_0 + \sum_{j=1}^{n} S_j - LP\sum_{j=1}^{n} p_j$$

$$\text{where } S_j = (1-t)\left[iC_{j-1} - rp_j LP\right] - tT_j \qquad (2.4)$$

$$\text{and } T_j = R_{j-1}d_{j-1} - R_j d_j - LPp_j$$

[1] Also, C_0 should be reduced by any excess premium ceded and the ultimate loss ratio modified by the impact that this might have on the retained losses.

for some tax rate t. If we call the first sum in (2.4) D_n and notice that $S_n = D_n - D_{n-1}$ we can use a standard trick for summing a series to get the following equation (see Appendix 1):

$$D_n = \sum_{j=1}^{n} S_j = C_0(u^n - 1) - (u-1)LP\sum_{j=1}^{n} u^{n-j}\sum_{k=1}^{j-1} p_k + LP[t - r(1-t)]\sum_{j=1}^{n} u^{n-j}p_j - t\sum_{j=1}^{n}\Delta R_j u^{n-j}$$

In this expression $u = 1+(1-t)i$ is the interest factor reduced by the tax rate and $\Delta R_j = d_jR_j - d_{j-1}R_{j-1}$. A few more algebraic manipulations (Appendices 1,2 and 3) give a simple equation for C_n.

$$C_n = (C_0 - LPd_0)u^n + R_nd_n \qquad (2.5)$$

If $n = N$, the year of the last payment, then $R_n = 0$. This also gives the break even loss ratio $L_b = C_0/d_0P$, which, in the absence of reinsurance, simplifies to $(1-E^*)/d_0$. E^* would be the underwriting expense ratio without a loading for the profit provision. This formula also gives a stronger expression for the minimum commission that the ceding company should demand from the re-insurer. At n=N it should be true that $C_0 = (1-c)W-$ $EW+cRW \triangleleft\!\leftarrow$

$d_0L(1-c)W$ or that $R \leftarrow 1-[(1-E)-d_0L(1-c)]/c$. Of course if L is too large the required R becomes too large and it becomes doubtful if there will be any quota share re-insurance.

3. Surplus

A related problem is determining if writing this policy will be profitable. This entails considering the return on the surplus, the investment return on underwriting and the underwriting profit. The fundamental equation to consider is:

$$Z_n = Z_0u^n + \sum_{k=1}^{n} S_k - LP\sum_{k=1}^{n} p_k + C_0$$
$$= Z_0u^n + C_n \qquad (3.1)$$

where Z_n is the surplus at the end of year n. The thing to notice about this formula is that the surplus increases because of two separate sources: the interest on the original surplus and the interest on the underwriting account. This can be a source of some confusion for an ongoing operation. After the first year Z_0 turns into Z_1, and for accounting purposes, this becomes the original surplus for a new year. It would not be correct to replace Z_0 with Z_1 in (3.1) because it would over-count the interest on the interest implicit in C_n. We must choose a zero year and then add the surplus contributions from future years by summing the C-contributions for each year from the zero year to the year n. For the purposes of calculating the income from the surplus Z_0

will not grow from underwriting until *n*=*N*, the year of the last payment and it is known for certain if there is a positive or negative contribution. Until that point is reached return on surplus for year *n* is just $(u-1)Z_0u^{n-1}$ and the return on the C_n contribution is calculated with (2.1). The surplus at the end of year *N* would be $Z_0u^N + C_N$.

The exception to this is the issue of capital infusion. For example, capital can be supplied by a surplus note, private investment or a stock issue. The timing can also be critical. If the *n-th* contribution , CC_n, is made at time T_n with initial surplus of Z_0 in year 1, the surplus for year *n* will be $Z_n = uZ_{n-1} + u^{1-T_n}CC_n$. Here $1 \trianglelefteq T_n \trianglelefteq 0$ is the fraction of the year that represents when the contribution is received by the company. Thus a contribution effective at June 30th of the year would make $T_n = ½$. This implies a general expression to use for computing surplus return: $Z_n = Z_0u^n + \sum_{k=1}^{n} u^{n+1-k-T_k}CC_k$. The income from surplus for year *n* would then be $(u-1)Z_{n-1} + (u^{1-T_n} - 1)CC_n$. Appendix 4 shows the from the equation for Z_n takes under the assumptions that the rate of return is different from year to year but that the rate for discounting reserves remains constant.

It almost goes without saying that if the capital contribution is from a surplus note, the calculated return should be reduced by the amount of interest payable on the note at the end of the year.

Exhibit 1 illustrates the flow into future years of a single years written premium. The last line uses the formula to calculate the cash flow fund as a check on the cash flow lines above.

4. Calendar Year Investment Income

This analysis began with the hypothesis that P is written premium and that the loss experience is policy year loss experience. At the end of a calendar year only half of the written premium is earned and only half of the losses have been incurred. The tax treatment of the reserves is on the reserves for losses that have been incurred. The derivation began with a single policy written at the beginning of the year. There isn't anything in the derivation that prevents us from regarding the premium earned during the year as coming from a single policy written at the beginning of the year for the amount of the earned premium at its end. The reserves will be accident year reserves, the p_j will be accident year settlement patterns and the tax treatment will be on accident year reserves. Throughout the assumption will be that the accident year outstanding and IBNR reserves for all years is accurate and therefore the accident year ultimate losses are equal to the calendar year incurred losses.

Assuming that premium writings stay constant from year to year, the calendar year investment income at year *n* would be the sum of equation (2.1) from one to *n*.

$$\sum_{k=1}^{n} I_k = (1-t)\sum_{k=1}^{n}\left[iC_{k-1} - rp_k LP\right]$$

$$= \sum_{k=1}^{n} S_k + t\sum_{k=1}^{n}\Delta R_k - tLP\sum_{k=1}^{n} p_k$$

$$= (C_0 - LPd_0)(u^n - 1) + LP(1-t)\sum_{k=1}^{n} p_k - LP(1-t)d_0$$

An alternate derivation of this formula and confirmation of its correctness would be to derive it directly from equation (2.5)

$$\sum_{k=1}^{n} I_k = \sum_{k=1}^{n} i(1-t)C_{k-1} - rLP(1-t)\sum_{k=1}^{n} p_k$$

$$= i(1-t)\sum_{k=1}^{n}(C_0 - LPd_0)u^{k-1} - rLP(1-t)\sum_{k=1}^{n} p_k + i(1-t)\sum_{k=1}^{n} R_{k-1}d_{k-1}$$

$$= \frac{(u-1)(C_0 - LPd_0)(u^n - 1)}{(u-1)} - rLP(1-t)\sum_{k=1}^{n} p_k + LPi(1-t)\sum_{k=1}^{n}\sum_{j=k}^{n} v^{j-k+1/2} p_j$$

$$= (C_0 - LPd_0)(u^n - 1) - rLP(1-t)\sum_{k=1}^{n} p_k + LPi(1-t)\sum_{j=1}^{n}\left(\sum_{k=1}^{j} v^{-k}\right)v^{j+1/2} p_j$$

$$= (C_0 - LPd_0)(u^n - 1) - rLP(1-t)\sum_{k=1}^{n} p_k - v^{-1}LP(1-t)\sum_{j=1}^{n}(1 - v^{-j})v^{j+1/2} p_j$$

$$= (C_0 - LPd_0)(u^n - 1) - rLP(1-t)\sum_{k=1}^{n} p_k - LP(1-t)d_0 + v^{-1/2}LP(1-t)\sum_{k=1}^{n} p_k$$

$$= (C_0 - LPd_0)(u^n - 1) + LP(1-t)\sum_{k=1}^{n} p_k - LP(1-t)d_0$$

This is all very interesting but it is rarely true that premium is written at the same level each year or that the ultimate loss ratios are the same from year to year. The advantage of simplified formulas is to provide insight into the process of how investment income is related to the underwriting process. There is still some practical use for formula (2.5) in constructing a pro-forma to predict future investment income under different written premium and re-insurance scenarios. We can find the cash fund in year *n* to support different levels of premium, reinsurance, interest rate, ultimate loss ratio and underwriting expense assumptions for all years prior to *n*. We could then use formula (2.1) to find the investment contribution each policy year makes to the current calendar year.

5. Refinements

A realistic scenario often includes the cession of re-insurance earned premium on a quarterly basis with the payment of commission based solely on the amounts ceded. This can result in an increase in the amounts of interest on the delay of premium payments and a decrease of the return on the commission not yet received. There can be other things that affect surplus such as capital infusions or surplus notes. If the capital infusion comes from a surplus note the interest on the note would be a deduction to the rate of return.

Accounting for these things can be simple enough in a spreadsheet format. We can regard the additional interest or additional outgo as adjustments to surplus. For example if written premium is ceded quarterly on an earned basis the interest on one dollar of premium written at the beginning of the year at the end of the year would be equal to:

$$i + 0.5 - \sum_{k=1}^{4} (1+i)^{1-k/4} \frac{2k-1}{32}$$

This amount would be added to surplus and form part of the next year's initial surplus.

In practice, the IRS rules for taxes are more complex than the simple allowance for the change in discounted reserves and the amounts paid shown in the model. For example there is a credit for 80% of the change in the unearned premium reserve and 70% of the dividends received are deductible. However it isn't the purpose of the model to follow the impact of taxes on income but only the impact that taxes have on investment income that is derived directly from underwriting. The choice of the rate of return should be based on the historical patterns of net return and thus would incorporate the special treatment given to dividends or realized capital gains. The details of the tax treatment for P&C companies can be found in [AG].

We can also generalize the expression for the cash fund, equation (2.5), by assuming that rates of return differ each year (and is = i_n for year n). A separate (constant) rate i is assumed for discounting reserves. Appendices 5-7 show the derivation of the more general equation to be

$$(C_0 - LPd_0)\prod_{m=1}^{n} u_m + \alpha_n R_n d_n + \beta_n (1+r)LP - LP(1-t)\gamma_n (1+r_n)p_n \qquad (5.1)$$

where u_m is the tax adjusted interest factor for year m and $\alpha_n = ivt + \alpha_{n-1} v u_n$

$\beta_n = (\alpha_n - t)p_n + \beta_{n-1} u_n$

$\gamma_n = \dfrac{(1+r_{n-1})p_{n-1} u_n \gamma_{n-1}}{(1+r_n)p_n} + 1$ where $\alpha_0 = 1$, $\beta_0 = 0$, $\gamma_0 = 0$ and $r_n = \sqrt{1+i_n} - 1$

This looks dauntingly complicated but the practical computation doesn't present any problems.[2] It will give us another equation for calculation of the breakeven loss ratio however, because it is not generally true that $\updownarrow_N = 0 = \bullet_N$. The new formula becomes

$$L_b = \frac{C_0}{P\left\{d_0 - \prod_{m=1}^{N} u_n^{-1}\left[\beta_N(1+r) - (1-t)\gamma_N p_N(1+r_N)\right]\right\}} \tag{5.2}$$

Whether this is an increase or decrease to the breakeven loss ratio calculated earlier depends on the average size of the rates of return. If they are small from year to year and less than the discount rate, $\updownarrow_N$ will be negative and $\bullet_N$ positive. The new breakeven ratio will be smaller. Large rates of return will make $\updownarrow_N$ positive and $\bullet_N$ close to zero. The breakeven ratio will be larger. An illustration of the calculation of $\mapsto$ $\updownarrow$ and $\bullet$ for small and large rates of return is shown in Exhibit 2.

Although Appendices 5-7 follow the logic of Appendices 1-3, Appendix 7 differs from Appendix 3 by proving a formula for $-t\lhd\lambda R_j$ by induction. Whenever the *u*'s are the same from year to year, and use a rate equal to the discount rate, it isn't to hard to see that the alphas are all identically equal to 1 and the beta and gamma factors combine to equal zero.

6. Summary

The cash flow equations for a deterministic model of future investment income were derived from a consideration of an initial cash fund that is derived from the underwriting process. The recurrence relations were based on the return from this fund and the tax implications of the change in the discounted reserves and the amounts paid. The term "reserves" as defined in the model is the premium times the estimated ultimate loss and loss adjustment expense ratio. At the beginning of the year this would include "policy reserves" or reserves for claims not yet incurred and the unearned premium reserves.

The analysis begins by following the cash flow to support a policy written at the beginning of the year by establishing a fund to pay claims from the premium written and the offsets to it represented by the re-insurance agreements and the underwriting expenses. The calculation of the investment return follows by applying the expected rate of return to the formula for the cash fund at the prior year.

The model is first generalized by assuming first that the premium earned during the year is equivalent to a single policy written at the beginning of the year, and then by assuming different rates of return to be applicable for different calendar years.

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2 Appendix 8 gives closed form solutions to the alpha , beta and gamma recurrences

References:

[KN] Graham, Ronald L; Knuth, Donald E and Patashnik, Oren, *Concrete Mathematics,* Addison-Wesley, 2nd Edition 1998.

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[F] Ferrari, J.Robert, "The Relationship of Underwriting, Investment, Leverage and Exposure to Total Return on Owners' Equity", PCAS LV, 1968

[K] Kellison, Stephen G., *The Theory of Interest,* Irwin, 2nd Edition 1991.

Exhibit 1 - Sheet 1

The Yearly Rate of Return and the Discount Rate Are All the Same

Return (discount) rate = 4.0%

	Year	1	2	3	4	5	6	7	8	9	10	11
(1)	Prior Surplus	2,213,780	4,172,496	3,984,224	3,738,464	3,488,869	3,268,365	3,093,634	2,969,421	2,892,791	2,856,773	2,853,048
(2)	LOB DWP (W)	7,000,000										
(3)	Premium for Excess (x)	146,661										
(4)	Ceded %	50.0%										
(5)	Net DWP [(W-x)(1-c)=P]	3,426,669										
(6)	Expense ratio (E)	20.0%										
(7)	Expenses (WE)	1,400,000										
(8)	Ultimate Loss ratio (L)	68.6%										
(9)	Commission rate	40.0%										
(10)	Commissions (cR)(W-x)	1,370,668										
(11)	**Beginning Loss Reserve**	**2,352,139**	**2,183,826**	**1,871,118**	**1,510,276**	**1,156,188**	**841,310**	**580,881**	**377,782**	**227,176**	**120,382**	**47,674**
(12)	% paid during year	7.2%	13.3%	15.3%	15.1%	13.4%	11.1%	8.6%	6.4%	4.5%	3.1%	2.0%
(13)	Paid on reserve (LP_R)	168,312	312,708	360,842	354,088	314,878	260,429	203,099	150,606	106,794	72,708	47,674
(14)	Interest on paid (LPr_R)	3,333	6,193	7,146	7,012	6,236	5,158	4,022	2,983	2,115	1,440	944
(15)	**Ending Loss Reserve (R_n)**	**2,183,826**	**1,871,118**	**1,510,276**	**1,156,188**	**841,310**	**580,881**	**377,782**	**227,176**	**120,382**	**47,674**	**0**
(16)	Reserve Discount factor (d_n)	0.8726	0.8887	0.9014	0.9123	0.9222	0.9319	0.9419	0.9530	0.9656	0.9806	1.0000
(17)	Discounted Reserve ($R_n d_n$)	1,905,585	1,662,907	1,361,435	1,054,792	775,870	541,319	355,850	216,495	116,246	46,748	0
(18)	Change in Disc'td Reserve	91,752	242,678	301,472	306,643	278,922	234,551	185,468	139,355	100,249	69,498	46,748
(19)	Tax Effect ($-tT_n$)	30,624	28,012	23,748	18,978	14,382	10,351	7,052	4,501	2,618	1,284	370
###	*Cash Fund (beginning)*	*1,997,337*	*1,905,585*	*1,662,907*	*1,361,435*	*1,054,792*	*775,870*	*541,319*	*355,850*	*216,495*	*116,246*	*46,748*
(21)	**Interest on Reserve: ($IC_{n-1} - rp_nLP$)**	76,560	70,031	59,370	47,445	35,956	25,877	17,631	11,251	6,545	3,210	926
(22)	**$(1-t)(IC_{n-1} - rp_nLP)-tT_n$**	**76,560**	**70,031**	**59,370**	**47,445**	**35,956**	**25,877**	**17,631**	**11,251**	**6,545**	**3,210**	**926**
###	*Cash Fund (end)*	*1,905,585*	*1,662,907*	*1,361,435*	*1,054,792*	*775,870*	*541,319*	*355,850*	*216,495*	*116,246*	*46,748*	*0*
(24)												
(25)	TAX Rate	40.0%	40.0%	40.0%	40.0%	40.0%	40.0%	40.0%	40.0%	40.0%	40.0%	40.0%
(26)												
(27)	Return on Initial Surplus**	53,131	54,406	55,712	57,049	58,418	59,820	61,256	62,726	64,231	65,773	67,351
(28)	**Net addition to surplus: (22)+(27)-(13)**	**-38,622**	**-188,272**	**-245,761**	**-249,594**	**-220,504**	**-174,731**	**-124,213**	**-76,629**	**-36,018**	**-3,725**	**20,603**
(29)	Ending Surplus: (28) + (1)*	4,172,496	3,984,224	3,738,464	3,488,869	3,268,365	3,093,634	2,969,421	2,892,791	2,856,773	2,853,048	2,873,651
(30)												
(31)	**$C_n = (C_0 - LPd_0)u^n + R_nd_n$**	1,905,585	1,662,907	1,361,435	1,054,792	775,870	541,319	355,850	216,495	116,246	46,748	0
(32)												
(33)	Investment Income (before taxes)	76,560	70,031	59,370	47,445	35,956	25,877	17,631	11,251	6,545	3,210	926
(34)	Cumulative Investment Income	76,560	146,591	205,961	253,406	289,362	315,239	332,870	344,121	350,666	353,876	354,802
(35)												
(36)	**Z_0u^n**	2,266,911	2,321,317	2,377,029	2,434,077	2,492,495	2,552,315	2,613,571	2,676,296	2,740,527	2,806,300	2,873,651
(37)	$Z_0u^n+C_n$	4,172,496	3,984,224	3,738,464	3,488,869	3,268,365	3,093,634	2,969,421	2,892,791	2,856,773	2,853,048	2,873,651

r = 6-month rate =	1.98%
u = 1+i(1-t) =	1.0240
d_0 =	0.8492
Breakeven Ult. L/R	68.6%

* This is equal to (28) + (1) + (20) for the first year

** (u-1)Z_0+ (u-1) x sum of the entries for prior years

Exhibit 1 - Sheet 2

The Yearly Rate of Return Differs from Year to Year

Discount rate = 4.0%

	Year	1	2	3	4	5	6	7	8	9	10	11
(1)	Prior Surplus	2,213,780	4,189,940	4,035,363	3,830,738	3,631,308	3,471,192	3,308,117	3,217,122	3,185,805	3,207,666	3,275,163
(2)	LOB DWP (W)	7,000,000										
(3)	Premium for Excess (x)	146,661										
(4)	Ceded %	50.0%		r = 6-month rate =		1.98%						
(5)	Net DWP [(W-x)(1-c)=P]	3,426,669		d_0 =		0.8492						
(6)	Expense ratio (E)	20.0%		β_N =		0.844	0.860 =$(1+r)\beta_N$					
(7)	Expenses (WE)	1,400,000		γ_N =		63.133	0.796 =$(1-t)(1+r_N)p_N\gamma_N$					
(8)	Ultimate Loss ratio (L)	72.2%		$1/\Pi u_j$ =		0.657						
(9)	Commission rate	40.0%		Breakeven Ult. L/R		72.2%						
(10)	Commissions (cR)(W-x)	1,370,668										
(11)	**Beginning Loss Reserve**	**2,475,100**	**2,297,989**	**1,968,934**	**1,589,228**	**1,216,629**	**885,291**	**611,248**	**397,531**	**239,052**	**126,675**	**50,166**
(12)	% paid during year	7.2%	13.3%	15.3%	15.1%	13.4%	11.1%	8.6%	6.4%	4.5%	3.1%	2.0%
(13)	Paid on reserve (LPp_n)	177,111	329,056	379,706	372,598	331,339	274,043	213,716	158,480	112,377	76,509	50,166
(14)	Interest on paid ($LPrp_n$)	3,507	6,517	7,520	7,379	6,562	5,427	4,232	3,139	2,225	1,515	993
(15)	**Ending Loss Reserve (R_n)**	**2,297,989**	**1,968,934**	**1,589,228**	**1,216,629**	**885,291**	**611,248**	**397,531**	**239,052**	**126,675**	**50,166**	**0**
(16)	Reserve Discount factor (d_n)	0.8726	0.8887	0.9014	0.9123	0.9222	0.9319	0.9419	0.9530	0.9656	0.9806	1.0000
(17)	Discounted Reserve (R_nd_n)	2,005,202	1,749,838	1,432,606	1,109,933	816,430	569,617	374,453	227,813	122,323	49,192	0
(18)	Change in Disc'td Reserve	96,549	255,364	317,232	322,673	293,503	246,813	195,164	146,640	105,490	73,131	49,192
(19)	Tax Effect ($-tT_n$)	32,225	29,477	24,990	19,970	15,134	10,892	7,421	4,736	2,755	1,351	390
(20)	Cash Fund (beginning)	1,997,337	1,909,747	1,673,082	1,376,328	1,073,813	798,610	555,357	365,263	222,723	120,135	48,694
(21)	Rate of Return (i_n)	5.0%	6.0%	6.5%	7.0%	7.5%	5.0%	6.0%	6.5%	7.0%	7.5%	7.5%
(22)	u_n	1.030	1.036	1.039	1.042	1.045	1.030	1.036	1.039	1.042	1.045	1.045
(23)	α_n	1.006	1.017	1.032	1.049	1.069	1.075	1.086	1.100	1.118	1.138	1.159
(24)	β_n	0.043	0.127	0.229	0.336	0.441	0.529	0.607	0.676	0.737	0.793	0.844
(25)	γ_n	1.000	1.555	2.397	3.539	5.149	7.489	10.901	16.238	24.806	38.986	63.133
(26)	**Interest on Reserve: ($i_nC_{n-1} - r_np_nLP$)**	95,493	104,857	96,604	83,523	68,335	33,163	27,003	18,673	11,724	6,193	1,805
(27)	**$(1-t)(i_nC_{n-1} - r_np_nLP)-tT_n$**	89,521	92,391	82,952	70,084	56,135	30,790	23,623	15,939	9,789	5,067	1,473
(28)	Cash Fund (end)	1,909,747	1,673,082	1,376,328	1,073,813	798,610	555,357	365,263	222,723	120,135	48,694	0
(29)	TAX Rate	40.0%	40.0%	40.0%	40.0%	40.0%	40.0%	40.0%	40.0%	40.0%	40.0%	40.0%
(30)	Return on Initial Surplus**	66,413	82,087	92,129	103,085	115,087	80,177	99,099	111,223	124,449	138,939	145,191
(31)	**Net addition to surplus: (27)+(30)-(13)**	**-21,177**	**-154,578**	**-204,625**	**-199,430**	**-160,116**	**-163,076**	**-90,994**	**-31,318**	**21,862**	**67,497**	**96,498**
(32)	Ending Surplus: (31) + (1)*	4,189,940	4,035,363	3,830,738	3,631,308	3,471,192	3,308,117	3,217,122	3,185,805	3,207,666	3,275,163	3,371,661
(33)												
(34)	$C_n = (C_0 - LPd_0)\Pi u_j + \alpha_nR_nd_n$											
(35)	$+ \beta_n(1+r)LP - (1-t)LP\gamma_n(1+r_n)p_n$	1,909,747	1,673,082	1,376,328	1,073,813	798,610	555,357	365,263	222,723	120,135	48,694	0
(36)	Investment Income (before taxes)	95,493	104,857	96,604	83,523	68,335	33,163	27,003	18,673	11,724	6,193	1,805
(37)	Cumulative Investment Income	95,493	200,350	296,954	380,477	448,812	481,975	508,978	527,651	539,375	545,568	547,373
(38)												
(39)	$Z_0\Pi u_j$	2,280,194	2,362,281	2,454,410	2,557,495	2,672,582	2,752,760	2,851,859	2,963,082	3,087,531	3,226,470	3,371,661
(40)	$Z_0\Pi u_j+C_n$	4,189,940	4,035,363	3,830,738	3,631,308	3,471,192	3,308,117	3,217,122	3,185,805	3,207,666	3,275,163	3,371,661

* This is equal to (20) + (1) + (31) for the first year

** $(\Pi u_j-1)Z_0$ - sum of prior year entries

Exhibit 2

Calculation of the Alpha and Beta Factors for Varying Interest Rates in the General Model

Tax Rate	40.0%	1+r =	1.024695
Disc't Rate	5.0%	β_N =	0.69380
$1/\Pi u_n$ =	0.793	γ_N =	195.97931

$(1+r)\beta_N/\Pi u_n$ = 0.56392

$(1-t)p_N(1+r_N)\gamma_N/\Pi u_n$ = 0.56796

$(1+r)\beta_N/\Pi u_n$ - $(1-t)p_N(1+r_N)\gamma_N/\Pi u_n$ = -0.00404

Calculation of Alpha and Beta for Low Rates of Return

n	1	2	3	4	5	6	7	8	9	10	11	12
i_n	4.0%	3.5%	3.7%	3.0%	2.5%	3.6%	3.9%	4.0%	2.5%	3.3%	2.0%	3.0%
u_n	1.024	1.021	1.022	1.018	1.015	1.022	1.023	1.024	1.015	1.020	1.012	1.018
r_n	2.0%	1.7%	1.8%	1.5%	1.2%	1.8%	1.9%	2.0%	1.2%	1.6%	1.0%	1.5%
p_n	23.6%	21.0%	16.1%	12.0%	8.7%	6.2%	4.3%	3.0%	2.1%	1.4%	1.0%	0.6%
α_n	0.994	0.991	0.993	0.989	0.986	0.992	0.994	0.994	0.986	0.990	0.983	0.989
β_n	0.14025	0.26740	0.36874	0.44600	0.50365	0.55123	0.58966	0.62164	0.64327	0.66427	0.67807	0.69380
γ_n	1.00000	2.15018	3.86408	6.29552	9.83520	15.02406	23.13747	34.94331	52.03719	80.29246	115.48100	195.97931

Tax Rate	40.0%	1+r =	1.024695
Disc't Rate	5.0%	β_N =	0.92492
$1/\Pi u_n$ =	0.565	γ_N =	1.13404

$(1+r)\beta_N/\Pi u_n$ = 0.53561

$(1-t)p_N(1+r_N)\gamma_N/\Pi u_n$ = 0.00239

$(1+r)\beta_N/\Pi u_n$ - $(1-t)p_N(1+r_N)\gamma_N/\Pi u_n$ = 0.53322

Calculation of Alpha and Beta for High Rates of Return

n	1	2	3	4	5	6	7	8	9	10	11	12
i_n	10.0%	8.8%	9.3%	7.5%	6.3%	9.0%	9.8%	10.0%	6.3%	8.3%	5.0%	7.5%
u_n	1.060	1.053	1.056	1.045	1.038	1.054	1.059	1.060	1.038	1.050	1.030	1.045
r_n	4.9%	4.3%	4.5%	3.7%	3.1%	4.4%	4.8%	4.9%	3.1%	4.0%	2.5%	3.7%
p_n	23.6%	21.0%	16.1%	12.0%	8.7%	6.2%	4.3%	3.0%	2.1%	1.4%	1.0%	0.6%
α_n	1.029	1.021	1.024	1.014	1.007	1.023	1.027	1.029	1.007	1.019	1.000	1.014
β_n	0.14834	0.28663	0.40305	0.49490	0.56628	0.63548	0.69962	0.76045	0.80172	0.85007	0.88157	0.92492
γ_n	1.00000	1.09869	1.13237	1.11453	1.09643	1.13736	1.15954	1.16608	1.10526	1.13598	1.08028	1.13404

Exhibit 3

Calculation of the Reserve Discount Factor

Interest rate = 4%

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
Year	Discount Factor	Incremt'l Payments	Cumulative Payments	Complement of Cumul.	Product	Cumul Product	Comp. Of Product	Reserve Discount Factor (d_n)
1	0.9806	7%	7%	93%	0.0702	0.0702	0.7790	0.8726
2	0.9429	13%	20%	80%	0.1254	0.1955	0.6536	0.8887
3	0.9066	15%	36%	64%	0.1391	0.3346	0.5146	0.9014
4	0.8717	15%	51%	49%	0.1312	0.4658	0.3833	0.9123
5	0.8382	13%	64%	36%	0.1122	0.5780	0.2711	0.9222
6	0.8060	11%	75%	25%	0.0892	0.6673	0.1819	0.9319
7	0.7750	9%	84%	16%	0.0669	0.7342	0.1150	0.9419
8	0.7452	6%	90%	10%	0.0477	0.7819	0.0673	0.9530
9	0.7165	5%	95%	5%	0.0325	0.8144	0.0347	0.9656
10	0.6889	3%	98%	2%	0.0213	0.8357	0.0134	0.9806
11	0.6624	2%	100%	0%	0.0134	0.8492	0.0000	1.0000
		100%			0.8492	$=d_0$		

Notes:

Col (2): 1.04^[Col (1) - 1/2]
Col (3): from Exhibit A
Col (4): Cumulative of Col (3)
Col (5): 1- Col (4)

Col (6): Col (2) x Col (3)
Col (7): Cumulative of Col (6)
Col (8): Sum of Col (7) - Col (7)
Col (9): 1.04^[Col (1)] x Col (8) / Col (5)

Exhibit 4

Mathematical Formulation of Exhibit 3

For: i = interest rate
p_j = % of ultimate paid in year j

$$Col\ (1) : k$$

$$Col\ (2) : (1+i)^{-k+1/2}$$

$$Col\ (3) : p_k$$

$$Col\ (4) : \sum_{j=1}^{k} p_j$$

$$Col\ (5) : 1 - Col\ (4) = \sum_{j=k+1}^{n} p_j$$

$$Col\ (6) : (1+i)^{-k+1/2}\, p_k$$

$$Col\ (7) : \sum_{j=1}^{k} (1+i)^{-j+1/2}\, p_j$$

$$Col\ (8) : \sum_{j=k+1}^{n} (1+i)^{-j+1/2}\, p_j$$

$$Col\ (9) : \frac{\sum_{j=k+1}^{n} (1+i)^{-j+k+1/2}\, p_j}{\sum_{j=k+1}^{n} p_j}$$

APPENDIX 1

$C_n = C_0 + \sum_{j=1}^{n} S_j - LP\sum_{j=1}^{n} p_j$ Now define $T_j = d_{j-1}R_{j-1} - d_jR_j - LPp_j$

where $R_j = LP\sum_{k=j+1}^{N} p_k$

where $S_j = (1-t)[iC_{j-1} - rp_jLP] - tT_j$ Letting $D_j = \sum_{k=1}^{j} S_k$ we get

$D_j - D_{j-1} = (1-t)[iC_{j-1} - rp_jLP] - tT_j$

$= (1\text{-}t)\left[i\left(C_0 + D_{j-1} - LP\sum_{k=1}^{j-1} p_k\right) - rp_jLP\right] + tLPp_j - t\Delta R_j$ (where $\Delta R_j = d_{j-1}R_{j-1} - d_jR_j$)

$= (u-1)D_{j-1} + \lambda_j$ if we let λ_j represent all of the terms that do not involve D and $u = 1 + (1-t)i$

Hence if we multiply both sides of this equation by u^{-j} and sum we get

$\sum_{j=1}^{n}\left(D_ju^{-j} - D_{j-1}u^{-(j-1)}\right) = D_nu^{-n} = \sum_{j=1}^{n} \lambda_ju^{-j}$ because $D_0 = 0$ by its definition. Thus

$D_n = \sum_{j=1}^{n} S_j = \sum_{j=1}^{n} \lambda_ju^{n-j} = S_1^* - S_2^* + S_3^* + S_4^*$

$S_1^* = (u-1)C_0\sum_{j=1}^{n} u^{n-j} = C_0(u^n - 1)$

$S_2^* = (u-1)LP\sum_{j=1}^{n} u^{n-j}\sum_{k=1}^{j-1} p_k = LP\left(\sum_{k=1}^{n-1} p_k\left(u^{n-k} - 1\right)\right) = LPu^nA_{n-1} - LP\sum_{k=1}^{n-1} p_k$ {See Appendix 2}

for $A_n = \sum_{k=1}^{n} p_ku^{-k}$

$S_3^* = LP[t - r(1-t)]\sum_{j=1}^{n} u^{n-j}p_j = LP[t - r(1-t)]u^nA_n$

$S_4^* = -t\sum \Delta R_ju^{n-j} = (1\text{-}t)LPu^n(1+r)A_n + R_nd_n - LPd_0\,u^n$ {See Appendix 3}

Adding together these simplified sums we get

$$\sum_{j=1}^{n} S_j = C_0(u^n - 1) - \left[LPu^n A_{n-1} - LP\sum_{k=1}^{n-1} p_k \right] + \left\{ LP\left[t - r(1-t)\right]u^n A_n \right\}$$

$$+ (1 - t)LPu^n\left[(1 + r)A_n - d_0\right] + R_n d_n - tLPd_0 u^n$$

$$= C_0(u^n - 1) + LP\left[t - r(1 - t) + (1 - t)(1 + r)\right]u^n A_n - LPu^n A_{n-1} + LP\sum_{k=1}^{n-1} p_k - d_0 LPu^n + R_n d_n$$

$$= C_0(u^n - 1) + LPu^n\left(A_n - A_{n-1}\right) + LP\sum_{k=1}^{n-1} p_k - d_0 LPu^n + R_n d_n$$

[since $t - r(1 - t) + (1 - t)(1 + r) = 1$]

$$= C_0(u^n - 1) + LPu^n(p_n u^{-n}) + LP\sum_{k=1}^{n-1} p_k - d_0 LPu^n + R_n d_n$$

$$= C_0(u^n - 1) + LP\sum_{k=1}^{n} p_k - d_0 LPu^n + R_n d_n$$

Adding this to the expression for C_n we get

$$C_n = C_0 u^n - d_0 LPu^n + R_n d_n$$

APPENDIX 2

Simplification of the S_2^* term in the expression for the sum of the surplus contributions

$$(u-1)LP\sum_{j=1}^{n} u^{n-j}\sum_{k=1}^{j-1} p_k$$

$$= (u-1)LP\sum_{j=2}^{n} u^{n-j}\sum_{k=1}^{j-1} p_k$$

$$= (u-1)LP\sum_{j=1}^{n-1} u^{n-j-1}\sum_{k=1}^{j} p_k = (u-1)LP\sum_{k=1}^{n-1} p_k\sum_{j=k}^{n-1} u^{n-j-1}$$

$$= (u-1)u^{n-1}LP\sum_{k=1}^{n-1} p_k u^{-k}\sum_{j=k}^{n-1} u^{k-j}$$

$$= (u-1)u^{n-1}LP\sum_{k=1}^{n-1} p_k u^{-k}\left(\frac{1-u^{-(n-k)}}{1-u^{-1}}\right) = LP\sum_{k=1}^{n-1} p_k u^{-k}\left(u^n - u^k\right)$$

$$= LPu^n A_{n-1} - LP\sum_{k=1}^{n-1} p_k \quad \text{where } A_n = \sum_{k=1}^{n} p_k u^{-k}$$

APPENDIX 3

Simplification of the S_4^* term in the expression for the sum of the surplus contributions

$$t\sum_{j=1}^{n} R_{j-1} d_{j-1} u^{n-j}$$

$$= tLP\sum_{j=1}^{n} u^{n-j} \sum_{k=j}^{n} p_k v^{k-j+1/2} + tLP\sum_{j=1}^{n} u^{n-j} \sum_{k=n+1}^{N} p_k v^{k-j+1/2}$$

$$= tu^n LP\sum_{k=1}^{n} p_k v^{k+1/2} \sum_{j=1}^{k} (uv)^{-j} + tu^n LP \sum_{k=n+1}^{N} p_k v^{k+1/2} \sum_{j=1}^{n} (uv)^{-j}$$

$$= tLPu^{n-1} v^{-1} \sum_{k=1}^{n} p_k \left[\frac{1-(uv)^{-k}}{1-(uv)^{-1}}\right] v^{k+1/2} + tLPu^{n-1} v^{-1} \sum_{k=n+1}^{N} p_k \left[\frac{1-(uv)^{-n}}{1-(uv)^{-1}}\right] v^{k+1/2}$$

$$= -\frac{(1+i)}{i} LPu^n \left[\sum_{k=1}^{n} p_k v^{k+1/2} - v^{1/2} \sum_{k=1}^{n} p_k u^{-k}\right] - \frac{(1+i)}{i} LPu^n v^{n+1} \sum_{k=n+1}^{N} p_k v^{k-n-1/2} \left[1-(uv)^{-n}\right]$$

$$= \frac{LPu^n}{i}\left[v^{-1/2} A_n - \sum_{j=1}^{n} p_j v^{j-1/2}\right] + \frac{R_n d_n}{i}\left[1-(uv)^n\right]$$

$$= \frac{LPu^n}{i}\left[(1+r)A_n - \left(\sum_{j=1}^{N} p_j v^{j-1/2} - \sum_{j=n+1}^{N} p_j v^{j-1/2}\right)\right] + \frac{R_n d_n}{i}\left[1-(uv)^n\right]$$

$$= \frac{LPu^n}{i}\left[(1+r)A_n - \left(d_0 - v^n \sum_{j=n+1}^{N} p_j v^{j-n-1/2}\right)\right] + \frac{R_n d_n}{i}\left[1-(uv)^n\right]$$

$$= \frac{LPu^n}{i}\left[(1+r)A_n - d_0\right] + \frac{(uv)^n R_n d_n}{i} + \frac{R_n d_n}{i}\left[1-(uv)^n\right]$$

$$= \frac{LPu^n}{i}\left[(1+r)A_n - d_0\right] + \frac{R_n d_n}{i}$$

But

$$t\sum_{j=1}^{n} R_j d_j u^{n-j}$$

$$= t\sum_{j=2}^{n+1} R_{j-1} d_{j-1} u^{n-j+1}$$

$$= ut\sum_{j=1}^{n} R_{j-1} d_{j-1} u^{n-j} + tR_n d_n - tR_0 d_0 u^n$$

$$= u\left\{\frac{LPu^n}{i}\left[(1+r)A_n - d_0\right]\right\} + \frac{uR_n d_n}{i} + tR_n d_n - tLPd_0 u^n$$

since $R_0 = LP$

Hence

$$-t\sum_{j=1}^{n} \Delta R_j u^{n-j}$$

$$= (u-1)\left\{\frac{LPu^n}{i}\left[(1+r)A_n - d_0\right]\right\} + \frac{(u-1)R_n d_n}{i} + tR_n d_n - tLPd_0 u^n$$

$$= (1-t)LPu^n\left[(1+r)A_n - d_0\right] + R_n d_n - tLPd_0 u^n$$

$$= (1-t)LPu^n (1+r)A_n + R_n d_n - LPd_0 u^n$$

APPENDIX 4

The Derivation of "Interest" Surplus When Interest Rates Vary From Year to Year

Let $Z_k = u_k Z_{k-1} + u_k^{1-T_k} CC_k$

where $u_k = 1 + (1-t)i_k$ is the interest factor for year k

and T_k is the timing of the capital infusion for year k

$$Z_k \prod_{j=1}^{k} u_j^{-1} - Z_{k-1} \prod_{j=1}^{k} u_j^{-1} u_k = u_k^{1-T_k} \prod_{j=1}^{k} u_j^{-1} CC_k$$

$$\Rightarrow Z_n \prod_{j=1}^{n} u_j^{-1} - Z_0 = \sum_{k=1}^{n} u_k^{1-T_k} \left[\prod_{j=1}^{k} u_j^{-1} \right] CC_k$$

$$\Rightarrow Z_n = Z_0 \prod_{j=1}^{n} u_j + \sum_{k=1}^{n} u_k^{1-T_k} \left[\prod_{j=1}^{k} u_j^{-1} \right] \left[\prod_{j=1}^{n} u_j \right] CC_k$$

$$= Z_0 \prod_{j=1}^{n} u_j + \sum_{k=1}^{n} u_k^{1-T_k} \left[\prod_{j=k+1}^{n} u_j \right] CC_k$$

where the product in the second term is interpreted as $= 1$ if $k + 1 > n$

APPENDIX 5

$C_n = C_0 + \sum_{j=1}^{n} S_j - LP\sum_{j=1}^{n} p_j$ Now define $T_j = d_{j-1}R_{j-1} - d_jR_j - LPp_j$

where $R_j = LP\sum_{k=j+1}^{N} p_k$

where $S_j = (1-t)[i_jC_{j-1} - r_jp_jLP] - tT_j$ Letting $D_j = \sum_{k=1}^{j} S_k$ we get

$D_j - D_{j-1} = (1-t)[i_jC_{j-1} - r_jp_jLP] - tT_j$

$= (1-t)\left[i_j\left(C_0 + D_{j-1} - LP\sum_{k=1}^{j-1} p_k\right) - r_jp_jLP\right] + tLPp_j - t\Delta R_j$ (where $\Delta R_j = d_{j-1}R_{j-1} - d_jR_j$)

$= (u_j - 1)D_{j-1} + \lambda_j$ if we let λ_j represent all of the terms that do not involve D and $u_j = 1 + (1-t)i_j$

Hence if we multiply both sides of this equation by $\prod_{m=1}^{j} u_m^{-1}$ and sum we get

$\sum_{j=1}^{n}\left(D_j\prod_{m=1}^{j} u_m^{-1} - D_{j-1}\prod_{m=1}^{j} u_m^{-1}u_j\right) = D_n\prod_{m=1}^{n} u_m^{-1} = \sum_{j=1}^{n}\lambda_j\prod_{m=1}^{j} u_m^{-1}$ because $D_0 = 0$ by its definition. Thus

$D_n = \sum_{j=1}^{n} S_j = \sum_{j=1}^{n}\lambda_j\prod_{m=j+1}^{n} u_m = S_1^* - S_2^* + S_3^* + S_4^*$

$S_1^* = C_0\sum_{j=1}^{n}(u_j - 1)\prod_{m=j+1}^{n} u_m = C_0\sum_{j=1}^{n}\left(\prod_{m=j}^{n} u_m - \prod_{m=j+1}^{n} u_m\right) = C_0\left(\prod_{m=1}^{n} u_m - 1\right)$

$S_2^* = LP\sum_{j=1}^{n}(u_j - 1)\prod_{m=j+1}^{n} u_m\sum_{k=1}^{j-1} p_k = LP\left[\sum_{k=1}^{n-1} p_k\left(\prod_{m=k+1}^{n} u_m - 1\right)\right] = LP\left(\prod_{m=1}^{n} u_m\right)A_{n-1}$

$- LP\sum_{k=1}^{n-1} p_k$ {See Appendix 6} for $A_n = \sum_{k=1}^{n} p_k\prod_{m=1}^{k} u_m^{-1}$

$S_3^* = LPt\sum_{j=1}^{n}\prod_{m=j+1}^{n} u_m p_j - LP(1-t)\sum_{j=1}^{n}\prod_{m=j+1}^{n} u_m p_j r_j$

$= LPt\left(\prod_{m=1}^{n} u_m\right)A_n - LP(1-t)\left(\prod_{m=1}^{n} u_m\right)A_n^* \quad \left(\text{for } A_n^* = \sum_{j=1}^{n} r_jp_j\prod_{m=1}^{j} u_m^{-1}\right)$

$S_4^* = -t\sum_{j=1}^{n}\Delta R_j\left(\prod_{m=1}^{k} u_m^{-1}\right) = (1-t)LP\prod_{m=1}^{n} u_m(A_n + A_n^*) + \alpha_nR_nd_n - LPd_0\prod_{m=1}^{n} u_m + LP(1+r)\beta_n$

$- LP(1-t)\gamma_n(1+r_n)p_n$ {See Appendix 7}

Adding together these simplified sums we get

$$\sum_{j=1}^{n} S_j = C_0\left(\prod_{m=1}^{n} u_m - 1\right) - \left[LP\prod_{m=1}^{n} u_m A_{n-1} - LP\sum_{k=1}^{n-1} p_k\right] + \left\{LPt\prod_{m=1}^{n} u_m A_n - (1-t)\prod_{m=1}^{n} u_m A_n^{\bullet}\right\}$$

$$+ (1-t)LP\prod_{m=1}^{n} u_m\left[(A_n + A_n^{\bullet}) - d_0\right] + \alpha_n R_n d_n - tLPd_0\prod_{m=1}^{n} u_m + LP(1+r)\beta_n$$

$$- (1-t)LP\gamma_n(1+r_n)p_n$$

$$= C_0\left(\prod_{m=1}^{n} u_m - 1\right) + LP[t + (1-t)]\prod_{m=1}^{n} u_m A_n - LP\prod_{m=1}^{n} u_m A_{n-1}$$

$$+ LP\sum_{k=1}^{n-1} p_k - d_0 LP\prod_{m=1}^{n} u_m + \alpha_n R_n d_n + LP(1+r)\beta_n - (1-t)LP\gamma_n(1+r_n)p_n$$

$$= C_0\left(\prod_{m=1}^{n} u_m - 1\right) + LP\prod_{m=1}^{n} u_m (A_n - A_{n-1}) + LP\sum_{k=1}^{n-1} p_k - d_0 LP\prod_{m=1}^{n} u_m + \alpha_n R_n d_n$$

$$+ LP(1+r)\beta_n - (1-t)LP\gamma_n(1+r_n)p_n$$

$$= C_0\left(\prod_{m=1}^{n} u_m - 1\right) + LP\prod_{m=1}^{n} u_m\left(p_n\prod_{m=1}^{n} u_m^{-1}\right) + LP\sum_{k=1}^{n-1} p_k - d_0 LP\prod_{m=1}^{n} u_m + \alpha_n R_n d_n$$

$$+ LP(1+r)\beta_n - (1-t)LP\gamma_n(1+r_n)p_n$$

$$= C_0\left(\prod_{m=1}^{n} u_m - 1\right) + LP\sum_{k=1}^{n} p_k - d_0 LP\prod_{m=1}^{n} u_m + \alpha_n R_n d_n + LP(1+r)\beta_n - (1-t)LP\gamma_n(1+r_n)p_n$$

Adding this to the expression for C_n we get

$$C_n = (C_0 - d_0 LP)\prod_{m=1}^{n} u_m + \alpha_n R_n d_n + LP(1+r)\beta_n - (1-t)LP\gamma_n(1+r_n)p_n$$

where $\alpha_n = ivt + v\alpha_{n-1}u_n$ and $\beta_n = \beta_{n-1}u_n + (\alpha_n - t)p_n$ $(\alpha_0 = 1$ and $\beta_0 = 0)$

$$\gamma_n = \frac{\gamma_{n-1}(1+r_{n-1})u_n p_{n-1}}{(1+r_n)p_n} + 1 \quad (\gamma_0 = 0)$$

Simplification of the S_2^* term in the expression for the sum of the surplus contributions

$$\text{Let } q_j = \prod_{m=j+1}^{N} u_m$$

$$LP\sum_{j=1}^{n}(u_j - 1)\prod_{m=j+1}^{n} u_m \sum_{k=1}^{j-1} p_k$$

$$= LP\sum_{j=2}^{n}\left(\frac{q_{j-1}}{q_j} - 1\right)\frac{q_j}{q_n}\sum_{k=1}^{j-1} p_k$$

$$= LP\sum_{j=1}^{n-1}\frac{(q_j - q_{j+1})}{q_n}\sum_{k=1}^{j} p_k = LP\sum_{k=1}^{n-1} p_k \sum_{j=k}^{n-1}\frac{(q_j - q_{j+1})}{q_n}$$

$$= LP\sum_{k=1}^{n-1}\left(\frac{q_k}{q_n} - 1\right)p_k$$

$$= LP\left(\prod_{m=1}^{n} u_m\right)\sum_{k=1}^{n-1} p_k\left(\prod_{m=1}^{k} u_m^{-1}\right) - LP\sum_{k=1}^{n-1} p_k$$

$$= LP\left(\prod_{m=1}^{n} u_m\right)A_{n-1} - LP\sum_{k=1}^{n-1} p_k \quad \text{where } A_n = \sum_{k=1}^{n} p_k\left(\prod_{m=1}^{k} u_m^{-1}\right)$$

APPENDIX 7

(See the defintions on the second page)

Assume that there is a closed form for $t\sum_{j=1}^{n}\frac{q_0 w_j}{q_n}\left(R_j d_j - R_{j-1}d_{j-1}\right)$ that is equal to

$$(1 - t)LP\frac{q_0}{q_n}(A_n + A_n^{\bullet}) + \alpha_n R_n d_n - LPd_0\frac{q_0}{q_n} + LP(1 + r)\beta_n - (1 - t)LP(1 + r_n)p_n\gamma_n$$

$$t\sum_{j=1}^{n+1}\frac{q_0 w_j}{q_{n+1}}\left(R_j d_j - R_{j-1}d_{j-1}\right)$$

$$= \frac{q_n}{q_{n+1}}\left((1 - t)LP\frac{q_0}{q_n}(A_n + A_n^{\bullet}) + \alpha_n R_n d_n + LP(1 + r)\beta_n - LPd_0\frac{q_0}{q_n} - (1 - t)LP(1 + r_n)p_n\gamma_n\right)$$

$$+ t\frac{q_0 w_{n+1}}{q_{n+1}}\left(R_{n+1}d_{n+1} - R_n d_n\right)$$

$$= (1 - t)LP\frac{q_0}{q_{n+1}}(A_n + A_n^{\bullet}) + \alpha_n u_{n+1}R_n d_n - LPd_0\frac{q_0}{q_{n+1}} + t\left(R_{n+1}d_{n+1} - R_n d_n\right)$$

$$+ LP(1 + r_n)\beta_n u_{n+1} - u_{n+1}(1 - t)LP(1 + r_n)p_n\gamma_n$$

$$= (1 - t)LP\frac{q_0}{q_{n+1}}\left[\left(A_{n+1} - w_{n+1}p_{n+1}\right) + \left(A_{n+1}^{\bullet} - w_{n+1}p_{n+1}r_{n+1}\right)\right]$$

$$+ R_{n+1}d_{n+1}\left[t - tv + v\alpha_n u_{n+1}\right] + LP(1 + r)p_{n+1}\left(v\alpha_n u_{n+1} - vt\right) - LPd_0\frac{q_0}{q_{n+1}} + LP(1 + r)\beta_n u_{n+1}$$

$$- LP(1 - t)\left[(1 + r_n)u_{n+1}p_n\gamma_n + (1 + r_{n+1})p_{n+1}\right]$$

$$= (1 - t)LP\frac{q_0}{q_{n+1}}\left(A_{n+1} + A_{n+1}^{\bullet}\right) + LP(1 + r)\left[p_{n+1}\left(\alpha_n v u_{n+1} - vt\right) + \beta_n u_{n+1}\right] +$$

$$R_{n+1}d_{n+1}\left[ivt + v\alpha_n u_{n+1}\right] - LPd_0\frac{q_0}{q_{n+1}} - LP(1 - t)\left[(1 + r_n)u_{n+1}p_n\gamma_n + (1 + r_{n+1})p_{n+1}\right]$$

$$= (1 - t)LP\frac{q_0}{q_{n+1}}(A_{n+1} + A_{n+1}^{\bullet}) + LP(1 + r)\beta_{n+1} + \alpha_{n+1}R_{n+1}d_{n+1} - LPd_0\frac{q_0}{q_{n+1}} - LP(1 + r_{n+1})p_{n+1}(1 - t)\gamma_{n+1}$$

$$\Rightarrow \alpha_{n+1} = ivt + v\alpha_n u_{n+1} \text{ and } \beta_{n+1} = p_{n+1}\left(\alpha_n v u_{n+1} - vt\right) + \beta_n u_{n+1}$$

$$\gamma_{n+1} = \frac{(1 + r_n)u_{n+1}p_n\gamma_n}{(1 + r_{n+1})p_{n+1}} + 1$$

If we let $\alpha_0 = 1$, $\beta_0 = 0$ and $\gamma_0 = 0$ then it is easy to verify that this equation holds for n = 1 (See below). The algebra above shows by induction that it is true for all n.

APPENDIX 7 (Continued)

The proof of the formula above for $n = 1$:

$$t\frac{q_0 w_1}{q_1}(R_1 d_1 - R_0 d_0) = t(R_1 d_1 - R_0 d_0)$$

$$= tLP\left(\sum_{j=2}^{N} v^{j-3/2} p_j - \sum_{j=1}^{N} v^{j-1/2} p_j\right)$$

$$= tLP\left[v^{-1}\left(\sum_{j=1}^{N} v^{j-1/2} p_j - v^{1/2} p_1\right) - d_0\right]$$

$$= tLP(1+i)d_0 - tLP(1+r)p_1 - tLPd_0$$

$$= itLPd_0 - tLP(1+r)p_1$$

The formula expression at $n = 1$ is:

$$(1-t)LP\frac{q_0}{q_1}(A_1 + A_1^*) + \alpha_1 R_1 d_1 + LP(1+r)\beta_1 - LPd_0\frac{q_0}{q_1} - LP(1-t)\gamma_1(1+r_1)p_1$$

$$= LP(1-t)(1+r_1)p_1 + (ivt + vu_1)[LP(1+i)d_0 - LP(1+r)p_1] - LP(1-t)\gamma_1(1+r_1)p_1 +$$
$$LP(1+r)[vu_1 - vt]p_1 - LPd_0 u_1$$

$$= itLPd_0 + LP(1+r)[vu_1 - vt - ivt - vu_1]p_1 \quad (\text{since } \gamma_1 = 1)$$

$$= itLPd_0 - tLP(1+r)p_1$$

Definitions used in the derivation above

$$R_{n+1}d_{n+1} = LP\sum_{k=n+2}^{N} v^{k-n-3/2} p_k$$
$$= LP\left[\sum_{k=n+1}^{N} v^{k-n-3/2} p_k - p_{n+1} v^{-1/2}\right]$$
$$= LPv^{-1}\left[\sum_{k=n+1}^{N} v^{k-n-1/2} p_k\right] - LPp_{n+1} v^{-1/2}$$
$$= v^{-1}R_n d_n - LP(1+r)p_{n+1} \Rightarrow$$
$$R_n d_n = vR_{n+1}d_{n+1} + LPv(1+r)p_{n+1}$$

$$\text{Let } q_j = \prod_{m=j+1}^{N} u_m \text{ and let } w_j = \prod_{m=1}^{j} u_m^{-1}$$
$$\Rightarrow \frac{q_{j-1}}{q_j} = u_j = \frac{w_{j-1}}{w_j} \Rightarrow \frac{q_0}{q_j} = \frac{1}{w_j}$$
$$\Rightarrow A_n = \sum_{j=1}^{n} w_j p_j \left(A_n^* = \sum_{j=1}^{n} r_j p_j \prod_{m=1}^{j} u_m^{-1} = \sum_{j=1}^{n} w_j r_j p_j \right)$$

APPENDIX 8

Solution to the alpha recurrence:

$$\alpha_j = \alpha_{j-1} v u_j + ivt \Rightarrow$$

$$\alpha_j v^{-j} \prod_{m=1}^{j} u_m^{-1} - \alpha_{j-1} v^{-(j-1)} \prod_{m=1}^{j-1} u_m^{-1} = ivtv^{-j} \prod_{m=1}^{j} u_m^{-1} \Rightarrow$$

$$\alpha_n v^{-n} \prod_{m=1}^{n} u_m^{-1} - 1 = it \sum_{j=1}^{n} v^{-j+1} \prod_{m=1}^{j} u_m^{-1} \Rightarrow$$

$$\alpha_n = v^n \prod_{m=1}^{n} u_m + it \sum_{j=1}^{n} v^{n-j+1} \prod_{m=j+1}^{n} u_m$$

Solution to the beta recurrence:

$$\beta_j = \beta_{j-1} u_j + (\alpha_{j-1} v u_j - vt) p_j \Rightarrow$$

$$\beta_j \prod_{m=1}^{j} u_m^{-1} - \beta_{j-1} \prod_{m=1}^{j-1} u_m^{-1} = \prod_{m=1}^{j} u_m^{-1} (\alpha_j - t) p_j \Rightarrow$$

$$\beta_n \prod_{m=1}^{n} u_m^{-1} = \sum_{j=1}^{n} (\alpha_j - t) p_j \prod_{m=1}^{j} u_m^{-1} \Rightarrow$$

$$\beta_n = \prod_{m=1}^{n} u_m \left[\sum_{j=1}^{n} \left(v^j \prod_{m=1}^{j} u_m + it \sum_{k=1}^{j} v^{j-k+1} \prod_{m=k+1}^{j} u_m \right) p_j \prod_{m=1}^{j} u_m^{-1} \right] - t \prod_{m=1}^{n} u_m \sum_{j=1}^{n} p_j \prod_{m=1}^{j} u_m^{-1}$$

$$= \prod_{m=1}^{n} u_m \left[\sum_{j=1}^{n} \left(v^j p_j + it \sum_{k=1}^{j} v^{j-k+1} p_j \prod_{m=1}^{k} u_m^{-1} \right) \right] - t \sum_{j=1}^{n} p_j \prod_{m=j+1}^{n} u_m$$

$$= \prod_{m=1}^{n} u_m \sum_{j=1}^{n} v^j p_j + it \sum_{j=1}^{n} v^j p_j \sum_{k=1}^{j} v^{-k+1} \prod_{m=k+1}^{n} u_m - t \sum_{j=1}^{n} p_j \prod_{m=j+1}^{n} u_m$$

Solution to the gamma recurrence:

Let $\varphi_k = (1 + r_k) p_k$ Then

$$\gamma_k \varphi_k = \gamma_{k-1} u_k \varphi_{k-1} + \varphi_k \Rightarrow$$

$$\gamma_k \varphi_k \prod_{m=1}^{k} u_m^{-1} - \gamma_{k-1} \varphi_{k-1} \prod_{m=1}^{k-1} u_m^{-1} = \prod_{m=1}^{k} u_m^{-1} \varphi_k \Rightarrow$$

$$\gamma_n \varphi_n \prod_{m=1}^{n} u_m^{-1} = \sum_{k=1}^{n} \prod_{m=1}^{k} u_m^{-1} \varphi_k \text{ or}$$

$$\gamma_n = \frac{\sum_{k=1}^{n} (1 + r_k) p_k \prod_{m=k+1}^{n} u_m}{(1 + r_n) p_n}$$