Stochastic GBM Methods for Modeling Market Prices

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Abstract

Motivation. Insurance companies and corporations require credible methods in order to measure and manage risk exposures that derive from market price fluctuations. Examples include foreign currency exchange, commodity prices and stock indices.

Method. This paper will apply Geometric Brownian Motion (GBM) models to simulate future market prices. The Cox-Ingersoll-Ross approach is used to derive the integral interest rate generator.

Results. Through stochastic simulations, with the key location and shape parameters derived from options market forward curves, the approach yields the full array of price outcomes along with their respective probabilities.

Conclusions. The method generates the requisite distributions and their parameters to efficiently measure capital risk levels as well as fair value premiums and best estimate loss reserves. The modeled results provide credible estimators for risk based and/or economic capital valuation purposes. Armed with these distributions of price outcomes, analysts can readily measure inherent portfolio leverage and more effectively manage these types of financial risk exposures.

Availability. An Excel version of this stochastic GBM method is available from the CAS website, *E-Forum* section under filename MPiR.xlsm.

Keywords. Dynamic risk models; capital allocation; geometric Brownian motion; options market volatility; stochastic process; Markov Process, Itō's lemma, economic scenario generator.

1. PRICE FORECASTING AND ECONOMIC CAPITAL MODELS

There are various methods actuaries may use to generate future contingent market prices. This paper provides the theoretical construct and detailed calculation methodology to model market prices for any asset class with a liquid exchange traded options market (i.e., foreign currency exchange, oil, natural gas, gold, silver, stocks, etc.).

The critical input parameters used in this approach are taken directly from the options market forward curves and their associated volatilities. For example, an insurer wants to determine the range of likely price movements over the next year for the British Pound (GBP) versus the U.S. Dollar (USD). The requisite mean and volatility input assumptions for this approach are readily available from real time financial market sources (i.e., Bloomberg, Reuters, etc.).

There are two fundamentally different approaches to modeling financial related risks, namely, fully integrated and modular.

The fully integrated approach applies an enterprise-wide stochastic model that requires complex economic scenario generator (ESG) techniques and the core inputs are aligned to either real-world or market-consistent parameters.

Real-world ESGs generally reflect current market volatilities calibrated via empirical time series better suited to long-term capital requirements. Market consistent ESGs reflect market option prices that provide an arbitrage-free process geared more toward derivatives and the analytics to manage other capital market instruments. Market consistent ESGs have fatter tails in the extreme right (i.e., adverse) side of the modeled distributions.

Outputs from the ESGs provide explicit yield curves that allow us to simulate fixed income "bond" returns. Interest rates (both real and nominal) are simulated as core outputs and the corresponding equity returns are derived as a function of the real interest rates.

Fully integrated models provide credible market price forecasts but they are complex and require highly experienced analysts to both calibrate the inputs and translate the modeled outputs. The findings derive from an apparent "black box" and are not always intuitive or easily explained to executive managers and third-party reviewers (i.e., rating agencies or regulators).

Proponents of the fully integrated approach assert that it provides an embedded covariance structure, reflecting the causes of dependence. However, a pervasive problem arises when using the fully integrated approach in that no matter how expert the parameterization of the ESG, the model by necessity will reflect an investment position on the future market performance.

Appendix A provides sample input vectors for a typical ESG. A cursory review of the input parameters confirms that any resulting simulation reflects the embedded investment position on the myriad of financial market inputs including short-term rates, long-term rates, force of mean reversion, variable correlations, jump-shift potential, etc.

The approach described in this paper is geared to analyze asset (and liability) risk components that are modeled individually. This is referred to as the modular approach. In this approach capital requirements are determined at the business unit or risk category level (e.g., market, credit and liquidity risk separately) and then aggregated by either simple summation of the risk components (assuming full dependence) or via covariance matrix tabulations (which reflect portfolio effects).

The main advantage of the modular approach is that it provides a simple but credible spreadsheet-based solution to economic capital estimation. Other advantages include ease of implementation, clear and explicit investment position derived from the market and covariance assumptions, and communication of basic findings.

Consider the financial risk exposure that derives from stock/equity investments. The expected returns originate from non-stationary distributions and the correlation parameters of the various equities likely derive from non-linear systems. Thus, it may be more appropriate to simulate stock prices with a model that eliminates any need to posit future returns but rather simply translates the range of likely outcomes defined by the totality of information embedded within open market trades. Selecting the location and scale parameters from the options markets data yields price forecasts which are devoid of any actuarial bias on the expected "state" of the financial markets. The results provide reliable measures of the range of price fluctuation inherent in these capital market assets.

Financial traders may scrutinize buy/sell momentum and promulgate their own view of the dependency linkages amongst and in between these asset variables, attempting to determine where arbitrage opportunities exist. The net sum of all of the option market trades collectively reflects an aggregate expectation. The market is deemed credible and vast amounts of trade data are embedded within these two key input parameters.

2. PRICE MODELING—THEORY

Markov analysis looks at sequences of events and analyzes the tendency of one event to be followed by another. Using this analysis, one can generate a new sequence of random but related events that will mimic the original. Markov processes are useful for analyzing dependent random events whereby likelihood depends on what happened last. In contrast, it would not be a good way to model coin flips, for example, because each flip of the coin has no memory of what happened on the flip before as the sequence of heads and tails is fully independent.

The Weiner process is a continuous-time stochastic process, $W(t)$ for $t \geq 0$ with $W(0) = 0$ and such that the increment $W(t) - W(s)$ is Gaussian (e.g., normally distributed) with mean = 0 and variance "$t - s$" for any $0 \leq s \leq t$, and the increments for non-overlapping time intervals are independent. Brownian motion (i.e., random walk with random step sizes) is the most common example of a Wiener process.

Changes in a variable such as the price of oil, for example, involve a deterministic component, "$a\Delta t$", which is a function of time and a stochastic component, "$b\Delta z$", which depends upon a random variable (here assumed to be a standard normal distribution). Let S be the price of oil at time = t and let dS be the infinitesimal change in S over the infinitesimal interval of time dt. Change in the random variable Z over this interval of time is dz. This yields a generalized function for determining the successive series of values in a random walk given by $dS = adt + bdz$, where "a" and "b" may be functions of S and t. The expected value of dz is equal to zero so thus the expected value of dS is equal to the deterministic component, "adt".

The random variable dz represents an accumulation of numerous random influences over the interval dt. Consequently, the Central Limit Theorem applies which infers that dz has a normal distribution and hence is completely characterized by mean and standard deviation.

The variance of a random variable, which is the accumulation of independent effects over an interval of time is proportional to the length of the interval, in this case dt. The standard deviation of dz is thus proportional to the square root of dt. All of this means that the random variable dz is equivalent to a random variable $\sqrt{W}(dt)$, where W is a standard normal variable with mean equal to zero and standard deviation equal to unity.

Itō's lemma[1] formalizes the fact that the random (Brownian motion) part of the change in the log of the oil price has a variance that is proportional to the square root of this time interval. Consequently, the second order (Taylor) expansion term of the change of the log of the oil price is proportional to the time interval. This is what allows the use of stochastic calculus to find the solutions. The formula for Itō's Lemma is as follows:

$$\Delta X = a(x,t)\Delta t + b(x,t)\Delta z \tag{2.1}$$

Itō's Lemma is crucial in deriving differential equations for the value of derivative securities such as options, puts, and calls in the commodity, foreign exchange and stock markets. A more intuitive explanation of Itō's Lemma that bypasses the complexities of stochastic calculus is given by the following thought experiment:

> Visualize a binomial tree that goes out roughly a dozen steps whereby the price at each step is determined by, drift +/- volatility. The average of returns at the end of these steps will be (drift - ½ volatility^2) x dt. This is as Itō's Lemma would expect. However, when you do this averaging to get that number, all of the outcomes (i.e., each of the individual returns) have the same weighting. It is as though you weighted each outcome by its beginning value or price. Since all of the paths started at the same price, it turns out being a simple average (actually, a probability-weighted average with equivalent weights).

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[1] Kiyoshi Itō (1951). On stochastic differential equations. *Memoirs, American Mathematical Society* **4**, 1–51.

Now run the experiment again, but this time by averaging each of the outcomes by their ending value, which will yield an average mean = (drift + ½ volatility^2) x *dt*. Note the change in the sign from - to +. Consequently, the formula has a minus sign if you use beginning value weights and a plus sign if you use ending value weights. Conceivably, somewhere in the middle of the process (or maybe the average drift of the process) is just the initial drift with no volatility adjustment. Why is this? When you weight by initial price, all of the paths share equal weightings – the bad performance paths carry the same weight as the good, in spite of the fact that they get smaller in relative size. Consequently they are bringing down the average return (thus the "minus ½ sigma^2"). The opposite happens when you use ending values as weights, whereby the top paths get really large versus the bottom paths and appear to artificially lift up the returns (in a manner similar to that often observed with some stock indices).

The "reality" is likely somewhere in between, where the number is the initial drift and thus, in this context, Itō's Lemma is just a weighted averaging protocol.

By inserting Itō's Lemma into the generalized formula yields a Geometric Brownian Motion (GBM) formula for price changes of the form:

$$\Delta S = \mu S \Delta t + \sigma S \Delta z; \text{ such that } S_{t+1} = S_t + S_t\,[\mu\Delta t + \sigma\varepsilon N(0,1)\sqrt{\Delta t}]. \tag{2.2}$$

μ is the expected price appreciation, which can be taken directly from the forward mean curves for any liquid market option (i.e., F/X, Oil, Gold, etc.).

σ is the implied volatility, which can also be taken directly from the option markets price data available on Bloomberg (for example).

S is typically assumed to follow a lognormal distribution and this process is used to analyze commodity and stock prices as well as exchange rates.

A critical input to this market price modeling approach is the interest rate assumption.

A general model of interest rate dynamics may be given by:

$$\Delta rt = k(b - r_t)\Delta t + \sigma r \gamma_t \Delta zt. \tag{2.3}$$

In this method we utilize the Cox-Ingersoll-Ross Model (CIR) as follows:

$r_i = r_i\text{-}1 + a(b\text{-}r_i\text{-}1)\Delta t + \sigma\sqrt{r_i\text{-}1}\ \varepsilon$

r_i = spot rate at time = *i*.

r_i-1 = spot rate at time = *i*-1.

a = speed of reversion = 0.01.

b = desired average spot rate at end of forecast: set to spot rate on *n*-year high-grade, corporate-zero, coupon bond at beginning of forecast; therefore, there is no expectation for a change in the level of yields over the forecast period.

σ = volatility of interest rate process = .85% (the historical standard deviation of the Citigroup Pension Discount Curve *n* year spot rate).

Δ*t* = period between modeled spot rates in months = 1.

ε = random sampling from a standard normal distribution.

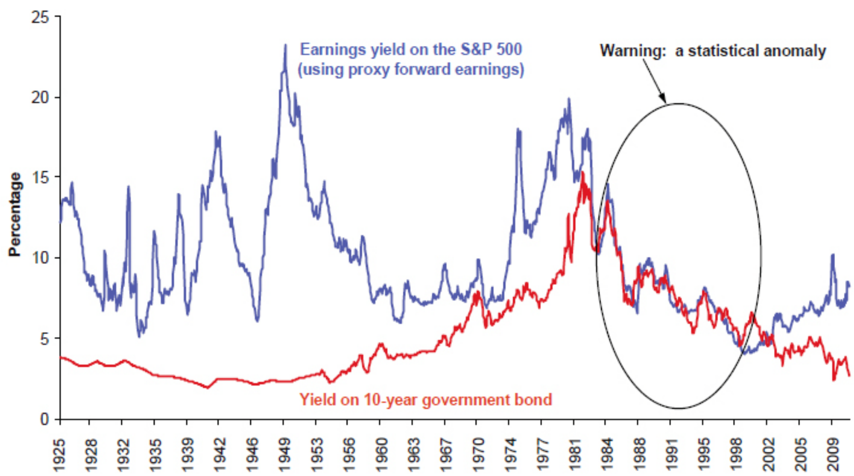
The CIR interest rate model characterizes the short-term interest rate as a mean-reverting stochastic process. Although the CIR model was initially developed to simulate continuous changes in interest rates, it may also be used to project discrete changes from one time period to another.

The CIR model is similar to our market price model in that it has two distinct components: a deterministic part $k(b\text{-}r_t)$ and a stochastic part $\sigma r \gamma_t$. The deterministic part will go in the reverse direction of where the current short-term rate is heading. That is, the further the current interest rate is from the long-term expected rate, the more pressure the deterministic part applies to reverse it back to the long-term mean.

The stochastic part is purely random; it can either help the current interest rate deviate from its long-term mean or the reverse. Since this part is multiplied by the square root of the current interest rate, if the current interest rate is low, then its impact is minimal, thereby not allowing the projected interest rate to become negative.

3. PRICE MODELING—APPLICATION AND PRACTICE

When implementing this modular approach to model these types of risks, there are key considerations that need be thought through by the actuary. The first and most important is correlation. For this paper, we are assuming independence for simplicity and clarity in the approach. A fully independent view does have value in that it defines a lower boundary region of the result and a fully dependent view defines an upper boundary. Correlation of financial variables is difficult because they are hard to estimate and can be unstable. For example, consider the chart below, which tracks the relationship between stocks and bonds over time.



Source: GMO as of January 2011.

Another key consideration is the form of the random walk variable. For this example, we are using a normal distribution to model the random walk of the results. The normal distribution is commonly used in financial modeling and does simplify the ideas shown. Depending on the use and application of the model, consideration should be given to this assumption and possible modifications.

The data for this sample exercise is from the forward call options for the British Pound (GBP) versus the U.S. Dollar (USD) currency pair from June 2010 through December 2011. This time interval was selected so that the user can compare the modeled results to the actual results.

GBP v USD Foreign Exchange Futures

Source: (Bloomberg)

Ticker	Month	Option Mean	Volatility
NRM0 Comdty	Jun-10	1.4558	14.890
NRN0 Comdty	Jul-10		14.920
NRQ0 Comdty	Aug-10		14.860
NRU0 Comdty	Sep-10	1.4557	14.850
NRV0 Comdty	Oct-10		
NRX0 Comdty	Nov-10		14.830
NRZ0 Comdty	Dec-10	1.4556	
NRF1 Comdty	Jan-11		
NRG1 Comdty	Feb-11		14.795
NRH1 Comdty	Mar-11	1.4555	
NRJ1 Comdty	Apr-11		
NRK1 Comdty	May-11		14.730
NRM1 Comdty	Jun-11	1.4554	
NRN1 Comdty	Jul-11		
NRQ1 Comdty	Aug-11		
NRU1 Comdty	Sep-11	1.4553	
NRV1 Comdty	Oct-11		
NRX1 Comdty	Nov-11		14.760

The first step is to complete the columns for the missing data fields with simple linear interpolation. Other interpolation options are available and should be reviewed when doing the analysis. In this case, a linear interpolation was selected due to the small changes expected in the mean market forward curve. When larger relative price movements are expected, then different interpolations may be used such as geometric means.

Interpolating the missing values generates the following table:

Ticker	Month	Option Mean	Volatility
NRM0 Comdty	Jun-10	1.4558	14.890
NRN0 Comdty	Jul-10	1.4558	14.920
NRQ0 Comdty	Aug-10	1.4557	14.860
NRU0 Comdty	Sep-10	1.4557	14.850
NRV0 Comdty	Oct-10	1.4556	14.840
NRX0 Comdty	Nov-10	1.4556	14.830
NRZ0 Comdty	Dec-10	1.4556	14.818
NRF1 Comdty	Jan-11	1.4556	14.807
NRG1 Comdty	Feb-11	1.4555	14.795
NRH1 Comdty	Mar-11	1.4555	14.773
NRJ1 Comdty	Apr-11	1.4555	14.752
NRK1 Comdty	May-11	1.4554	14.730
NRM1 Comdty	Jun-11	1.4554	14.735
NRN1 Comdty	Jul-11	1.4554	14.740
NRQ1 Comdty	Aug-11	1.4553	14.745
NRU1 Comdty	Sep-11	1.4553	14.750
NRV1 Comdty	Oct-11		14.755
NRX1 Comdty	Nov-11		14.760

The CIR interest rate model is then applied in this example as follows:

$$r(i) = (ab - (a+y) \text{ x } r(i\text{-}1))dt + sr^{g}dZ$$
$$a = 0.25$$
$$b = 0.06$$
$$y = 0$$
$$s = 0.05$$
$$g = 0.50$$
$$dt = 1/12$$
$$r(0) = 0.0028 \text{ (1 month LIBOR).}$$

The above parameterization was provided by life actuarial advisors. Derivation of the CIR parameters is beyond the scope of this paper.

Adding the interest rate calculation expands the table as follows:

Month	Market Forward GBP/USD	Implied Volatility	Interest Rate	Z
Jun-10	1.4558	14.89%		
Jul-10	1.4558	14.92%	0.28%	0.00%
Aug-10	1.4557	14.86%	0.40%	0.00%
Sep-10	1.4557	14.85%	0.52%	0.00%
Oct-10	1.4556	14.84%	0.63%	0.00%
Nov-10	1.4556	14.83%	0.74%	0.00%
Dec-10	1.4556	14.82%	0.85%	0.00%
Jan-11	1.4556	14.81%	0.96%	0.00%
Feb-11	1.4555	14.80%	1.06%	0.00%
Mar-11	1.4555	14.77%	1.17%	0.00%
Apr-11	1.4555	14.75%	1.27%	0.00%
May-11	1.4554	14.73%	1.37%	0.00%
Jun-11	1.4554	14.74%	1.46%	0.00%
Jul-11	1.4554	14.74%	1.56%	0.00%
Aug-11	1.4553	14.75%	1.65%	0.00%
Sep-11	1.4553	14.75%	1.74%	0.00%

Where Z is $N(0,1)$.

This currency model has the following basic structure:

Currency price (end of month) = currency price (beginning of month) x (random walk) x (1 + drift rate adjustment).

The first two elements are typical of standard GBM models. The third component adjusts the model so that the mean of the modeled currencies match the market forward curve. By implementing this adjustment factor, the model is transformed to be price taking. That is, the GBM model is modified to realign the simulated forward means with the current options market expectation[2].

[2] The GBM model may be adjusted to use different forward curves than the market aggregate expectation, but then the model would by definition be taking a market pricing position on the variable. However, if that is the case use caution since that analysis may be construed as offering investment advice. Please note the relevant actuarial statements of practice related to investment advice.

Next we introduce the Brownian motion component.

random walk = exp((*r*(*i*)- ½ x σ2)*dt* + σ (*dt*)½*dZ*).

Where *dZ*, *dt*, *r(i)* are from the interest rate calculation, and σ is the implied volatility of the currency prices from the Bloomberg table.

Adding these calculations to the table yields the following:

Month	Market Forward GBP/ USD	Implied Vol.	Interest Rate	*Z*	Price in Month		Weiner	Drift Rate Adj.	Modeled Mean	Target v. Modeled Mean Difference
					Beg	End				
Jun-10	1.4558	14.89%				1.4558				
Jul-10	1.4558	14.92%	0.28%	0.00%	1.4558	1.4548	0.9993	0.00%	1.4548	0.0006715
Aug-10	1.4557	14.86%	0.40%	0.00%	1.4548	1.4539	0.9994	0.00%	1.4553	0.0003007
Sep-10	1.4557	14.85%	0.52%	0.00%	1.4539	1.4532	0.9995	0.00%	1.4560	(0.0001773)
Oct-10	1.4556	14.84%	0.63%	0.00%	1.4532	1.4527	0.9996	0.00%	1.4568	(0.0007546)
Nov-10	1.4556	14.83%	0.74%	0.00%	1.4527	1.4522	0.9997	0.00%	1.4577	(0.0014195)
Dec-10	1.4556	14.82%	0.85%	0.00%	1.4522	1.4519	0.9998	0.00%	1.4589	(0.0022343)
Jan-11	1.4556	14.81%	0.96%	0.00%	1.4519	1.4518	0.9999	0.00%	1.4602	(0.0031548)
Feb-11	1.4555	14.80%	1.06%	0.00%	1.4518	1.4517	1.0000	0.00%	1.4616	(0.0041686)
Mar-11	1.4555	14.77%	1.17%	0.00%	1.4517	1.4518	1.0001	0.00%	1.4633	(0.0053047)
Apr-11	1.4555	14.75%	1.27%	0.00%	1.4518	1.4520	1.0001	0.00%	1.4651	(0.0065532)
May-11	1.4554	14.73%	1.37%	0.00%	1.4520	1.4524	1.0002	0.00%	1.4670	(0.0078706)
Jun-11	1.4554	14.74%	1.46%	0.00%	1.4524	1.4528	1.0003	0.00%	1.4690	(0.0092570)
Jul-11	1.4554	14.74%	1.56%	0.00%	1.4528	1.4534	1.0004	0.00%	1.4712	(0.0107382)
Aug-11	1.4553	14.75%	1.65%	0.00%	1.4534	1.4541	1.0005	0.00%	1.4735	(0.0123158)
Sep-11	1.4553	14.75%	1.74%	0.00%	1.4541	1.4549	1.0005	0.00%	1.4760	(0.1403333)

The final step is to determine the Drift Rate Adjustment values, which is accomplished with a recursive iteration technique. The first drift rate adjustment calculation is found in the last column ("Target vs. Modeled Mean Difference"). The formula in that column is equal to: (Market Forward Price) / (Modeled Mean) – 1.

The modeled mean is the average of the month ending prices from the simulation results. The first value shown is input into the Drift Rate Adjustment field, and then the GBM model is rerun to calculate the next adjustment factor, and so on until all the monthly forward means are aligned and the differences are all zero.

This can be seen in the following table, shown mid-adjusting:

Month	**Market Forward GBP/ USD**	**Implied Vol.**	**Interest Rate**	*Z*	**Price in Month**		**Weiner**	**Drift Rate Adj.**	**Modeled Mean**	**Target v. Modeled Mean Difference**
					Beg	**End**				
Jun-10	1.4558	14.89%				1.4558				
Jul-10	1.4558	14.92%	0.28%	0.00%	1.4558	1.4558	0.9993	0.07%	1.4558	0.0000000
Aug-10	1.4557	14.86%	0.40%	0.00%	1.4558	1.4544	0.9994	(0.04%)	1.4557	0.0000000
Sep-10	1.4557	14.85%	0.52%	0.00%	1.4544	1.4530	0.9995	(0.05%)	1.4557	0.0000000
Oct-10	1.4556	14.84%	0.63%	0.00%	1.4530	1.4516	0.9996	(0.06%)	1.4557	0.0000000
Nov-10	1.4556	14.83%	0.74%	0.00%	1.4516	1.4502	0.9997	(0.07%)	1.4556	0.0000000
Dec-10	1.4556	14.82%	0.85%	0.00%	1.4502	1.4487	0.9998	(0.08%)	1.4556	0.0000000
Jan-11	1.4556	14.81%	0.96%	0.00%	1.4487	1.4485	0.9999	0.00%	1.4569	(0.0009225)
Feb-11	1.4555	14.80%	1.06%	0.00%	1.4485	1.4485	1.0000	0.00%	1.4584	(0.0019386)
Mar-11	1.4555	14.77%	1.17%	0.00%	1.4485	1.4486	1.0001	0.00%	1.4600	(0.0030773)
Apr-11	1.4555	14.75%	1.27%	0.00%	1.4486	1.4488	1.0001	0.00%	1.4618	(0.0043286)
May-11	1.4554	14.73%	1.37%	0.00%	1.4488	1.4491	1.0002	0.00%	1.4637	(0.0056489)
Jun-11	1.4554	14.74%	1.46%	0.00%	1.4491	1.4496	1.0003	0.00%	1.4657	(0.0070384)
Jul-11	1.4554	14.74%	1.56%	0.00%	1.4496	1.4501	1.0004	0.00%	1.4679	(0.0085230)
Aug-11	1.4553	14.75%	1.65%	0.00%	1.4501	1.4508	1.0005	0.00%	1.4702	(0.0101040)
Sep-11	1.4553	14.75%	1.74%	0.00%	1.4508	1.4516	1.0005	0.00%	1.4727	(0.0118254)

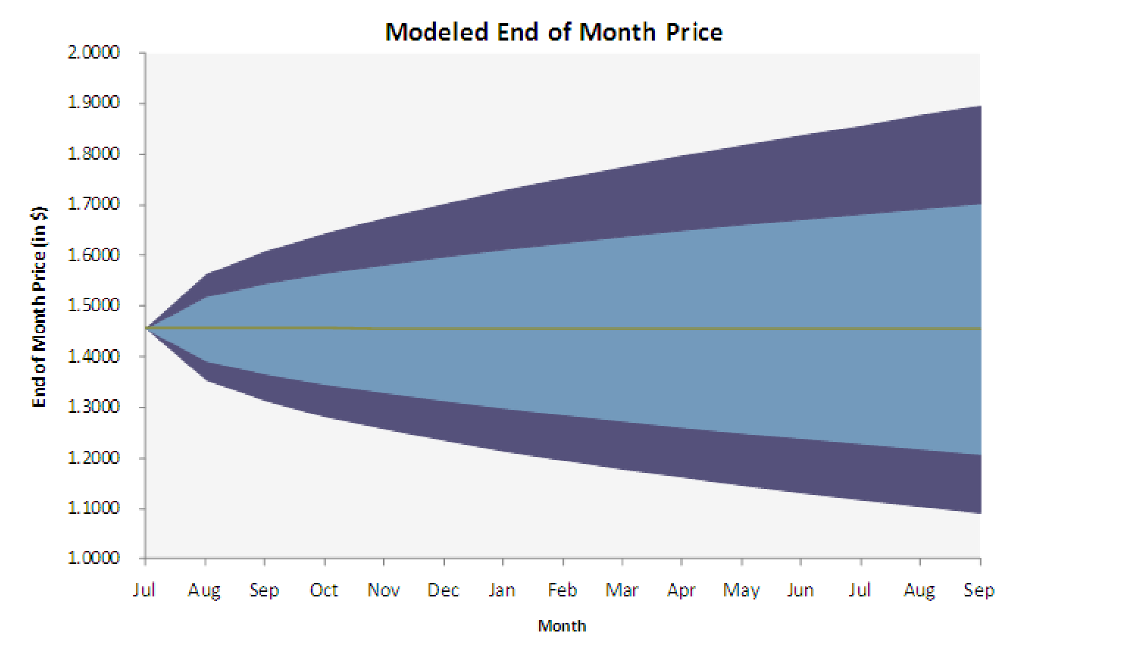
It is also possible to derive the drift rate adjustment values directly from an analytic approach applied to second differences but the recursive iterative technique was used here for ease of explanation.

After completing the drift rate adjustment process, the results are summarized as follows:

Month	Market Forward GBP/ USD	Implied Vol.	Interest Rate	*Z*	Price in Month		Weiner	Drift Rate Adj.	Modeled Mean	Target v. Modeled Mean Difference
					Beg	End				
Jun-10	1.4558	14.89%				1.4558				
Jul-10	1.4558	14.92%	0.28%	0.00%	1.4558	1.4558	0.9993	0.07%	1.4558	0.0000000
Aug-10	1.4557	14.86%	0.40%	0.00%	1.4558	1.4544	0.9994	(0.04%)	1.4557	0.0000000
Sep-10	1.4557	14.85%	0.52%	0.00%	1.4544	1.4530	0.9995	(0.05%)	1.4557	0.0000000
Oct-10	1.4556	14.84%	0.63%	0.00%	1.4530	1.4516	0.9996	(0.06%)	1.4557	0.0000000
Nov-10	1.4556	14.83%	0.74%	0.00%	1.4516	1.4502	0.9997	(0.07%)	1.4556	0.0000000
Dec-10	1.4556	14.82%	0.85%	0.00%	1.4502	1.4487	0.9998	(0.08%)	1.4556	0.0000000
Jan-11	1.4556	14.81%	0.96%	0.00%	1.4487	1.4472	0.9999	(0.09%)	1.4566	0.0000000
Feb-11	1.4555	14.80%	1.06%	0.00%	1.4472	1.4457	1.0000	(0.10%)	1.4555	0.0000000
Mar-11	1.4555	14.77%	1.17%	0.00%	1.4457	1.4441	1.0001	(0.11%)	1.4555	0.0000000
Apr-11	1.4555	14.75%	1.27%	0.00%	1.4441	1.4425	1.0001	(0.13%)	1.4555	0.0000000
May-11	1.4554	14.73%	1.37%	0.00%	1.4425	1.4409	1.0002	(0.13%)	1.4554	0.0000000
Jun-11	1.4554	14.74%	1.46%	0.00%	1.4409	1.4394	1.0003	(0.14%)	1.4554	0.0000000
Jul-11	1.4554	14.74%	1.56%	0.00%	1.4394	1.4378	1.0004	(0.15%)	1.4554	0.0000000
Aug-11	1.4553	14.75%	1.65%	0.00%	1.4378	1.4362	1.0005	(0.16%)	1.4553	0.0000000
Sep-11	1.4553	14.75%	1.74%	0.00%	1.4362	1.4344	1.0005	(0.17%)	1.4553	0.0000000

This modified GBM model has generated a 15-month market aligned foreign exchange price forecast. Each of the month ending values are the means from a probability density function unique to that point in time.

The graph below depicts the modeled end of month prices for GBP/USD.



The apparent horizontal line is the mean forward curve for this currency pair. The area bounded by the light shading represents +/- 1 Standard Deviation and roughly accounts for two-thirds of the outcomes. The area bounded by the darker shading is determined as the 5th and 95th percentile amounts over time. Note the modest asymmetry whereby price appreciation is expected to be greater than price depreciation over time. This asymmetry is even more pronounced out in the extreme tails as summarized in the table that follows.

This table relates the modeled prices to their confidence levels modeled as of July, August, September, and the subsequent quarter ends:

Confidence Level	Modeled End of Month Price						
	Jul-10	**Aug-10**	**Sep-10**	**Dec-10**	**Mar-10**	**Jun-10**	**Sep-10**
0.01%	1.4558	1.2396	1.1606	1.0223	0.9232	0.8566	0.7815
0.05%	1.4558	1.2626	1.1903	1.0502	0.9832	0.8905	0.8372
10.00%	1.4558	1.3762	1.3431	1.2787	1.2321	1.1938	1.1583
20.00%	1.4558	1.4025	1.3806	1.3350	1.3006	1.2720	1.2463
30.00%	1.4558	1.4219	1.4072	1.3763	1.3525	1.3317	1.3118
40.00%	1.4558	1.4386	1.4304	1.4132	1.3991	1.3858	1.3735
50.00%	1.4558	1.4544	1.4531	1.4485	1.4429	1.4384	1.4333
60.00%	1.4558	1.4703	1.4754	1.4849	1.4893	1.4937	1.4949
70.00%	1.4558	1.4876	1.5000	1.5249	1.5404	1.5549	1.5643
80.00%	1.4558	1.5081	1.5294	1.5710	1.6027	1.6286	1.6518
90.00%	1.4558	1.5370	1.5708	1.6409	1.6945	1.7379	1.7813
99.50%	1.4558	1.6252	1.7010	1.8607	1.9980	2.1148	2.2281
99.90%	1.4558	1.6613	1.7553	1.9512	2.1243	2.2844	2.4654

This provides the requisite estimators for risk-based or economic capital valuation purposes. For example, under Solvency II type risk level constraints, the 99.50% confidence level estimate at December is $1.8607. Consequently, the 1:200 stress level risk capital charge for this risk component is required to provide for the net losses that derive from a 28% weakening of the U.S. dollar (= 1.8607/1.4558).

Note: Actuaries must use caution in the display and communication of results from this modified GBM approach. Recall that we seek to provide an unbiased view of the range of future price outcomes. That is, we have not taken an independent view rather we have simply translated the aggregate market expectation.

In the U.S., professionals are licensed specifically to give investment advice to individuals and companies. Although actuaries may present the quantitative results of the GBM model and its effects, use caution in providing any qualitative summarization of the findings. Providing qualitative assessments of the company's expected future performance may be construed as giving unqualified investment advice.

4. CONCLUSIONS

The method generates probability distribution functions and their parameters to efficiently measure capital risk levels as well as fair value premiums and best estimate loss reserves. The model yields credible estimates of either risk-based or economic capital requirements or both. Equipped with these distributions of price outcomes, analysts can readily measure inherent portfolio leverage and more effectively manage these types of financial risk exposures.

Acknowledgment

The authors acknowledge that this methodology evolved from an initial project that modeled future natural gas prices, which was performed by their actuarial colleague Joe Kilroy. Analytic and editorial assistance has been provided by Jillian Hagan.

Appendix A

This exhibit provides a sample of the types of complex inputs required to run economic scenario generators.

Appendix A

ESG Prototype: Model Parameters

US Economy : Sample Parameters

Valuation Date 2010.12

Projection Period	50	time steps
Time Step	1.000	in years
Real Estate Time Step	1.000	in years

Current Risk Free Term Structure

Current 3-mo rate	0.14%	per year
Current 1-yr rate	0.29%	
Current 2-yr rate	0.62%	
Current 5-yr rate	1.93%	
Current 10-yr rate	3.29%	
Current 30-yr rate	4.42%	
50-yr Selection	4.60%	

Observed Term Structure (linearly interpolated between key rates)

1-yr	2-yr	3-yr	4-yr	5-yr	6-yr	7-yr	8-yr	9-yr	10-yr
0.29	0.62	1.06	1.49	1.93	2.20	2.47	2.75	3.02	3.29
11-yr	12-yr	13-yr	14-yr	15-yr	16-yr	17-yr	18-yr	19-yr	20-yr
3.35	3.40	3.46	3.52	3.57	3.63	3.69	3.74	3.80	3.86
21-yr	22-yr	23-yr	24-yr	25-yr	26-yr	27-yr	28-yr	29-yr	30-yr
3.91	3.97	4.02	4.08	4.14	4.19	4.25	4.31	4.36	4.42
31-yr	32-yr	33-yr	34-yr	35-yr	36-yr	37-yr	38-yr	39-yr	40-yr
4.43	4.44	4.45	4.46	4.47	4.47	4.48	4.49	4.50	4.51
41-yr	42-yr	43-yr	44-yr	45-yr	46-yr	47-yr	48-yr	49-yr	50-yr
4.52	4.53	4.54	4.55	4.56	4.56	4.57	4.58	4.59	4.60

Real Rate Parameters

Long INT Reversion Mean	0.0432	Long INT Volatility	2.33%
Long INT Reversion Speed	0.3516		
Short INT Reversion Speed	0.1382	Short INT Volatility	2.18%

Inflation Parameters

Initial Inflation	0.0148		
INF Mean	0.0259	INF Volatility	0.0215
INF Reversion Speed	0.3852		

Medical Inflation Parameters

Initial MED INF	0.0324
MED INF Mean	0.0271
MED INF Volatility	0.0088
MED INF Reversion Speed	0.0709

Large and Small Stock Parameters

					Prob
Stage0 Mean LS Return	9.00%	Stage0 LS Volatility	10.12%	stage: 1	0.9760
Stage1 Mean LS Return	-26.16%	Stage1 LS Volatility	27.12%	stage: 2	0.8507
Stage0 Mean SS Return	8.16%	Stage0 SS Volatility	13.86%	stage: 1	0.9760
Stage1 Mean SS Return	3.60%	Stage1 SS Volatility	57.50%	stage: 2	0.9000

Dividend Parameters

DIV Reversion Mean	4.17%	DIV Volatility	0.85%
DIV Reversion	0.13		
Initial DIV	1.83%		

Real Estate Parameters

RE Reversion Mean	2.22%	RE Volatility	2.82%
RE Reversion Speed	0.87		
Initial RE	4.62%		

Correlation Parameters

Correl LS, SS Regime Switch	90%
Correl LS, SS Return	90%
Correl DIV, LS	-25%
Correl INF, DIV	-50%
Correl Short, Long INT	68%
Correl INF, Short Real INT	2%
Correl INF, MED INF	72%

Dependence method to use?

Rank Dependence

5. REFERENCES

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Abbreviations and notations

CIR, Cox-Ingersoll-Ross
ESG, economic scenario generator
GBM, geometric Brownian motion
GBP, British pound sterling
USD, United States dollar

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