Catastrophe Pricing: Making Sense of the Alternatives

Ira Robbin, Ph.D.

Abstract:
This paper examines different ways of pricing catastrophe (CAT) coverage for reinsurance treaties and large insurance accounts. While all the methods use CAT loss simulation model statistics, they use different statistics and different algorithms to arrive at indicated prices. This paper will provide the reader with the conceptual foundations and practical insights for understanding alternative approaches.

Keywords: Catastrophe Pricing, Risk Measure, Coherence, Capital Allocation

1. INTRODUCTION

This paper will examine different ways of pricing the property catastrophe (CAT) coverage provided by reinsurance treaties and by insurance policies written on large accounts. All the approaches utilize CAT loss simulation models to estimate key statistics. However, they differ in the choice of statistics and their ways of translating statistics into premiums.

For the type of business under discussion, all widely-accepted methods use a Return on Risk-adjusted Capital (RORAC) approach. Indicated prices are defined as those that hit the specified target return on an amount of required capital that has been adjusted for risk. Where the methods differ is in how they arrive at the amount of required capital. They employ different risk measures and they use risk measures in different ways.

One objective of this paper is to bridge the gap of understanding between abstract formulas and black-box simulation software. It will provide definitions and also demonstrate how to apply them. It will show, in particular, how to properly define relevant risk measures on sets of discrete data such as those found in a deck of simulated CAT losses by year. The paper will feature a comparison of pricing indications for a few hypothetical accounts. In the end, the reader should have a better understanding of the formulas and a balanced framework for evaluating the alternatives.

1.1 Existing Literature

In 1990, Kreps [13] published an influential paper on Marginal Capital methods for pricing reinsurance treaties. He used Marginal Variance as a metric for determining Marginal Capital. Mango [15] observed that applying Marginal Variance or Marginal Standard Deviation to CAT pricing led to pricing that was Order Dependent. He proposed an application of game theory to eliminate the order dependency.

In the late 1990's, the Value at Risk (VaR) concept moved from the financial literature into rating agency capital requirement models. It was then but a short step before Incremental VaR (also referred to as Marginal VaR) approaches to required capital made their way into CAT pricing algorithms. However, theoreticians, such as Artzner, Delbaen, Eber, and Heath [4], had proposed a set of fundamental axiomatic properties called *coherence* and observed that VaR is not coherent. Others, including Acerbi and Tashce [1] and Wang [24], also objected to VaR, citing its incoherence along with other objections. Order dependence is also a problem for Marginal VaR. Meyers, Klinker, and Lalonde [19] also highlighted the point that the Marginal VaR method does not calibrate automatically with the portfolio and introduced an overall adjustment factor to achieve such calibration.

Rockafellar and Uryasev [22], Acerbi and Tasche [1] and others have promoted use of Tail Value at Risk (TVaR) as a risk metric, citing its coherence as a key advantage. As Uryasev and Rockafellar [22] explained, TVaR may not be the same as Conditional Tail Expectation (CTE) in discrete cases.[1] This paper will explain and demonstrate the difference.

Kreps [14] developed a general riskiness leverage model and also proposed co-statistics as an alternative to incremental statistics. Co-statistics provide a method for allocating portfolio capital based on account contributions to portfolio results. This approach has great intuitive appeal, but the application of the contribution concept to tail statistics such as VaR and TVaR does not produce perfectly well-behaved risk measures. As will be shown later with a simple example, Co-VaR is unstable in that modest changes in the inputs can lead to wild swings in the resulting statistic[2]. A new result in this paper is that Co-TVaR fails subadditivity, a defining property of coherence. This is perhaps a bit surprising since TVaR is coherent and one might have hoped co-measures would inherit coherence properties.

Other authors such as Bodoff [6] and Wang [24] have questioned the exclusive focus on the tail inherent in VaR and TVaR. Bodoff proposed a Percentile Allocation approach and Wang proposed use of Distortion Measures.

Venter [23] and Goldfarb [12] have published excellent survey articles that provide useful

---

[1] Many authors incorrectly assume TVaR and CTE are the same, but this is only true for continuous distributions. For example, D'Arcy [9] wrote, "…Tail-Value-at-Risk (TVaR), which is also termed Tail Conditional Expectation (TCE), takes the average of all values above a particular percentile….TVaR only provides the average loss if a loss in excess of the TVaR threshold were to occur."

[2] From informal discussions, the author is aware that this behavior is known to several analysts. However, it does not seem to have been previously noted in the literature.

background and discussion. What is clear from a review of existing literature is that the field is not settled. Practitioners use approaches decried by theoreticians and shrug off anomalies. Yet, the experience of 2010-2012 suggests that sub-optimal approaches were used that may have inadvertently promoted "de-worsification".[3] Overall, there is no want of proposals and opinions.

1.2 Organization of the Paper

The discussion will begin in Chapter 2 with a definition of the indicated account premium based on required CAT capital. Chapter 3 will present an overview of risk measure theory and then examine several commonly used risk measures. Chapter 4 will discuss the different algorithms used to set CAT capital for an account.

Then in Chapter 5 several particular algorithms will be reviewed. Finally in Chapter 6, simulation will be used to generate hypothetical years of CAT loss data for a portfolio and two sample accounts. Then risk measures and algorithms will be applied to this set of simulated data to arrive at prices based on different procedures. Chapter 7 will provide several general observations and Chapter 8 will have a summary and conclusion.

2. INDICATED PREMIUM ALGORITHMS

Let X denote the CAT Loss for a particular account and let P(X) denote the indicated premium prior to any loading for expenses. We may write the indicated premium as the sum of the expected CAT Loss, E[X] plus a risk load, RL(X).

$$P(X) = \mathrm{E}[X] + \mathrm{RL}(X) \qquad \text{Equation (2.1)}$$

We define some basic properties that an indicated pricing algorithm formula should obey.

Table 1

<u>**Premium Algorithm Basic Properties**</u>

1. **Monotonic: If $X_1 \leq X_2$, then $P(X_1) \leq P(X_2)$.**
2. **Pure: If $X \equiv \alpha$ a constant, then RL(X) =0 and P(X)=E[X] = α.**

[3] "De-worsification" is a term used (see article by Brodsky [7]) to describe overzealous diversification to the point that business is written with minimal or no profit load in order to diversify a portfolio.

3. **Bounded: If X ≤ K, then P(X) ≤ K.**
4. **Continuous(Stable): P(X) is a continuous function[4] of X.**

We say a premium calculation algorithm is **coherent** if it also satisfies:

Table 2

Premium Algorithm Coherence Properties

1. **Scalable: $P(\lambda X) = \lambda \cdot P(X)$**
2. **Translation Invariant: $P(X+\alpha) = P(X)+\alpha$.**
3. **Subadditive: $P(X_1 + X_2) \leq P(X_1) + P(X_2)$**

Other authors first define coherence as a set of properties of risk measures, but from our perspective it seems more natural to define it first with respect to the indicated premiums.

2.1 RORAC Pricing Formula

Write C(X) to stand for the Required CAT Capital for the account. Applying the RORAC approach to pricing, the risk load is given by applying a target return, r_{Target}, against required capital. Thus the indicated premium is given as:

$$P(X) = E[X] + r_{Target} \cdot C(X) \quad \text{Equation (2.1.1)}$$

Note Equation 2.1.1 produces a premium equal to the Expected Loss when the required capital is zero.

3. RISK MEASURES

The different premium calculations use different formulas for computing Required CAT Capital for an account. They employ different functions, called risk measures, to quantify risk. The risk measures are also used in different algorithmic procedures.

3.1 Risk Measures

We define a risk measure simply as a mapping from real-valued random variables to the non-negative real numbers. It is desirable that a risk measure, ρ, obey basic properties that correspond to

[4] Continuity is with respect to the L^1 topology.

the basic premium algorithm properties in Table 1:

Table 3

Risk Measure Basic Properties

1. **Monotonic: If $X_1 \le X_2$, then $E[X_1]+\rho(X_1) \le E[X_2]+\rho(X_2)$**
2. **Pure: If $X \equiv \alpha$ a constant, then $\rho(X)=0$**
3. **Bounded: If $X \le K$, then $\rho(X) \le K$.**
4. **Continuous(Stable): $\rho(X)$ is a continuous function of X.**

Note monotonicity for risk measures is more complicated than the corresponding property for indicated premiums. The reason is that if one random variable always exceeds another, it need not have more risk. However, under Property #1 in Table 3, the sum of its measured risk plus its expected value should exceed the corresponding sum for the smaller random variable. The idea of purity in a risk measure is that, if the outcome is known and fixed, a pure risk metric will say there is no risk.[5] Some popular risk measures such as VaR are not pure. However, unless a risk measure is pure, it may produce a pricing algorithm that does not satisfy Basic Property 2 in Table 1 and thus may generate a strictly positive risk load even if the loss is fixed and constant. Boundedness requires that the measured risk be no greater than the largest possible loss, if there is such a largest loss. Finally, the property of continuity means that small changes in the points or probabilities do not lead to large changes in the measured value of risk. As we will see later, some commonly used premium algorithms are not stable.

Going beyond the basic properties, we now turn to the special properties of coherence. We say a risk measure is **coherent** if it is scalable, translation invariant, and subadditive.

Table 4

Coherence Properties of Risk Measures

1. **Scalable: $\rho(\lambda X) = \lambda \cdot \rho(X)$**
2. **Translation Invariant: $\rho(X+\alpha) = \rho(X)$ when $0 \le \alpha$.**
3. **Subadditive: $\rho(X_1 + X_2) \le \rho(X_1) + \rho(X_2)$**

[5] Panjer and Jing [20] present a similar property called "riskless allocation" under which a risk with no uncertainty gets a zero capital allocation.

These definitions are arranged so they are consistent with the comparable coherent premium calculation properties.

Some (see Panjer and Jing [20]) believe scalability is a fundamental property expressing the idea that currency conversion should not impact measured risk. Unfortunately, some commonly-used measures of risk, such as variance, do not obey the scaling property. Acerbi and Tasche [1] refuse to refer to a function as a risk measure if it does not obey coherence properties. While we agree with the points these authors have made, we choose to use a very broad definition of a risk measure. This is for convenience. We need a ready way to refer to the various functions that are used, rightly or wrongly, to measure risk in pricing CAT covers.

3.1.1 Sign Conventions

Our approach to sign conventions is the same as that taken in most papers on property and casualty insurance. However, in reviewing the academic and financial literature, a reader may find it confusing because sign conventions are the reverse of what we have used. For example, many authors, including Artzner, Delbaen, Eber and Heath [4] as well as Acerbi and Tashce [1], state the property of translation invariance as $\rho(X + \alpha) = \rho(X) - \alpha$.

From this perspective, the random variable X represents the net outcome: so a positive result is favorable. Adding a constant reduces the measured amount of risk. For our work, the random variable X represents the CAT loss; so larger positives are less favorable.

Many of the other financial and academic authors also allow the risk measure to be negative, while for our purposes we always want the risk measure to be non-negative. Again most of the writers focusing on property and casualty insurance assume, implicitly or explicitly, that risk measures are non-negative.[6]

3.1.2 Risk Measure Definitions on Distributions with Mass Points

In a CAT pricing context, it is important to understand just how to apply the definition of any particular risk measure to a loss distribution that has mass points. There are three reasons for this. First, it is common practice to use a CAT simulation model to generate thousands of simulated years of results. So any measure needs to be defined on such a set of discrete sample points. Second, the impact of catastrophic events on underlying building values may produce real mass points. For

---

[6] In Wang [24] the translation invariance property is presented with all positive signs and in Venter [23] all the risk measures are non-negative.

example, if a risk has a $100 million beachfront hotel in a hurricane prone locale and no other property nearby, there could be a natural mass point at $100 million. Finally, limits and layering could also give rise to mass points.

3.2 Specific Risk Measures

We will now list several risk measures, discuss them briefly, and then show how they work on a small set of hypothetical sample loss data.

Table 5

<u>**Specific Risk Measures**</u>

1. **Variance:** $\mathrm{Var}(X) = E[(X-\mu)^2]$
2. **Semivariance:** $\mathrm{Var}^+(X) = E[(X-\mu)^2 | X \geq \mu] \cdot \mathrm{Prob}(X \geq \mu)$
3. **Standard Deviation:** $\sigma = \sqrt{\mathrm{Var}(X)}$
4. **Semi Standard Deviation:** $\sigma^+ = \sqrt{\mathrm{Var}^+(X)}$
5. **Value at Risk:** $\mathrm{VaR}(\theta) = \inf\ \{x \mid F(x) \geq \theta\}$ **for** $0 < \theta < 1$
6. **Tail Value at Risk:**

$$\mathrm{TVaR}(\theta) = \frac{1}{1-\theta}\begin{bmatrix} E[X|X > VaR(\theta)] \cdot (1 - F(\mathrm{VaR}(\theta))) \\ + \\ \{1 - \theta - (1 - F(\mathrm{VaR}(\theta))\} \cdot \mathrm{VaR}(\theta) \end{bmatrix}$$

7. **Excess Tail Value at Risk:** $\mathrm{XTVaR}(\theta) = \mathrm{TVaR}(\theta) - \mu$
8. **Distortion Risk Measure:** $E^*[X] = E[X^*]$
   where F*(x) = g(F(X)) for g a distortion function
9. **Excess Distortion Risk Measure:** $E^*[X] - \mu$
   where E*[X] is the mean under a distortion risk measure
   and μ = E[X]

3.2.1 Variance and Standard Deviation

Variance and Standard Deviation are well-known. Both are pure risk measures in the sense we have defined that term. However, Variance is not scalable and is therefore not coherent, but Standard Deviation is.

Semivariance has been advocated by Fu and Khury [11] and SemiStandard Deviation does obey the scaling property (Property 1 in Table 5). A major appeal of both is that they do not count favorable deviations as part of the risk. Exhibit 1, Sheet 1 shows the computation of these measures in an example with twenty sample points.

3.2.2 VaR, TVaR, and XTVaR

VaR is a well-known metric that was developed in financial and investment risk analysis settings. VaR(θ) is the θ^{th} percentile. It is intuitively the "best of the worst", the least unfavorable outcome from the worst (1- θ) % of outcomes. This may not be precisely correct when there are mass points and the cumulative distribution jumps discontinuously. To make the definition work in the general case, the "infimum" formulation is used[7]. VaR is not pure: its value on a constant is equal to that constant. VaR is not subadditive and therefore not coherent.

TVaR is intuitively the average loss in the worst (1- θ) % of outcomes. As the definition in Table 5 indicates, the computation of this average entails evaluating two terms. The first term is the product of the Conditional Tail Expectation (CTE) where CTE $=E[X|X > VaR(\theta)]$ and the probability weight corresponding to the tail of X values strictly larger than VaR(θ). The second term is the product of VaR(θ) times the residual probability needed so that the total of the two probabilities equals (1- θ). The sum of these two terms is then divided by (1 -θ) to obtain TVaR. For example, suppose VaR(99.0%) = 150 and assume the probability of X values strictly larger than 150 is 0.8%. It follows the mass at 150 is equal to or larger than 0.2%. Also assume the CTE is 200. Then TVaR(99.0%) equals 190 since 190 =(200*0.8% + 150*0.2%)/1.0%.

In the general case, the residual probability will be non-zero only when there is a strictly positive mass point at VaR(θ). When there is no mass point, TVaR is equal to CTE. However, in the discrete case, 1-F(VaR(θ)), the probability of exceeding VaR(θ), may be less than 1- θ. If that scenario, CTE is not truly an average of the worst (1-θ) of outcomes and the second term is needed. Our definition follows that of several authors (Rockafellar and Uryasev [22], and Acerbi and Tasche [1]) by including a portion of the mass point at VaR(θ) in the definition of TVaR.

When evaluating VaR and TVaR on a sample of trial data as might be generated by a simulation model, the formulas can be stated in a more straightforward way using the rankings of ordered data.

---

7 The infimum is this context is the lower bound of the tail.

Table 6

Ranking Definitions of VaR and TVaR

Let $X_1 \geq X_2 \geq \ldots \geq X_n$ be an ordering of n trails of X.

If k=(1-θ)n, then:

1. **VaR(θ) = X_κ**
2. **TVaR(θ) =** $\frac{1}{k}\left(\sum_{j=1}^{k} X_j\right)$

For instance, if there are 20 trials, the largest trial is VaR(95%) since 1= (1-.95)*20 and the smallest trial is VaR(0%) since 20= (1-0)*20. Also observe that TVaR(0%) is the average of all 20 sample points. Under these definitions VaR(75%) is the 5th largest point and TVaR(75%) is the average of the 5 largest points.[8] Exhibit 1, Sheet 2 shows the computation of these measures using the "ranking" definitions in the simple example of twenty samples points. Note the repetition of some of the sample point values does not cause any difficulty for the ranking definitions.

TVaR is not pure, but it is subadditive. As Acerbi and Tasche [1] and others have noted, the CTE may fail subadditivity in the discrete case[9]. XTVaR captures the average amount by which the worst (1-θ)% of loss outcomes exceeds the mean. Note TVaR is automatically larger than the mean so XTVaR is non-negative. In cases where X is constant, TVaR will be equal to the expectation, while XTVaR will be zero. Thus XTVaR is a pure risk measure, while TVaR is not.

3.2.3 Distortion Risk Measures

Wang [24] has proposed use of distortion risk measures and shown that they are coherent when g is continuous. We have not seen the excess risk measure in the literature, but we list it as an obvious extension that leads to a coherent and pure risk measure.

One particular distortion is the Wang shift[10]:

$$\mathbf{g(u) = \Phi[\Phi^{-1}(u) - \lambda]} \qquad \text{Equation (3.2.3.1)}$$

---

[8] One alternative is to use the average of the 4 largest points out of 20 for TVaR(75%) and more generally use the average of the k-1, instead k largest points, as the definition for TVaR(θ). In our view, the definition with k points is better, for one, because it results in TVaR(0) being equal to the average.

[9] Acerbi and Tasche[1] use the terminology Tail Conditional Expectation (TCE) where we use CTE.

[10] Wang also introduced the Proportional Hazards transform [26].

Here Φ is the standard unit normal and λ is the Wang shift parameter.

Exhibit 1, Sheet 3 has an example with columns showing how the original cumulative distribution is transformed. To pick a particular row, the second ranking point out of twenty sample points has an empirical cumulative distribution (cdf) of 0.950. The standard unit normal inverse of 0.950 is 1.645. The Wang shift in the example is 0.674. Subtracting this from 1.645 yields 0.971 and the standard normal has a cdf of 0.834 at 0.971. So the cdf is transformed from 95.0% to 83.4%. New transformed mass densities are derived by subtracting the transformed cdfs in sequence. Using the resulting transformed densities moves the mean from 10.0 to 16.7; so the resulting excess mean is 6.7.

4. ALGORITHMIC PROCEDURES

There are three different ways of using risk measures to arrive at a capital calculation algorithm. In this chapter, we will review these and define properties of such algorithms.

First is a **Standalone** capital computation. This simply involves looking at the value of a selected risk measure on an account's own CAT loss distribution. Even if this is not the final selected approach, it is usually a good idea to know the standalone value for any account as it provides a baseline for comparison. Some analysts would stop there and say the standalone capital from a coherent pure risk measure is the right answer. A variation of the standalone approach is the market equilibrium approach. This method attempts to find the theoretical price that should be charged for a given risk in a hypothetical fair and efficient market in equilibrium. While this may seem the direct opposite of a standalone approach, it can be rightly classed as a variant of a standalone concept in that the indicated price for an account does not depend on other risks in the portfolio.

Other methods do reflect the portfolio in their computations. The second general algorithm, the **Marginal** approach, determines required capital for an account by computing how it changes required capital for the portfolio. The third general algorithm, **Real Allocation,** entails use of a risk measure as an allocation base to allocate portfolio capital. This methodology may use one risk measure to compute required portfolio capital and then possibly another one as an allocation base.

Let X stand for the CAT Loss for a particular account and let R denote the portfolio CAT Loss excluding that account. Write C(X) to stand for the Account CAT Capital for the account. Equations for the three general procedures are:

Table 7

General CAT Capital Calculation Procedures

1. **Standalone :** $\mathbf{C(X)=\rho(X)}$
2. **Marginal:** $\mathbf{C(X\,|\,R)=\rho(X+R)-\rho(R)}$
3. **Real Allocation:** $$\mathbf{C(X\,|\,R)=\left(\frac{\rho_2(X)}{\sum\limits_{Y\in(X+R)}\rho_2(Y)}\right)\cdot\rho_1(X+R)}$$

We say the Marginal and Real Allocation procedures are **Portfolio Dependent** because the required capital amounts depend on the portfolio as well as on the account. We define several properties of Portfolio Dependent Algorithms.

Table 8

Portfolio Dependent Capital Properties

1. **Standalone Capital Cap:** A portfolio dependent algorithm is capped by standalone capital if $\mathbf{C(X\,|\,R)\le C(X)}$.
2. **Automatically Calibrated:** A portfolio dependent algorithm is automatically calibrated if $$\sum \mathbf{C(X|R)=C(R)}$$
3. **Order Dependent:** A portfolio dependent algorithm is order dependent if $\mathbf{C(X_1\,|\,R+X_2)\neq C(X_1\,|\,R)}$ for some X_1 and X_2..

.

4.1 Calibration and Consolidation Benefits or Penalties

With both Standalone and Marginal approaches, there is a potential mismatch between the sum of individual account required capital amounts and the required capital for the portfolio taken as a whole. In other words, they are not automatically calibrated. If the portfolio requires less capital than the sum of the account capital requirements, as would be true for a subadditive measure, the difference will be called the **consolidation benefit.** However, not all risk measures are subadditive and therefore it is possible to have a **consolidation penalty**. So, even though it is a near universal truth in insurance that risk is reduced by pooling, that may not necessarily be the case with respect to

CAT coverage. Further, when subadditivity fails with a marginal algorithm, it is possible that required standalone capital is less than the indicated marginal capital. Such an algorithm would not satisfy the standalone cap.

Exhibit 2 has an example showing VaR is not subadditive. In the exhibit there are 20 trials with results for an account, "A", and a Reference Portfolio. The next column shows the computed sum, "A+ Ref", for each trial. Then there is a block labeled "Ordered Loss Data" showing the resulting rank-ordered losses for risk A, the Reference Portfolio, and the sum. The ordering is done separately for each. Since the 5th largest loss for account A is 4 and the 5th largest loss for the Reference Portfolio is 34, it follows that $VaR_A(75\%)$ is 4.and $VaR_{Ref}(75\%)$ is 34. Yet the 5th largest loss for the combined portfolio after A is added is 39. So the Marginal VaR for account A is 5 (= 39-34), which exceeds its Standalone VaR.

There are two general options for achieving calibration if desired when it does not occur automatically. A simple and direct idea is to apply a calibration factor to each initial required capital amount.[11] The other approach is to use game theoretic methods or other approaches to "auction off" any portfolio consolidation benefit (or penalty). Another view is that there is no need to reconcile the sum of the individual account required capital amounts and required capital for the portfolio. The key point is that users should be aware of the calibration properties of any proposed method and make adjustments or not according to their own philosophy.

4.2 Incoherence and Order Dependence

Marginal methods are also prone to generating required capital amounts that fail one or more of the coherence properties. They inherit incoherence if they are based on an incoherent measure, and they are also subject to order dependence[12] and other vulnerabilities arising from the nature of the incremental process.

For example, the scaling property does not necessarily hold if capital is based on Marginal VaR. This means that capital for a 50% share of a treaty is not necessarily twice the capital needed for a 25% share. See Exhibit 3 for an example demonstrating this.

TVaR has been shown to be coherent. However, if one is not careful with definitions and uses TCE (Tail Conditional Expectation) instead of TVaR, the resulting capital formula is not necessarily

[11] Meyers, Klinker, and LaLonde [19] proposed such a factor to calibrate capital requirements based on the Marginal VaR.

[12] Mango [15] showed the order dependence of marginal variance and marginal standard deviation.

even monotonic. In other words, adding an account may <u>decrease</u> the conditional tail expectation. Exhibit 4 demonstrates this.

4.3 Portfolio Allocation Methodologies

True allocation methodologies are less subject to calibration and order dependence issues than marginal methods. There are several different approaches to real allocation. Some are effectively a reprise of the methods used to set account capital; only in this context they are used to allocate a portfolio total already determined.

Table 9

<u>Allocation Procedures</u>

1. **Stand Alone**: Allocate portfolio capital in proportion to account Stand-alone risk measures

2. **Marginal:** Allocate portfolio capital in proportion to account marginal impact on portfolio risk measure.
   - Game Theory Modified Marginals: Adjust Marginals via Game Theory so allocations are not order dependent.

3. **Co-Measure**: Allocate portfolio capital using a co-measure.

4. **Percentile Allocation**: Allocate surplus based on allocations of each percentile of portfolio loss.

4.4 Standalone and Marginal Allocations

The first method uses an allocation base equal to the standalone value of a risk measure. This measure could be the same or different than the one used to compute required portfolio capital. The second approach uses allocations proportional to the account marginal increments. The order in which an account is added to a portfolio can have a strong influence on its marginal impact. This can lead to pricing anomalies. Mango [15] has developed a refinement of the marginal allocation approach in which game theoretic averaging over all orderings produces a more stable allocation.

4.5 Contribution Statistics

The third allocation concept is based on **Contribution Statistics** (Co-Statistics). The Co-Statistic for an account is conceptually the amount it contributes to the portfolio statistic, where the computation is based on an examination of the scenarios that determine the portfolio statistic.

Given that the portfolio had an adverse result, how much of it was due to a particular account? If portfolio capital is determined by a particular risk measure, the associated account co-measures are automatically additive and consistent with the portfolio total.

4.5.1. Co-VaR Instability

Though intuitively appealing, Co-Statistics may exhibit troublesome behavior. One significant problem is that Co-VaR can be unstable.[13] Specifically, this means Co-VaR can change dramatically due to small changes in the data. Even more troubling, different sets of simulations can yield quite different answers and increasing the number of trials may or may not yield convergence.

How does this arise? Consider the simulated events for the portfolio put in descending order. For each event, the contribution due to a specific treaty is also known from the simulation. However, depending on how the events line up at the portfolio level, the Co-stats for the treaty may vary considerably. The treaty may contribute a sizeable portion of loss to the event corresponding to the portfolio 100-year VaR, yet it may add nothing to many of the events close by in the list. Or the opposite may be true.

Exhibit 5 shows an example in which the portfolio 100-year VaR is $405 and the associated Co-VaR for the account is $20. But for the events on either side, the account loss is $0. Averaging over a band of nearby points (the set {6, 0, 20, 0, 4} in the example in Exhibit 5) will produce a more stable and meaningful answer. This will be called the **Co-Stat Band** approach. How large a neighborhood to include is a matter of judgment.

4.5.2. Co-TVaR Fails to be Subadditive

To calculate Co-TVaR for Treaty A when it is added to a given Reference Portfolio, the first step is to rank-order events by the sum of combined Reference Portfolio plus Treaty A losses. This is shown in Exhibit 6, Sheet 1 where the ordered total is displayed in the far right column. Then the contribution of Treaty A to the sum is posted in the column labeled Co-A. To obtain Co-TVaR of the 75th percentile, we average the Co-A amounts for the first 5 out of 20 of the events ordered on the sum[14] and arrive at Co-TVaR = (1+7+0+4+3)/5= 3.

It is known that TVaR is coherent and that has helped spark interest in using Co-TVaR as an

---

[13] The behavior of co-VaR is important because VaR is often used to define required capital at the portfolio level. Perhaps the most natural way to allocate VaR-based portfolio capital would be to use co-VaR.

[14] In cases where several events have the same total loss equal to the VaR of the target percentile, results from all orderings of those events should be averaged to get the resulting co-stat.

allocation metric. However, as we demonstrate by example in Exhibit 6, Co-TVaR is not coherent because it is not necessarily subadditive. This is a new result that has not been previously presented in the literature. Continuing with the example, we already know Co-TVaR of Treaty A is 3.0 and Sheet 2 and Sheet 3 of Exhibit 6 show the Co-TVaR of Treaty B is 3.0 and the Co-TVaR for the combination of the two, Treaty A+B, is 11.0. To be subadditive, the Co-TVaR for Treaty A+B would needs to be less than 6.0.

4.6 Percentile Allocation

Bodoff [6] has proposed the concept of percentile allocation. This starts with the observation that capital is needed to cover not just the large event losses, but also all losses up to and including those due to large events. For the first dollar of capital that is needed to cover losses, an allocation is done proportional to how often an account will use that first dollar relative to the other accounts. This entails looking at all events, seeing which ones tap the first dollar of capital, and then looking at account losses for each of those events. Event probabilities are then used to allocate the first dollar of capital. After the first dollar is allocated, the same procedure is followed to allocate the second dollar and so forth. This method is fairly stable and is not order dependent.

5. PRACTICAL APPLICATION ISSUES

Before going further, we need to address several practical issues that directly impact which methods can realistically be used to price CAT business.

5.1 Reference Portfolios

The first is that all the portfolio approaches are fundamentally impractical in their pure form. This is because reality differs from the one-by-one pricing paradigm implicit in portfolio approaches. Under this paradigm, the model is run, a quote is generated, and the account is bound in an instant during which no other changes take place to the portfolio. In the real world, time is not frozen. The portfolio changes as other accounts expire or are bound[15] in the time lag between quoting and binding a particular account. So the indicated price at the time of binding could be different than the indicated price at the time of quoting.

In principle, one should rerun and derive updated indications for all existing unbound quotes

[15] Or possibly those that are authorized but not yet bound. Since the authorized share of treaty may not be the final share, a more careful treatment would use an estimate of the expected final share.

each time the portfolio changes. But this is unworkable. It would slow down the process of providing quotes to underwriters and put a huge strain on models and analysts during peak renewal season. Further, even if the modeling could be done, the market will not accept quotes that change after they are made. Thus, in practice, it is likely very few companies attempt to use the actual up-to-date portfolio as the base for their pricing calculations. None that we are aware of revise existing open quotes. Rather most of them use a defined Reference Portfolio. This is fixed for a week or a month or a quarter. The possible conceptual advantage in using a portfolio-based pricing algorithm may be partially undercut if the Reference Portfolio differs materially from the real one.

In some shops, a renewal account is run against a Reference Portfolio that includes the expiring policy. While that is technically wrong and can lead to inaccurate results, the alternative entails setting up a custom Reference Portfolio for each account. That can become somewhat cumbersome.

5.2 Impracticality of Game Theoretic Averaging

In theory, averaging over all possible orders of accounts is the ideal solution to order dependence anomalies. However, in practice that is generally unworkable. While Mango found a closed form solution for a particular risk measure so that one did not need to actually run impact statistics for each ordering, such closed form solutions do not exist for many risk measures. The number of different orderings explodes in factorial fashion with the number of accounts so that performing all the brute-force computations is usually not feasible. This is one reason the order dependence issue is often ignored and Marginal statistics are computed against the Reference Portfolio with each account added on a last-in basis.

6. DEMONSTRATION

We will now demonstrate various pricing algorithms on hypothetical CAT loss data. We start with 50 events as shown in Exhibit 7. Each has a 2% annual probability of occurring. We also show the loss amounts for a Reference Portfolio and for two risks, denoted Treaty A and Treaty B. Both these risks have the same theoretical mean loss; however Treaty A losses are much more volatile. Total Annual Treaty A losses are theoretically independent of the Annual Reference Portfolio losses as Treaty A was constructed so that it has zero loss for events that generate losses for the Reference Portfolio. Under this construction, events with IDs from 1 to 25 have zero loss for Treaty A and non-zero loss for the Reference Portfolio. The situation is reversed for events with IDs from 26 to 50. Treaty B was constructed so that it has significant correlation with the Reference Portfolio because it has losses on many of the events that give rise to Reference Portfolio losses. It was also constructed to be theoretically independent of Treaty A. To summarize Treaty A is independent and volatile, while Treaty B is relatively well-behaved but correlated.

We then simulated 1,000 trial years and created a matrix with 1,000 rows and 50 columns to record which events if any occurred in each year. An excerpt of that matrix is shown in Exhibit 8. This sample size of 1,000 is too small to drive sampling error down to the fine decimals. This is true, even though with 2% annual probabilities each event should turn up in roughly 50 of the 1,000 trial years. We computed linear and rank correlations on the simulated data for each of our two risks. These are consistent with our construction. In the 1,000 trial years, Treaty A has a 1% linear correlation with the Reference Portfolio, while Treaty B has a 60% linear correlation. The corresponding sample rank correlations over these trials are -2% and 93% for treaties A and B respectively.

Indicated capital amounts and indicated premiums were then computed using Standalone and Incremental algorithms with the VaR, TVaR, and XTVaR risk measures. Required capital amounts and premium indications were also computed using Real Allocation based on the standalone amounts and the co-statistics of these risk measures. For Co-VaR, we used a banded approach for stability. Results are detailed in Exhibit 9 and a pricing comparison for the two risks is shown on Sheet 3 of Exhibit 9.

As might be expected, the incremental portfolio and co-statistic methods produce very modest risk loads for Treaty A. Indications using Co-XTVaR in particular have very little risk load. The standalone pricing for Treaty A is much higher. It is up to the reader to decide whether it makes sense to charge negligible risk load on this volatile account because it diversifies the portfolio or to go with substantial risk load because it is risky on a standalone basis. The situation is reversed for Treaty B. It has significant risk load under all the portfolio based methods, but standalone pricing is lower than it was for Treaty A.

7. GENERAL OBSERVATIONS

This chapter contains several key general observations that impact catastrophe pricing.

7.1 Sampling Error

In the background one should always be aware that simulation models are being used to generate the results and that simulation statistics are prone to sampling error. When adding a small account to a large initial portfolio, such sampling error may mask or distort the impact of adding the new account. The resulting relative error in pricing may be quite significant. As well, the pricing of relatively small changes in treaty layers may be prone to inaccuracy due to sampling error.

The event probabilities in simulation models are often quite small. Many events have return periods on the order of 10,000 years. To get even a roughly accurate estimate that has a representative number of such events, one needs to run the model with the number of trials set as a multiple of the return period of rare events in the models.

Beyond that, there are other pricing pitfalls stemming from sampling error. One of the most common is pricing a layer by taking differences of other layers that were priced using results from several different simulations. Statistical fluctuations could lead to an inconsistent result. All layers of interest need to be run in one set of simulation trials to achieve more accurate differentials in layer prices. Of course, sampling error can be reduced by running more simulation trials.[16] The analyst needs to balance the extra time and cost against the need for accuracy.

7.2 Tails

A key area of distinction concerns how much influence the tail has on the answer. Under some methods, capital allocation is determined solely by the tail. "Tail-Only" advocates implicitly or explicitly believe the tail should be regarded as the one true indicator of risk. Others argue that any event that could potentially consume capital needs to be considered; not just the tail events that would consume all the capital.

Beyond the philosophical divide, one should be aware that tail dependence tends to increase the volatility in the results. This is partly due to statistical sampling error. Further, as models change over time, the tails often move quite a bit more than the expected CAT losses.

For strict empiricists, an additional concern is that there is effectively no empirical way to validate the distribution of events in the extreme tail. What is the average size of portfolio losses that happen less frequently on average than once in every 250 years? It would take more than twenty samples of 1,000 simulated years to obtain a tentative estimate of this average. The simulation

[16] The author is advocating the analyst estimate beforehand how many simulated years are needed to achieve a desired level of accuracy in the estimated value of the risk measure. This is not the same as the flawed procedure of running sets of trials and stopping when results seem to stabilize. When the trial sets are too small to reliably capture the big but rare events, results from two or three sets of trials may be fairly stable. However, such "stable" results could underestimate the true tail of losses.

modeling software can do this without any problem, but our available empirical history is insufficient to validate the estimate.

This line of thinking also highlights the high-end cut-off problem. The size and frequency of mega-events included in a model may have an inordinate impact on any particular tail metric. Whether there are enough extreme events in a model is subject to some debate.[17] However, once very rare events are put into a model, it is not clear on what philosophical grounds we can exclude geologically significant asteroid strikes or comet impacts or events relating to ice ages. Many practicing analysts feel that discussion of such calamities is a vast digression and that such events should not be in the model as they could unduly impact tail metrics.

7.3 Diversification Benefit

If required capital for the portfolio is less than the sum of the standalone requirements, then there is effectively a diversification benefit to the portfolio as a whole. Many pricing methods would translate this into a reduction in the overall level of pricing of the whole portfolio as compared to the total pricing that would otherwise result if each account were priced on a standalone basis. For example, if the portfolio required 10% less capital than the sum of the standalone capital requirements, indicated standalone account pricing for each account could be given a 10% haircut to arrive at the final indication reflecting the portfolio diversification benefit.

Another sense of diversification benefit arises at the account level even if there is no overall benefit to the portfolio. Some methods effectively surcharge risks in peak zones, and provide discounts for those written in off-peak zones. It had been an accepted truism for many that CAT pricing indications should reward such geographic diversification. But the experience of recent years (2010-12) has led some to question the uncritical acceptance of diversification and warn of the dangers of "de-worsification".[18]

Our comparisons indicate that some algorithms lead to bargain-basement prices for exposures in zones where the company has little existing exposure. This highlights the issue: how much of a discount in the indicated price is justifiable in order to achieve the benefits of diversification?

---

[17] No vendor we are aware has said it included an event as large as the Japanese earthquake of March 2011 in its event set for events associated with faults in that region of Japan.

[18] Brodsky [7] quoted the noted CAT modeling expert Karen Clark as saying , "… some reinsurers may want diversification a little too much …" and "How much do you want to write of this underpriced risk just to get diversification?".

7.4 Treatment of Premiums, Commissions, and Loss-sensitive Features

As argued in Robbin and DeCouto [21], capital should be based on the distribution of Bounded Underwriting Loss, B, where

$$B = \text{Max}(0, \text{Loss} + \text{Commission} - \text{Premium}) \qquad \text{Equation (7.6.1)}$$

Notice there is a floor in the definition which prevents any negative result. In effect this prevents reduction in capital for scenarios where a profit is made.

In practice, capital is often based on just the loss and no adjustment is made for the premium or expense. The advantage is that this is straightforward to implement. An excess metric, such as XTVaR, captures the amount by which loss is above the mean and that partly incorporates the more complete approach. However, the reader is cautioned that the more sophisticated methodology is needed to model reinstatements and loss-sensitive features.

8. SUMMARY AND CONCLUSION

We started with the theory and basic equations of CAT pricing within the RORAC framework and saw that the crux of the issue comes down to how to set capital for an account. We looked at basic properties of risk measures, examined several specific ones, and demonstrated that some fail key theoretical properties. We have also systematically reviewed the Standalone, Marginal, and Real Allocation algorithms for computing CAT capital. We have shown the theoretical formulas, demonstrated how they work with accessible discrete examples, and illustrated why some need to be modified to be practical. We emphasized the "ranking" definitions as a clear way to define and compute the VaR and TVaR metrics. We also explained why TVaR on a discrete data set of CAT loss data is not the same as the Conditional Tail Expectation, and why a banded version of Co-VaR is necessary. We presented the new result that Co-TVaR is not subadditive.

Our work led to a concrete comparison of how various alternatives performed on two hypothetical accounts. A key goal of our presentation was to provide an understandable illustration of the process for generating the answers. We started with an event loss table showing account and portfolio losses for each event in a modest set of hypothetical CAT events. We then used this to generate simulated random results by year. With this generated data, we computed capital under several alternatives. Our examples showed that incremental or co-statistic tail based pricing could lead to pricing barely above expectation for a non-correlated, but risky, account.

If nothing else, we would urge any reader considering different CAT pricing methodologies to examine their properties and see how they work on simple examples. We hope this paper helps to increase understanding of the alternatives. We also hope it motivates readers to investigate, test, and achieve insights beyond those we have offered.

Exhibit 1
Sheet 1

Risk Measure Definitions -Discrete Example
Variance, Standard Dev, SemiVariance and SemiStnd Dev

Statistic	Value	Statistic	Value
Trials	20	Variance	88.4
Average	10.0	Standard Dev	9.4
		Semivariance	64.2
		SemiStnd Dev	8.0

Ordered Loss Data			
Rank	Loss	Variance Contribution	Semivariance Contribution
1	40.0	900	900
2	26.0	256	256
3	18.0	64	64
4	14.0	16	16
5	14.0	16	16
6	14.0	16	16
7	14.0	16	16
8	10.0	0	0
9	8.0	4	0
10	8.0	4	0
11	6.0	16	0
12	6.0	16	0
13	6.0	16	0
14	4.0	36	0
15	4.0	36	0
16	2.0	64	0
17	2.0	64	0
18	2.0	64	0
19	2.0	64	0
20	0.0	100	0

Exhibit 1
Sheet 2

Risk Measure Definitions -Discrete Example
VaR, TVaR and XTVaR

Statistic	Value	Statistic	Value
Trials	20	Rank for VaR	5.0
Average	10.0	VaR	14.0
Percentage	75.00%	TVaR	22.4
		XTVaR	12.4

Ordered Loss Data

Rank	Loss	VaR Percentage	Conditional Tail Avg
1	40.0	95%	40.0
2	26.0	90%	33.0
3	18.0	85%	28.0
4	14.0	80%	24.5
5	14.0	75%	22.4
6	14.0	70%	21.0
7	14.0	65%	20.0
8	10.0	60%	18.8
9	8.0	55%	17.6
10	8.0	50%	16.6
11	6.0	45%	15.6
12	6.0	40%	14.8
13	6.0	35%	14.2
14	4.0	30%	13.4
15	4.0	25%	12.8
16	2.0	20%	12.1
17	2.0	15%	11.5
18	2.0	10%	11.0
19	2.0	5%	10.5
20	0.0	0%	10.0

Exhibit 1
Sheet 3

Risk Measure Definitions -Discrete Example
Transformed Mean and XSTransformed Mean

Statistic	Value	Statistic	Value
Trials	20	Wang Shift Parameter	0.674
Average	10.0	Transformed Mean	16.7
Percentage	75.00%	XS Transformed Mean	6.7

Ordered Loss Data				
Rank	Loss	Empirical CDF	Transformed CDF	Transformed Density
1	40.0	100.0%	100.0%	16.6%
2	26.0	95.0%	83.4%	10.6%
3	18.0	90.0%	72.8%	8.7%
4	14.0	85.0%	64.1%	7.5%
5	14.0	80.0%	56.6%	6.6%
6	14.0	75.0%	50.0%	6.0%
7	14.0	70.0%	44.0%	5.4%
8	10.0	65.0%	38.6%	4.9%
9	8.0	60.0%	33.7%	4.5%
10	8.0	55.0%	29.2%	4.2%
11	6.0	50.0%	25.0%	3.8%
12	6.0	45.0%	21.2%	3.5%
13	6.0	40.0%	17.7%	3.2%
14	4.0	35.0%	14.5%	2.9%
15	4.0	30.0%	11.5%	2.7%
16	2.0	25.0%	8.9%	2.4%
17	2.0	20.0%	6.5%	2.1%
18	2.0	15.0%	4.4%	1.8%
19	2.0	10.0%	2.5%	1.5%
20	0.0	5.0%	1.0%	1.0%

Exhibit 2

Failure of VaR Subadditivity

Statistic	Value		Mean	VaR
Trials	20	Risk A Standalone	2.50	4.00
Percentage	75.00%	Reference Portfolio	25.00	34.00
Rank	5	Sum	27.50	38.00
		Combined Portfolio	27.50	39.00
		Consolidation Benefit	0.00	-1.00
		Marginal VaR for A		5.00

Loss Data by Trial				**Ordered Loss Data**			
Trial	A	Ref	A+Ref	Rank	A	Ref	A+Ref
1	0.00	12.00	12.00	1	8.00	37.00	41.00
2	0.00	37.00	37.00	2	8.00	36.00	40.00
3	4.00	36.00	40.00	3	7.00	35.00	40.00
4	0.00	35.00	35.00	4	6.00	34.00	40.00
5	6.00	34.00	40.00	5	4.00	34.00	39.00
6	2.00	17.00	19.00	6	4.00	32.00	37.00
7	1.00	16.00	17.00	7	4.00	31.00	35.00
8	8.00	32.00	40.00	8	3.00	30.00	34.00
9	0.00	27.00	27.00	9	2.00	27.00	30.00
10	0.00	14.00	14.00	10	2.00	27.00	27.00
11	3.00	27.00	30.00	11	1.00	26.00	26.00
12	4.00	15.00	19.00	12	1.00	23.00	24.00
13	0.00	20.00	20.00	13	0.00	20.00	20.00
14	4.00	30.00	34.00	14	0.00	18.00	20.00
15	8.00	31.00	39.00	15	0.00	17.00	19.00
16	2.00	18.00	20.00	16	0.00	16.00	19.00
17	1.00	23.00	24.00	17	0.00	16.00	17.00
18	0.00	26.00	26.00	18	0.00	15.00	16.00
19	7.00	34.00	41.00	19	0.00	14.00	14.00
20	0.00	16.00	16.00	20	0.00	12.00	12.00

Exhibit 3
Sheet 1

Failure of Marginal VaR Scalability - Risk A + Portfolio

Statistic	Value		Mean	VaR
Trials	20	Risk A Standalone	2.50	4.00
Percentage	75.0%	Reference Portfolio	25.00	34.00
Rank	5	Sum	27.50	38.00
		Combined Portfolio	27.50	37.00
		Consolidation Benefit	0.00	1.00
		Marginal VaR for A		3.00

Loss Data by Trial				**Ordered Loss Data**			
Trial	A	Ref	A+Ref	Rank	A	Ref	A+Ref
1	4.00	12.00	16.00	1	8.00	37.00	41.00
2	0.00	37.00	37.00	2	8.00	36.00	40.00
3	0.00	36.00	36.00	3	7.00	35.00	40.00
4	0.00	35.00	35.00	4	6.00	34.00	39.00
5	6.00	34.00	40.00	5	4.00	34.00	37.00
6	2.00	17.00	19.00	6	4.00	32.00	36.00
7	1.00	16.00	17.00	7	4.00	31.00	35.00
8	8.00	32.00	40.00	8	3.00	30.00	34.00
9	0.00	27.00	27.00	9	2.00	27.00	30.00
10	0.00	14.00	14.00	10	2.00	27.00	27.00
11	3.00	27.00	30.00	11	1.00	26.00	26.00
12	4.00	15.00	19.00	12	1.00	23.00	24.00
13	0.00	20.00	20.00	13	0.00	20.00	20.00
14	4.00	30.00	34.00	14	0.00	18.00	20.00
15	8.00	31.00	39.00	15	0.00	17.00	19.00
16	2.00	18.00	20.00	16	0.00	16.00	19.00
17	1.00	23.00	24.00	17	0.00	16.00	17.00
18	0.00	26.00	26.00	18	0.00	15.00	16.00
19	7.00	34.00	41.00	19	0.00	14.00	16.00
20	0.00	16.00	16.00	20	0.00	12.00	14.00

Exhibit 3
Sheet 2

Failure of Marginal VaR Scalability - Risk 2*A + Portfolio

Statistic	Value		Mean	VaR
Trials	20	Risk 2A Standalone	5.00	8.00
Percentage	75.0%	Reference Portfolio	25.00	34.00
Rank	5	Sum	30.00	42.00
		Combined Portfolio	30.00	38.00
		Consolidation Benefit	0.00	4.00
		Marginal VaR for 2A		4.00

Loss Data by Trial				Ordered Loss Data			
Trial	2A	Ref	2A+Ref	Rank	2A	Ref	2A+Ref
1	8.00	12.00	20.00	1	16.00	37.00	48.00
2	0.00	37.00	37.00	2	16.00	36.00	48.00
3	0.00	36.00	36.00	3	14.00	35.00	47.00
4	0.00	35.00	35.00	4	12.00	34.00	46.00
5	12.00	34.00	46.00	5	8.00	34.00	38.00
6	4.00	17.00	21.00	6	8.00	32.00	37.00
7	2.00	16.00	18.00	7	8.00	31.00	36.00
8	16.00	32.00	48.00	8	6.00	30.00	35.00
9	0.00	27.00	27.00	9	4.00	27.00	33.00
10	0.00	14.00	14.00	10	4.00	27.00	27.00
11	6.00	27.00	33.00	11	2.00	26.00	26.00
12	8.00	15.00	23.00	12	2.00	23.00	25.00
13	0.00	20.00	20.00	13	0.00	20.00	23.00
14	8.00	30.00	38.00	14	0.00	18.00	22.00
15	16.00	31.00	47.00	15	0.00	17.00	21.00
16	4.00	18.00	22.00	16	0.00	16.00	20.00
17	2.00	23.00	25.00	17	0.00	16.00	20.00
18	0.00	26.00	26.00	18	0.00	15.00	18.00
19	14.00	34.00	48.00	19	0.00	14.00	16.00
20	0.00	16.00	16.00	20	0.00	12.00	14.00

Exhibit 4

Conditional Tail Expectation is Not Monotonic

Statistic	Value		Mean	VaR	TVaR	CTE
Trials	20	Risk A Standalone	2.50	4.00	6.60	7.25
Percentage	75.0%	Reference Portfolio	25.00	34.00	35.20	36.00
Rank	5	Sum	27.50	38.00	41.80	43.25
		Combined Portfolio	27.50	34.00	35.40	35.75
		Consolidation Benefit	0.00	4.00	6.40	7.50
		Marginal Measure		0.00	0.20	-0.25

Loss Data by Trial				**Ordered Loss Data**			
Trial	A	Ref	A+Ref	Rank	A	Ref	A+Ref
1	8.00	12.00	20.00	1	8.00	37.00	37.00
2	0.00	37.00	37.00	2	8.00	36.00	36.00
3	0.00	36.00	36.00	3	7.00	35.00	35.00
4	0.00	35.00	35.00	4	6.00	34.00	35.00
5	1.00	34.00	35.00	5	4.00	34.00	34.00
6	2.00	17.00	19.00	6	4.00	32.00	34.00
7	7.00	16.00	23.00	7	4.00	31.00	34.00
8	0.00	32.00	32.00	8	3.00	30.00	33.00
9	4.00	27.00	31.00	9	2.00	27.00	32.00
10	4.00	14.00	18.00	10	2.00	27.00	31.00
11	6.00	27.00	33.00	11	1.00	26.00	26.00
12	8.00	15.00	23.00	12	1.00	23.00	24.00
13	0.00	20.00	20.00	13	0.00	20.00	23.00
14	4.00	30.00	34.00	14	0.00	18.00	23.00
15	3.00	31.00	34.00	15	0.00	17.00	20.00
16	2.00	18.00	20.00	16	0.00	16.00	20.00
17	1.00	23.00	24.00	17	0.00	16.00	20.00
18	0.00	26.00	26.00	18	0.00	15.00	19.00
19	0.00	34.00	34.00	19	0.00	14.00	18.00
20	0.00	16.00	16.00	20	0.00	12.00	16.00

Exhibit 5

Instability of Co-VaR

Rank	VaR Percentage	Portfolio Loss	Risk A Loss
1			
98	99.02%	$422	$6
99	99.01%	$408	$0
100	99.00%	$405	$20
101	98.99%	$395	$0
102	98.98%	$390	$4
10,000			

Exhibit 6
Sheet 1

Co-TVaR Subadditivity Failure
Risk A Co-TVaR Calculation

Stat	Value
Trials	20
Pct	75.0%
Rank	5

Results	A	Ref	A+Ref
Mean	2.50	25.00	27.50
VaR	4.00	33.00	36.00
TVaR	6.60	35.80	38.00
Co-TVaR	3.00	35.00	38.00

Loss Data by Trial				**Separately Ordered**			**Co-Stats**			
Trial	A	Ref	A+Ref	Rank	A	Ref	Trial	Co- A	Co-Ref	A+Ref
1	2.00	8.00	10.00	1	8.00	39.00	7	1.00	39.00	40.00
2	0.00	38.00	38.00	2	8.00	38.00	3	7.00	32.00	39.00
3	7.00	32.00	39.00	3	7.00	35.00	2	0.00	38.00	38.00
4	0.00	35.00	35.00	4	6.00	34.00	14	4.00	33.00	37.00
5	2.00	14.00	16.00	5	4.00	33.00	6	3.00	33.00	36.00
6	3.00	33.00	36.00	6	3.00	33.00	4	0.00	35.00	35.00
7	1.00	39.00	40.00	7	3.00	32.00	19	0.00	34.00	34.00
8	2.00	16.00	18.00	8	2.00	30.00	15	3.00	30.00	33.00
9	8.00	25.00	33.00	9	2.00	27.00	11	6.00	27.00	33.00
10	0.00	11.00	11.00	10	2.00	26.00	9	8.00	25.00	33.00
11	6.00	27.00	33.00	11	2.00	25.00	12	8.00	22.00	30.00
12	8.00	22.00	30.00	12	1.00	23.00	18	0.00	26.00	26.00
13	0.00	20.00	20.00	13	1.00	22.00	17	1.00	23.00	24.00
14	4.00	33.00	37.00	14	1.00	20.00	16	2.00	18.00	20.00
15	3.00	30.00	33.00	15	0.00	18.00	13	0.00	20.00	20.00
16	2.00	18.00	20.00	16	0.00	16.00	8	2.00	16.00	18.00
17	1.00	23.00	24.00	17	0.00	16.00	20	1.00	16.00	17.00
18	0.00	26.00	26.00	18	0.00	14.00	5	2.00	14.00	16.00
19	0.00	34.00	34.00	19	0.00	11.00	10	0.00	11.00	11.00
20	1.00	16.00	17.00	20	0.00	8.00	1	2.00	8.00	10.00

Exhibit 6
Sheet 2

Co-TVaR Subadditivity Failure

Risk B Co-TVaR Calculation

Stat	Value
Trials	20
Pct	75.0%
Rank	5

Results	B	Ref	B+Ref
Mean	2.50	25.00	27.50
VaR	4.00	33.00	36.00
TVaR	6.40	35.80	38.40
Co-TVaR	3.00	35.40	38.40

Loss Data by Trial				**Separately Ordered**			**Co-Stats**			
Trial	B	Ref	B+Ref	Rank	B	Ref	Trial	Co- B	Co-Ref	B+Ref
1	0.00	8.00	8.00	1	9.00	39.00	7	5.00	39.00	44.00
2	0.00	38.00	38.00	2	7.00	38.00	2	0.00	38.00	38.00
3	4.00	32.00	36.00	3	7.00	35.00	14	4.00	33.00	37.00
4	2.00	35.00	37.00	4	5.00	34.00	4	2.00	35.00	37.00
5	2.00	14.00	16.00	5	4.00	33.00	3	4.00	32.00	36.00
6	1.00	33.00	34.00	6	4.00	33.00	19	1.00	34.00	35.00
7	5.00	39.00	44.00	7	3.00	32.00	11	7.00	27.00	34.00
8	1.00	16.00	17.00	8	2.00	30.00	9	9.00	25.00	34.00
9	9.00	25.00	34.00	9	2.00	27.00	6	1.00	33.00	34.00
10	0.00	11.00	11.00	10	2.00	26.00	15	0.00	30.00	30.00
11	7.00	27.00	34.00	11	1.00	25.00	12	7.00	22.00	29.00
12	7.00	22.00	29.00	12	1.00	23.00	18	2.00	26.00	28.00
13	1.00	20.00	21.00	13	1.00	22.00	17	1.00	23.00	24.00
14	4.00	33.00	37.00	14	1.00	20.00	16	3.00	18.00	21.00
15	0.00	30.00	30.00	15	1.00	18.00	13	1.00	20.00	21.00
16	3.00	18.00	21.00	16	0.00	16.00	8	1.00	16.00	17.00
17	1.00	23.00	24.00	17	0.00	16.00	20	0.00	16.00	16.00
18	2.00	26.00	28.00	18	0.00	14.00	5	2.00	14.00	16.00
19	1.00	34.00	35.00	19	0.00	11.00	10	0.00	11.00	11.00
20	0.00	16.00	16.00	20	0.00	8.00	1	0.00	8.00	8.00

Exhibit 6
Sheet 3

Co-TVaR Subadditivity Failure

Combined Risk A + Risk B Co-TVaR Calculation

Stat	Value
Trials	20
Pct	75.0%
Rank	5

Results	A+B	Ref	A+B +Ref
Mean	5.00	25.00	30.00
VaR	8.00	33.00	40.00
TVaR	12.80	35.80	42.20
Co-TVaR	11.00	31.20	42.20

Loss Data by Trial				Separately Ordered			Co-Stats			
Trial	A+B	Ref	A+B +Ref	Rank	A+B	Ref	Trial	Co-A+B	Co-Ref	A+B +Ref
1	2.00	8.00	10.00	1	17.00	39.00	7	6.00	39.00	45.00
2	0.00	38.00	38.00	2	15.00	38.00	3	11.00	32.00	43.00
3	11.00	32.00	43.00	3	13.00	35.00	9	17.00	25.00	42.00
4	2.00	35.00	37.00	4	11.00	34.00	14	8.00	33.00	41.00
5	4.00	14.00	18.00	5	8.00	33.00	11	13.00	27.00	40.00
6	4.00	33.00	37.00	6	6.00	33.00	2	0.00	38.00	38.00
7	6.00	39.00	45.00	7	5.00	32.00	12	15.00	22.00	37.00
8	3.00	16.00	19.00	8	4.00	30.00	6	4.00	33.00	37.00
9	17.00	25.00	42.00	9	4.00	27.00	4	2.00	35.00	37.00
10	0.00	11.00	11.00	10	3.00	26.00	19	1.00	34.00	35.00
11	13.00	27.00	40.00	11	3.00	25.00	15	3.00	30.00	33.00
12	15.00	22.00	37.00	12	2.00	23.00	18	2.00	26.00	28.00
13	1.00	20.00	21.00	13	2.00	22.00	17	2.00	23.00	25.00
14	8.00	33.00	41.00	14	2.00	20.00	16	5.00	18.00	23.00
15	3.00	30.00	33.00	15	2.00	18.00	13	1.00	20.00	21.00
16	5.00	18.00	23.00	16	1.00	16.00	8	3.00	16.00	19.00
17	2.00	23.00	25.00	17	1.00	16.00	5	4.00	14.00	18.00
18	2.00	26.00	28.00	18	1.00	14.00	20	1.00	16.00	17.00
19	1.00	34.00	35.00	19	0.00	11.00	10	0.00	11.00	11.00
20	1.00	16.00	17.00	20	0.00	8.00	1	2.00	8.00	10.00

Exhibit 6
Sheet 4

Co-TVaR Subadditivity Failure Summary

Statistic	Value
Trials	20
Pct	75%
Rank	5

Results	A	B	A+B	Ref	A+B +Ref
Mean	2.50	2.50	5.00	25.00	30.00
VaR	4.00	4.00	8.00	33.00	40.00
TVaR	6.60	6.40	12.80	35.80	42.20
Co-TVaR	3.00	3.00	11.00	31.20	42.20

Loss Data by Trial

Trial	A	B	A+ B	Ref	A+ B +Ref
1	2.00	0.00	2.00	8.00	10.00
2	0.00	0.00	0.00	38.00	38.00
3	7.00	4.00	11.00	32.00	43.00
4	0.00	2.00	2.00	35.00	37.00
5	2.00	2.00	4.00	14.00	18.00
6	3.00	1.00	4.00	33.00	37.00
7	1.00	5.00	6.00	39.00	45.00
8	2.00	1.00	3.00	16.00	19.00
9	8.00	9.00	17.00	25.00	42.00
10	0.00	0.00	0.00	11.00	11.00
11	6.00	7.00	13.00	27.00	40.00
12	8.00	7.00	15.00	22.00	37.00
13	0.00	1.00	1.00	20.00	21.00
14	4.00	4.00	8.00	33.00	41.00
15	3.00	0.00	3.00	30.00	33.00
16	2.00	3.00	5.00	18.00	23.00
17	1.00	1.00	2.00	23.00	25.00
18	0.00	2.00	2.00	26.00	28.00
19	0.00	1.00	1.00	34.00	35.00
20	1.00	0.00	1.00	16.00	17.00

Exhibit 7
Sheet 1

Event Loss table

Event ID	Annual Probability	Treaty A Loss	Treaty B Loss	Reference Portfolio Loss
1	2%	0	500	125,000
2	2%	0	1,000	100,000
3	2%	0	1,000	90,000
4	2%	0	2,000	80,000
5	2%	0	2,500	75,000
6	2%	0	3,000	70,000
7	2%	0	3,000	60,000
8	2%	0	3,000	50,000
9	2%	0	3,000	50,000
10	2%	0	2,500	40,000
11	2%	0	2,000	38,000
12	2%	0	1,000	36,000
13	2%	0	1,000	34,000
14	2%	0	1,000	32,000
15	2%	0	1,000	30,000
16	2%	0	2,000	25,000
17	2%	0	2,500	20,000
18	2%	0	3,000	15,000
19	2%	0	3,000	10,000
20	2%	0	3,000	5,000
21	2%	0	3,000	5,000
22	2%	0	2,500	4,000
23	2%	0	2,000	3,000
24	2%	0	1,000	2,000
25	2%	0	500	1,000

Exhibit 7
Sheet 2

Event Loss table

Event ID	Annual Probability	Treaty A Loss	Treaty B Loss	Reference Portfolio Loss
26	2%	12,500	0	0
27	2%	10,000	0	0
28	2%	7,500	0	0
29	2%	5,000	0	0
30	2%	4,000	0	0
31	2%	3,000	0	0
32	2%	2,000	0	0
33	2%	1,000	0	0
34	2%	900	0	0
35	2%	800	0	0
36	2%	700	0	0
37	2%	600	0	0
38	2%	500	0	0
39	2%	400	0	0
40	2%	300	0	0
41	2%	200	0	0
42	2%	100	0	0
43	2%	90	0	0
44	2%	80	0	0
45	2%	70	0	0
46	2%	60	0	0
47	2%	50	0	0
48	2%	50	0	0
49	2%	50	0	0
50	2%	50	0	0

Exhibit 8

Simulation of 1,000 Trial Years for 50 events

Event ID	1	2	3	4	5
Treaty A Loss	0	0	0	0	0
Treaty B Loss	500	1,000	1,000	2,000	2,500
Treaty C Loss	500	0	1,000	0	2,500
Reference Portfolio Loss	125,000	100,000	90,000	80,000	75,000
Ref + A + B	125,500	101,000	91,000	82,000	77,500
Simulated Annual Prob	1.80%	1.60%	1.90%	2.60%	2.40%
Prob Occ	2.00%	2.00%	2.00%	2.00%	2.00%

Event Indicator Matrix

Trial Year	**Number of events**	**Event ID 1**	**Event ID 2**	**Event ID 3**	**Event ID 4**	**Event ID 5**
1	0					
2	1					
3	0					
4	2					
5	2					
6	2					
7	1					
8	1		1			
9	1					
10	1			1		
11	1			1		
12	1					
13	1					
14	1					
15	0					
16	2					
17	0					
18	0					
19	0					
20	0					

Exhibit 9
Sheet 1

Capital and Pricing Indications
Treaty A

Percentage	95.0%
Target Return	15.0%

	E[L]	Linear Correlation	Rank Correlation
Reference Portfolio	19,651		
Treaty A	1,060	1%	-2%

Capital Calculation Method	Treaty A Standalone Capital	Reference Portfolio Capital	Reference + Treaty A Capital	Treaty A Required Capital	Treaty A Indicated Premium	Treaty A Indicated Risk Load	Risk Load % of Premium
Standalone VaR	7,590	N/A	N/A	7,590	2,198	1,139	52%
Standalone TVaR	11,255	N/A	N/A	11,255	2,748	1,688	61%
Standalone XTVaR	10,195	N/A	N/A	10,195	2,589	1,529	59%
VaR Increment	N/A	100,000	102,000	2,000	1,360	300	22%
TVaR Increment	N/A	130,800	132,206	1,406	1,271	211	17%
XTVaR Increment	N/A	111,149	111,495	346	1,112	52	5%
Allocation via Standalone VaR	7,590	100,000	102,000	7,196	2,139	1,079	50%
Allocation via Standalone TVaR	11,255	100,000	102,000	10,318	2,608	1,548	59%
Allocation via Standalone XTVaR	10,195	100,000	102,000	9,437	2,475	1,416	57%
Band Co-VaR Allocation	N/A	N/A	102,000	3,237	1,545	486	31%
Co-TVaR Allocation	N/A	N/A	102,000	1,239	1,246	186	15%
Co-XTVaR Allocation	N/A	N/A	102,000	1,519	1,288	228	18%

Exhibit 9
Sheet 2

Capital and Pricing Indications
Treaty B

Percentage	95.0%
Target Return	15.0%

	E[L]	Linear Correlation	Rank Correlation
Reference Portfolio	19,651		
Treaty B	1,054	60%	93%

Capital Calculation Method	Treaty B Standalone Capital	Reference Portfolio Capital	Reference + Treaty B Capital	Treaty B Required Capital	Treaty B Indicated Premium	Treaty B Indicated Risk Load	Risk Load % of Premium
Standalone VaR	4,000	N/A	N/A	4,000	1,654	600	36%
Standalone TVaR	5,480	N/A	N/A	5,480	1,876	822	44%
Standalone XTVaR	4,426	N/A	N/A	4,426	1,718	664	39%
VaR Increment	N/A	100,000	104,000	4,000	1,654	600	36%
TVaR Increment	N/A	130,800	134,010	3,210	1,536	482	31%
XTVaR Increment	N/A	111,149	113,305	2,156	1,377	323	23%
Allocation via Standalone VaR	4,000	100,000	104,000	4,000	1,654	600	36%
Allocation via Standalone TVaR	5,480	100,000	104,000	5,403	1,864	810	43%
Allocation via Standalone XTVaR	4,426	100,000	104,000	4,408	1,715	661	39%
Band Co-VaR Allocation	N/A	N/A	104,000	2,588	1,442	388	27%
Co-TVaR Allocation	N/A	N/A	104,000	2,491	1,428	374	26%
Co-XTVaR Allocation	N/A	N/A	104,000	2,571	1,440	386	27%

Exhibit 9
Sheet 3

Capital and Pricing Indications Summary
Treaty A vs Treaty B Comparison

Percentage	95.0%
Target Return	15.0%

	Simulated Value E[L]	Linear Correlation	Rank Correlation
Reference Portfolio	19,651		
Treaty A	1,060	1%	-2%
Treaty B	1,054	60%	93%

Capital Calculation Method	Treaty A Required Capital	Treaty A Indicated Premium	Risk Load % of Premium	Treaty B Required Capital	Treaty B Indicated Premium	Risk Load % of Premium
Standalone VaR	7,590	2,198	52%	4,000	1,654	36%
Standalone TVaR	11,255	2,748	61%	5,480	1,876	44%
Standalone XTVaR	10,195	2,589	59%	4,426	1,718	39%
VaR Increment	2,000	1,360	22%	4,000	1,654	36%
TVaR Increment	1,406	1,271	17%	3,210	1,536	31%
XTVaR Increment	346	1,112	5%	2,156	1,377	23%
Standalone VaR Allocation	7,196	2,139	50%	4,000	1,654	36%
Standalone TVaR Allocation	10,318	2,608	59%	5,403	1,864	43%
Standalone XTVaR Allocation	9,437	2,475	57%	4,408	1,715	39%
Band Co-VaR Allocation	3,237	1,545	31%	2,588	1,442	27%
Co-TVaR Allocation	1,239	1,246	15%	2,491	1,428	26%
Co-XTVar Allocation	1,519	1,288	18%	2,571	1,440	27%

BIBLIOGRAPHY

[1] C. Acerbi and D. Tasche, "Expected Shortfall: a natural coherent alternative to Value at Risk", 2001.

[2] D. Appell, "Quantifying Risk Load for Property Catastrophe Exposure", Presentation *Casualty Actuarial Society Ratemaking Seminar,* 2010.

[3] P. Artzner, "Application of Coherent Risk Measures to Capital Requirements in Insurance", *North American Actuarial Journal*, Volume 3 Number 2, p 11-25.

[4] P. Artzner, F. Delbaen, J. Eber, and D. Heath, "Coherent Measures of Risk," *Math. Finance 9* (1999), no. 3, p 203-229.

[5] M. Barges, H. Cossette, and E. Marceau, "TVaR-based capital allocation with copulas", *Insurance: Mathematic and Economics*, Vol 45 Issue 3 December 2009, p 348-361.

[6] N. Bodoff, "Capital Allocation by Percentile Layer," *Variance*, 2009, Volume 3/ Issue 1, p 13-30.

[7] M. Brodsky, "Diversification or Deworsification?", *Risk and Insurance*, June 1, 2011.

[8] R. Butsic, "Allocating the Cost of Capital", *Casualty Actuarial Society Forum,* 2002 Fall.

[9] S. D'Arcy, "Capital Allocation in the Property-Liability Insurance Industry", *Variance*, 2011, Volume 5/Issue 2, p 141-157.

[10] M. Denault, "Coherent Allocation of Risk Capital", *The Journal of Risk*, Volume 4, Number 1, Fall 2001.

[11] L. Fu and C.K. Khury, "Optimal Layers for Catastrophe Reinsurance", *Variance,* 2010, Volume 4/Issue 2, p 193-208.

[12] R. Goldfarb, "Risk Adjusted Performance Measures for P&C Insurers", *Casualty Actuarial Society Study Note,* 2010.

[13] R. Kreps, "Reinsurer Risk Loads from Marginal Surplus Requirements", *Proceedings of the Casualty Actuarial Society* Vol LXXVII, 1990, p. 196-203.

[14] R. Kreps, "Riskiness Leverage Models", *Proceedings of the Casualty Actuarial Society*, Vol XCII, 2005, p. 31-60.

[15] D. Mango, "An Application of Game Theory: Property Catastrophe Risk Load", *Proceedings of the Casualty Actuarial Society* Vol LXXXV, 1998, 157-186.

[16] D. Mango, "Insurance Capital as a Shared Asset", *Casualty Actuarial Society Forum*, 2006 Fall, p. 573-576.

[17] D. Mango and J. Sandor, "Dependence Models and the Portfolio Effect", *Casualty Actuarial Society Forum*, 2002 Winter, p 57-72.

[18] G. Meyers, "Setting Capital Requirements with Coherent Measures of Risk –Part 1 and Part 2". *Actuarial Review*, August and November, 2002.

[19] G. Meyers, F. Klinker and G. Lalonde, "The Aggregation and Correlation of Reinsurance Exposure," *Casualty Actuarial Society Forum*, 2003.

[20] H. Panjer and J. Jing, "Solvency and Capital Allocation", *International Symposium on Insurance Regulatory Environment and Actuarial Practice in China*, October 2001.

[21] I. Robbin and J. DeCouto, "Coherent Capital for Treaty ROE Calculations", *Casualty Actuarial Society Forum*, Spring 2005.

[22] R. Rockafellar and S. Uryasev, "Conditional value-at-risk for general loss distributions", *Journal of Banking and Finance* 26 (2002), p 1443-1471.

[23] G.Venter, "A Survey of Capital Allocation Methods with Commentary," ASTIN Colloquia, 2003.

[24] S. Wang, "A Risk Measure That Goes Beyond Coherence", 2002.

[25] S. Wang, V. Young, and H. H. Panjer, "Axiomatic Characterization of Insurance Prices", *Insurance Mathematics and Economics,* 21 (1997), pp 173-182.

[26] S. Wang, "Implementation of Proportional Hazards Transforms in Ratemaking", *Proceedings of the Casualty Actuarial Society,* Vol LXXXV, 1998, p. 940-979.

[27] J.Wirch and M. Hardy, "Distortion Risk Measures: Coherence and Stochastic Dominance", *International Congress on Insurance: Mathematics and Economics*, July 15-17, 2002.

[28] N. Zhao, "Modeling Dependence in Catastrophe Risk Portfolio and Optimizing Capital Loading", *ETH Thesis*, 2010.

Abbreviations and notations

CAT, Catastrophe
cdf, Cumulative Distribution Function
Co-Statistic or Co-Stat, Contribution Statistic
CTE, Conditional Tail Expectation
PML, Probable Maximum Loss
RORAC, Return on Risk-Adjusted Capital
TVaR, Tail Value at Risk
VaR, Value at Risk
XTVaR, Excess Tail Value at Risk

Biography of the Author

Ira Robbin is currently at AIG after having previously held positions at P&C Actuarial Analysts, Endurance, Partner Re, CIGNA PC, and INA working in several corporate, pricing, research, and consulting roles. He has written papers and made presentations on many topics including risk load, capital requirements, ROE, Coherent Capital, price monitors, and One-year Reserve Risk. He has a PhD in Math from Rutgers University and a Bachelor degree in Math from Michigan State University.

Disclaimers

Opinions expressed in this paper are solely those of the author. They may or may not be consistent with the views of the authors' current or prior clients or employers. No warranty is given that any formula or assertion is accurate. No liability is assumed whatsoever for any losses, direct or indirect, that may result from use of the methods described in this paper or reliance on any of the views expressed therein.

Acknowledgments

The author gratefully acknowledges the contributions of Joshua London, David Clark, and Neil Bodoff. Their insightful commentary, illustrative spreadsheets, and constructive reviews were critical in helping the author draft and improve this paper. The author also wishes to thank the review committee members for their thoughtful comments and corrections.