ABSTRACT OF THE DISCUSSION OF THE PAPERS READ AT THE PREVIOUS MEETING.

A SUGGESTION FOR A MODIFIED FORM OF AMORTIZATION, WITH A BRIEF MEMORANDUM OF THE APPLICABILITY OF THAT PRINCIPLE TO THE BONDS OF MISCELLANEOUS COMPANIES—S. HERBERT WOLFE.

VOL. III, PAGE 4.

WRITTEN DISCUSSION.

MR. WILLIAM BREIBY:

Mr. Wolfe in his paper attacks the present system of amortizing bonds, because, he says it "is based upon hypotheses which never exist and in consequence produce theoretical results only," and also hints that the work involved in using that method is unduly complicated. He then proposes as a substitute, amortization as if all bonds were purchased to yield the same rate of interest.

He says further, in the first paragraph following his Table III, that, in order to extinguish the premiums scientifically according to the effective rate corresponding to the purchase price, the amounts he shows in column (3), Table III, called the "Amortization Factor" (which is, of course, the amortization factor applying in the first period only), "must be reinvested the instant . . . received, and the amount . . . must be used to purchase securities which will yield its effective rate shown in column (4)."

This statement, I believe, can be shown to be wrong. The successive "Amortization Factors," where a bond is bought at a premium, are actual repayments of principal, and we are not concerned with the rate, or time, at which they are re-invested, so far as affecting the yield on, or values of, the particular block of funds originally invested and the successive balances of principal outstanding. Though this is seen by the usual text-book analysis of the annuity certain in the formula:

$$k = (g - i)\,a_{\overline{n}|},$$

where k is the rate of premium or discount, g, the rate of "coupon," i, the rate of effective interest yield, and $a_{\overline{n}|}$ is calculated at rate i, a schedule showing the actual figures may better help to refresh the

memories of those who have not for some time past looked at a text-book on "Interest and Annuities Certain":

TABLE *A*.

BOND OF $1,000,000.00 PAR, PAYING 2½% ($25,000) SEMI-ANNUALLY, MATURING AT END OF FIVE YEARS. TO YIELD 2¼% SEMI-ANNUALLY, I. E., YIELDING ¼% LESS EACH PERIOD THAN THE RATE OF COUPON.

Amortized Value $= (v^n + .025 \cdot a_{\overline{n}|}) \times 1{,}000{,}000$, v^n and $a_{\overline{n}|}$ Calculated at Rate .0225.

Half-year Periods n.	(1) Amortized Value or Principal Outstanding.	(2) Semi-annual Interest on Outstanding Principal (1) × .0225.	(3) Semi-annual Amortization Item or Repayment of Principal $25,000—(2).
10	$1,022,165.54	$22,998.725	$2,001.275
9	1,020,164.27	22,953.696	2,046.304
8	1,018,117.97	22,907.654	2,092.346
7	1,016,025.62	22,860.576	2,139.424
6	1,013,886.19	22,812.439	2,187.561
5	1,011,698.63	22,763.219	2,236.781
4	1,009,461.86	22,712.892	2,287.108
3	1,007,174.74	22,661.432	2,338.568
2	1,004,836.18	22,608.814	2,391.186
1	1,002,444.99	22,555.012	2,444.988
0	1,000,000.00	.....	.....

Sum of principal repayments	$ 22,165.541
Cost price of bonds	1,022,165.54
Balance = redemption value	$1,000,000.00

Of course, the sum of column (3) equals the premium on the bond, so, as the schedule clearly shows, in each of the $25,000 of coupons as paid is included the repayment of principal of the successive amounts shown in (4). It need hardly be said that, after determining the yield, the schedule could be prepared without previously calculating the whole of column (1), by the operations indicated at the head of columns (2) and (3) and successive subtraction of the figures in (3) from those in (1).

In text-books and discussions on amortization the case of bonds bought below par is often discharged with the statement that the rules are similar to those applying to bonds bought at a premium, but as the operations and the results are rather reversed, I give here a schedule showing the figures for a bond bought at a discount.

Here, of course, the "Extra Interest" is that portion of the difference between the amount payable at maturity and the cost price which can be considered as having accrued for the interval indicated. There is no need to re-invest any sum during this period at any rate in order to carry out the system of amortization. The $20,000 is received each period, together with the certain ac-

crual of value to the purchase cost. There can be no question that at the maturity there will have been received $20,000 at the end of each period, and a lump sum at maturity, such that the yield will have been semi-annually 2¼ per cent. on the original cost and outstanding principal amounts.

TABLE *B*.

BOND OF $1,000,000.00 PAR, PAYING 2% ($20,000) SEMI-ANNUALLY, MATURING AT END OF FIVE YEARS. TO YIELD 2¼% SEMI-ANNUALLY I. E., YIELDING ¼% MORE EACH PERIOD THAN THE RATE OF COUPON.

Amortized Value $= (v^n + .02 \cdot a_{\overline{n}|}) \times 1{,}000{,}000$ or $(1 - .0025 \cdot a_{\overline{n}|}) \times 1{,}000{,}000$.

Half-year Periods n.	(1) Amortized Value or Principal Outstanding.	(2) Semi-annual Interest on Outstanding Principal (1) × .0225.	(3) Semi-annual Amortization Item or Extra Interest, being Accrual of Principal (2)—$20,000.
10	$ 977,834.46	$22,001.275	$2,001.275
9	979,835.73	22,046.304	2,046.304
8	981,882.04	22,092.346	2,092.346
7	983,974.38	22,139.424	2,139.424
6	986,113.81	22,187.561	2,187.561
5	988,301.37	22,236.781	2,236.781
4	990,538.15	22,287.108	2,287.108
3	992,825.26	22,338.568	2,338.568
2	995,163.83	22,391.186	2,391.186
1	997,555.01	22,444.988	2,444.988
0	1,000,000.00	.....	.....

Sum of accruals of principal	$ 22,165.541
Cost price of bonds	977,834.46
Sum = redemption value	$1,000,000.00

As in the first table the columns (2) and (3) could, of course, have been developed without the preliminary calculation of all the values in (1), by adding the successive values in (3) to those in (1). The sum of (3) of course equals the discount at which the bond was bought. As in this case we have a bond with a "coupon" rate ¼ per cent. more than the yield, the discount equals the premium in Table *A*, where the yield exceeded the "coupon" rate by ¼ per cent. The formula demonstrating this fact being the same as before quoted: $k = (g - i)a_{\overline{n}|}$, k being rate of premium, or discount: in one case being plus, and in the other minus.

Thus, I do not see how the principle underlying the present method of amortization of bonds fails to apply to actual conditions existing, nor how it produces theoretical results only.

In case it were necessary that the outstanding principal should remain throughout at the original investment, and the amount earned thereon periodically should be at that effective rate of yield,

Mr. Wolfe's statement, that the amortization factor must be reinvested the instant received, is correct when extended to include the continued reinvestment of the "amortization" elements. If it were, however, required to have on hand at the maturity of the bond the exact amount originally invested, together with interest compounded thereon at the effective rate of yield, it would be necessary to reinvest immediately the total of each coupon payment at the same effective rate. This is of course readily seen by the equality

$$(v^n + g\bar{a}_{\overline{n}|})(1+i)^n = 1 + g\bar{s}_{\overline{n}|}.$$

For life insurance the net premiums are calculated on the assumption that they are invested at compound interest as soon as collected, and accumulated in that way to the maturity of the insurances. So that an insurance company operating under such an assumption must avoid investing too large a proportion of its funds in securities bought at a premium with an interest yield just in excess of that assumed in the premium calculations, unless, of course, there were assurance that the principal repayments could be reinvested to yield at least the assumed rate. Where, however, assuming the security behind the bonds is ample, bonds are bought at a discount, such possible difficulty is not presented. On the other hand, in such a company insurances are constantly maturing, and theoretically the company needs constant repayment of some of the principal.

As to the work involved in deriving the amortized values under the present method:

First it is necessary to ascertain the interest yield, which is usually established to the nearest 1/100 per cent. by proportion between the tabular values, as given in a bond value table or book, for the usual rates of yield at intervals of 5/100 per cent. By observing the differences between values at the successive rates it can readily be seen that the yield so obtained is as correct as needed, and probably seldom differs in the second place of decimals per cent. from that derived by formula.

After so determining the yield the amortized values corresponding to that rate are derived by proportion between the tabular values. Even if the established rate should be wrong by as much as 1/100 per cent., the values so derived are sufficiently accurate and average with others, and as the period to maturity reduces, the values so derived approach absolute accuracy. The difference between values at different rates approaches nr as n becomes small, where r is the difference in rates and n the number of periods.

Where bonds are bought on dates other than interest due dates, values are interpolated between the tabular values by proportion to correspond to such time, and then the rate of yield is determined from such interpolated values as above described. Similarly the December 31st values are established by proportion between the

tabular. Values for dates other than interest due dates so obtained give correct results on the assumption of simple interest within an interest period.

Where later purchases are made of the same issue of bonds already held, the usual practice is to add the purchase price of the new lot to the amortized value of the old lot brought down to date of new purchase, or else to bring them both down according to their respective yields to some common date, such as December 31st, and then ascertain a new yield based on the sum.

All of this work requires only a knowledge of arithmetic, and instructions can easily be given to clerks to carry out the work. Where the records are kept on cards, and the values entered from year to year, as is usual where many bond transactions take place, the cards could readily be sorted so that the bond value book could be used to best advantage with a minimum turning of pages.

Where bonds are bought to yield a rate beyond published tables a knowledge of the algebra of bond calculations is, of course, required, but the increased yield would well compensate for the increased cost of calculation, and if opportunity to invest at such high rates should prevail for any time published tables would soon be extended to include them.

To ascertain the actual interest earned during the year the items of "interest received" must be modified by the amortization items, as shown in the Gain and Loss Exhibits of most of the New York life insurance companies.

Whether the bond values entered in the periodic statements of a company are the amortized values or not, the company's official in charge of investments should have on hand schedules showing the amortized values, in order to determine readily whether at any time it is advantageous to sell the securities. The fact that where securities are sold causes a yield to have been experienced different from that where held until maturity, does not vitiate the principle of amortization; because the profit or loss on sale would be a distinct profit or loss as compared with the then amortized value, which is really the cost price extended to the time of sale. The bookkeeping in case of a sale could be carried out by extending the amortized value from the last statement date to date of sale, entering the difference in either "Increase for accrual of discount," or "Decrease for amortization of premiums," as the case may be, and then entering the difference between that amortized value and the selling price in either "profit (or loss) on sale of bonds"; or else by entering the difference between the selling price and the amortized value at the last statement date in "profit (or loss) on sale of bonds."

As to Mr. Wolfe's proposed method. If all the bonds were bought to yield approximately the same rate, amortization by using annuities certain based on a single rate of interest would probably be

sufficiently accurate for statement purposes. But without the amortized values for particular bonds there would not be the means to learn readily whether it were advantageous to sell as changes took place in the market values. As a matter of fact the yields on the different lots of bonds are spread over quite a range, as Mr. Wolfe indicates in referring to bonds yielding rates beyond present practical bond value tables, and amortization by means of annuities certain based on the average rate earned on all would probably seldom give results approaching a correct amortization.

The use of a single rate table would give results which would not alow of any valuable interpretation or analysis, and there could not be ascertained therefrom the actual interest earned in a year as outlined above. Further, when it was found that a change in the assumed interest rate would have to be made, the change in the aggregate bond values in that year would be violent, though the change in conditions had probably been gradual; and I fail to see how such a system would be better than one of market values, or of carrying the bonds at cost price until maturity.

Following Table I, Mr. Wolfe points out that the amorization of bonds is complicated because bonds are not always bought on coupon dates. The same complication arises also under the proposed plan. His table gives the annuity amounts for periods which are likewise multiples of 6 months, and values would have to be gotten up for all intervening times, and some adjustment would have to be made to get December 31 values. As soon as the values in the single table cannot be used directly most of the advantage claimed for the proposed system disappears. Further, as bonds are bought on different dates, and with various maturity and interest dates, they could not readily be grouped so as to have one factor apply to a group comprising several lots.

In the third paragraph under Table III he says, "why not adopt some method . . . which permits the use of a standard table . . . dealing only with the purchase prices."

In applying the table of annuities certain to the first case given above, the work would be as follows:

Premium to be extinguished		$22,165.54
Dividing that by the amount opposite 5 in Mr. Wolfe's Table IV of annuities certain	$11.075,708	
Gives what would be his first amortization factor for the case...		2,001.28
Balance being the Premium Outstanding		20,164.26
Dividing that by the amount opposite 4½ in Table IV	$9.853,993 =	2,046.30
Leaving balance		$18,117.96
etc., etc., etc.,		

i. e., reproducing the values in column (3), and the premiums in column (1) in Table *A* above. Just how this proposed system deals

only with the purchase price any more than the usual amortization method it is hard to see.

As to extending the amortization plan to the so-called "miscellaneous" companies: Many of such companies, as going concerns, are probably not more subject to the need of disposing of large blocks of their assets than savings banks, and an amortization plan could probably be used. Where, however, owing to the nature of the business, a company is liable to an "underwriting loss" such as under Company *E* in Mr. Wolfe's Table VI, I do not think it advisable to permit the use of an amortization method. Where a company's status is subject to sudden change, dependent on immediate circumstances or conditions, its assets should be stated in any statement at values most likely to be realized immediately in case of necessity. It is conceivable that a company doing a business subject to violent loss, even though carrying multiple lines, could be so impaired as not to be in shape to reinsure.

Before permitting "miscellaneous" companies to amortize their bonds, it might be instructive to learn how far such companies have in the past found it necessary or desirable to sell some of their bonds because of the nature of their business.

Though I feel that in a discussion of this paper it is not necessary to raise the question of the relative merits of a market value method and an amortization method, I will venture to say in closing that the most valuable plan is that which best shows the going qualities of a company; and as market values are indicative more of the immediate time or recent past, whereas amortized values are of the nature of "going" values, the latter are the ones to use.

Note: Since writing the above I have read Mr. Alexander T. Maclean's criticism of Mr. Wolfe's paper as published in the *Economic World* of Dec. 2, 1916, and find his remarks to be along the same general lines as those in this discussion.

MR. VIRGIL M. KIME:

The method of valuation of the bonds of companies amortizing their bonds is, if purchased above or below par, on the basis of the purchase price, adjusted so as to bring the value to par at maturity, and so as to yield meantime the effective rate of interest at which the purchase was made.

When a bond is purchased at a premium, the application of the amortization principle consists in the application of a portion of each gross interest payment from the debtor corporation to the liquidation of the principal of the investment and a portion to the payment of interest on the unliquidated investment. The total investment decreases from interest period to interest period, the balance of principal at maturity being the amount then payable.

The following table illustrates the application of the principle to a $10,000 ten-year bond investment bearing 5 per cent. interest

purchased at a premium of $1,256.47 to yield 3½ per cent. interest on the investment.

$10,000—5 PER CENT PAYABLE SEMI-ANNUALLY 10 YEARS.
To Yield 3½ Per Cent. Payable Semi-annually.

Period.	Payment at End of Period.		Principal at End of Period.
0 Yr.	Interest	Principal	$11,256.47
.5	$196.98	$53.02	11,203.45
1.	196.06	53.94	11,149.51
1.5	195.12	54.88	11,094.63
2.	194.16	55.84	11,038.79
2.5	193.18	56.82	10,981.97
3.	192.18	57.82	10,924.15
3.5	191.17	58.83	10,865.32
4.	190.15	59.85	10,805.47
4.5	189.09	60.91	10,744.56
5.	188.03	61.97	10,682.59
5.5	186.95	63.05	10,619.54
6.	185.84	64.16	10,555.38
6.5	184.72	65.28	10,490.10
7.	183.57	66.43	10,423.67
7.5	182.42	67.58	10,356.09
8.	181.23	68.77	10,287.32
8.5	180.03	69.97	10,217.35
9.	178.80	71.20	10,146.15
9.5	177.56	72.44	10,073.71
10.	176.29	73.71	10,000.00

It will be noted that the interest is for each period 1¾ per cent. of the principal at the end of the preceding period.

The transaction consists in the investment at the beginning of the ten-year period of $11,256.47, in the reduction of the investment by a part payment thereof from half-year to half-year in the amount indicated in the table, and in the final payment of the balance of principal at the end of ten years of $10,000. I cannot see that the disposition to be made of any portion of the principal when it is repaid enters into the problem. A company might in a similar manner invest in farm mortgage security in the sum of $10,000 to bear at 5 per cent. per annum, payable semi-annually; $5,000 to mature in 2½ years and $5,000 to mature in 5 years. The yield on the investment would, it seems, be determined by the yield on $2,500 for 2½ years and for 5 years respectively. The manner in which the $2,500 repaid at the end of 2½ years is invested or disposed of would hardly affect the yield on the investment. Applying the same principle, the disposition of the series of principal repayments, $53.02, $53.94, etc., would not affect the yield of the investment, while invested in the bond.

If the problem be to invest $11,256.47 for ten years and to determine the yield on the full amount including the diminishing bond

principal half yearly amounts falling to be reinvested, we must then take into consideration the reinvestment rate, or as it is described in the *Text Book of The Institute of Actuaries, Part I,* the reproductive rate. The problem involved in amortization is, however, I believe, that of the determination of the yield under the investment while invested in the bond and not thereafter. If that be the case, the usual method of bond amortization seems the proper one to apply.

In the absence of statutory provision and where the bond investments are made at similar remunerative rates, Mr. Wolfe's suggestion of the use of one interest rate to be used in amortization could be applied at a considerable saving in labor. In fact, Mr. Joseph Burn, in his *Stock Exchange Investments,* suggests that all sinking funds be computed at some average rate of interest and that such average rate of interest be one which would result in a somewhat more stringent treatment, that is a more rapid reduction of principal, than would be shown by taking true rates for every individual investment.

When a bond is purchased below par, it would seem that the principal invested is as a matter of fact being accumulated at the rate of interest yielded by the bond investment, so that the question of the average rate yielded by a company on its investments would not enter.

We are indebted to Mr. Wolfe for a discussion of the application of a sinking fund method to the amortization of bond premium. Beyond question, I should agree with his method were the problem thereof determining the yearly interest yield on an invested fund to be held intact over a given period.

The Society should be very gratified to have printed in its *Proceedings* Mr. Wolfe's able discussion of the application of the amortization plan to "miscellaneous" companies. Where the catastrophe element is properly provided for, the investments of a "miscellaneous" company are in the nature of fixed assets to be held until maturity just as in the case of a life insurance company. A method of relieving such companies from the burdens incident to the taking into account of market value fluctuations is equitable and fair. The use of average market values spread over a period is not desirable because of fluctuations therein and because the method of valuation is one not intrinsically connected with the bond investment.

Even in cases of reinsurance, as pointed out by Mr. Wolfe, the company assuming the risks should be permitted to carry the assets of the ceding company at the investment value and should not be required to use market values as of the date of reinsurance. Market values would depend upon variations in the market altogether foreign to the intrinsic quality of the investment.

Oral Discussion.

Mr. Albert H. Mowbray: Mr. Chairman, one point has occurred to me. Mr. Wolfe proposes, as I understand it, to value the bonds at a uniform rate to be determined from a study of recent earnings. He says on page 7: "The plan is feasible if we can assign a proper value to i_x." Then he says the company's average interest rate doesn't remain level, but, he says, "However, by taking the average interest rate earned during the past five years, conservatively adjusted with an arbitrary factor to allow for the rising or falling tendency, we can obtain i_x sufficiently correct for all practical purposes." That, it seems to me, may be very well "for all practical purposes," but when you have passed beyond a certain period, when you have come to estimate what the company's average earnings will be, you necessarily take account of this valuation of these bonds. It seems to me this method will by being carried forward tend to carry forward a false interest rate developed from itself, and that your actual interest earnings may rise or fall considerably according to the purchase of new bonds and yet the showing be considerably distorted by the fact of an amortization resting upon an interest earning assumed from a past earning.

Turning to the second part of the paper: He divides the multiple line companies into two groups, those that are subject to the catastrophe hazard and those that are not, and of course he classes workmen's compensation properly in with the group that is subject to catastrophe hazard, and yet it does not seem to me necessarily that that precludes that group of companies from consideration as to the amortization principle. For, at least in workmen's compensation, and I think in some other lines, you may have a catastrophe loss which does not have to be paid immediately but by the operation of the compensation law is spread over a considerable period of time, which would give a company considerable leeway in handling its investments and would not necessarily require an immediate sale of a considerable volume of them. So that it does not necessarily follow that a catastrophe loss of that kind would get a company in trouble when it was using the amortization principle in valuation.

Mr. Benedict D. Flynn: There are objections to the use of Mr. Wolfe's method of obtaining amortized values of bonds which it seems to me outweigh the advantages pointed out by him. Before discussing them, however, I would like to emphasize the fact that the effect of Mr. Wolfe's method is simply to modify the amount of the interest payment under the bond which goes to interest on the one hand and to amortization of premium of the bond or accrual of discount on the other. For instance, a premium bond the effective rate for which is adjusted upward to reflect current interest earnings will show higher amortized values and larger interest return than if the amortized values for the bond which were established at date of purchase are adhered to, for the reason that

a smaller part of the interest payment will go to repayment of principal and a larger amount to interest.

It appeals to me that if a company adopts a plan of amortization of its bonds, the plan should give a set of amortized values which would not be susceptible to change. Mr. Wolfe's plan of modifying the effective rate to agree with the average interest rate earned during a preceding short period appears to me to give too much leeway to the company in the application of the amortization plan. One can easily imagine instances where the total amortized values of a company's bonds would be materially changed, possibly upward, if the change in interest rates were important. The possibility of a rapidly growing company, or a company with a distinctive class of bonds, experiencing a material change in the total value of its assets if the effective rate of interest were changed, say, from four to five per cent., constitutes a serious objection to the use of the plan.

Another objection is based upon the fact that an insurance company which carries the amortized value as the book value of its bond, makes the assumption when it purchases its bond that the purchase price with the resulting effective interest rate gives a certain status to the bond and that so long as interest payments are made under the bond when due, the company can consider the amortized value from year to year upon the effective rate of interest as the book value in its valuation of assets. The effective rate of interest under the bond reflects the rating of the character of the security given to it in financial circles at the time of purchase. The insurance company, in order to eliminate the effect upon the value of its assets which would result from widely fluctuating market quotations year after year, accepts this rating and adopts a set of amortized values which shall be used as book values until maturity of the bond. It seems to me that any modification of amortized values during the period of the bond is a step in the wrong direction and weakens the position of the insurance company in its original assumption.

I would like to endorse strongly Mr. Wolfe's plea for the privilege of amortization of bonds by the miscellaneous casualty insurance companies. The two possible dangers which might require forced liquidation of assets are so remote, as pointed out by Mr. Wolfe, that with proper supervision the amortization of bonds could be permitted to the miscellaneous company with just as great safety and just as great usefulness as to the life insurance company. The outgo of funds in a casualty insurance company does not fluctuate to any material extent. Possible catastrophes are provided for by proper reinsurance arrangements. Further, the character of the business of certain casualty lines, such as workmen's compensation, involves in claim payments long term contingencies—life annuities which may run thirty or forty or more years. It is my firm belief that the bonds which are of the proper grade to be included in an amortization plan should be valued upon such a basis in every casualty insurance company.

SOME PRINCIPLES OF COMPENSATION MERIT RATING—E. H. DOWNEY.

VOL. III, PAGE 26.

ORAL DISCUSSION.

MR. S. BRUCE BLACK: Mr. President, I think the majority of those who would discuss this paper have been very busy since the first of the year and have not had very much opportunity to give it proper study.

Two objects of schedule rating, as Mr. Downey has brought out, are, first, to induce measures being taken to prevent accidents, and, second, to measure the differences in hazards between individual risks in the same rate class. In the past, schedule-rating plans have been constructed with very little regard to actually measuring the loss producing characteristics or particular conditions of the risks insured. The persons who have constructed the schedule-rating plans have been engineers chiefly who have had more knowledge of the cost of correcting an unsafe condition in the plant than they have had of the loss-producing ability of such a condition of the plant. The result is that schedule-rating plans have tended to decrease materially the premium income collected from the total business.

The "Actuarial Sub-Committee" of the recent rate conference found it necessary to put a loading of 9 per cent. into the basis rates to take care of the probable reduction in the premium income that would result from the use of the Industrial Compensation Rating Schedule, for an insurance company cannot give anything as an inducement to accident prevention that it does not collect. There is a question as to whether it is proper for any schedule-rating plan to produce such a premium decrease which must be made up by an increase in the basis rate itself. We must be sure, in assuming such a public function as the collection of a fund by a tax upon all industry to be redistributed as inducements to accident prevention, that we are properly assessing that cost.

In the present compensation rates, we make a flat assessment of 9 per cent. on all classifications subject to schedule rating. A majority of schedule-rating plans do not collect as charges because of unsafe conditions as much money as they give out as credits or reductions to those who have better than ordinary conditions. A flat assessment ignores completely the particular characteristics of a particular class of industry. The schedule-rating plan may produce neither charge nor credit on a certain broad group of industries, while on another broad group it may give a reduction of

20 per cent. It is of questionable propriety to assess an average of 9 per cent. over all classes because that happens to be the average reduction resulting from the use of the schedule. If we are to continue the use of such a schedule-rating plan which requires the collection of a loading in the base rate itself, we must make our rate determination a little more complicated and adjust our base rate to the particular effect upon the particular class of industry of the use of such a schedule.

An alternative is the use of a schedule-rating system that will adjust itself to the differences in industries and have the same effect upon all classes.

Mr. Downey, using to a certain extent the principle upon which is based the Coal Mine Schedule used by the Associated Companies, proposes a system that will adjust the item values in the rating schedule to different classes of industry. He would also base the values in the schedule entirely upon accumulated accident statistics of cost. It is true that such statistics do not exist or rather, have not been collected, but before a schedule-rating plan can be constructed that will both measure deviations from average hazard and offer all inducement to accident prevention which can be given by the money that can be collected in a plan that can collect as much as it gives out, there must be a statistical basis. As Mr. Downey brings out, with appropriate emphasis, a schedule-rating plan must be based upon statistics, and must be subject to statistical control.

NOTE ON AN APPLICATION OF BAYES' RULE IN THE CLASSIFICATION OF HAZARDS IN EXPERIENCE RATING—ARNE FISHER.

VOL. III, PAGE 43.

ORAL DISCUSSION.

MR. EDWIN W. KOPF: Mr. President, I wonder if our statistical members appreciate the significance of one sentence in the concluding paragraph of Mr. Fisher's article?

"It is up to the statisticians to take a more conciliatory attitude towards the introduction of mathematical methods in statistics instead of taking a suspicious, if not actually acrimonious and ignoring attitude towards the lonely little band of students who attempt to reach a mutual understanding with the mathematicians."

I had occasion to discuss Mr. Fisher's work with a number of teaching statisticians recently. I found that they had only read Professor Edwin Bidwell Wilson's review* of Mr. Fisher's initial volume on the mathematical theory of probabilities, and frankly admitted not having read the book itself. They acknowledged, also, that they had not made themselves acquainted with the newer conceptions of statistical analytics advanced by the Scandinavians, the German school under Lexis, and by the English biometricians. This lack of ready acquaintance with the really "practical" aids to statistical analysis and to the testing of the reliability of statistical conclusions, is deplorable in teachers of statistics. It may partly be ascribed to the absence of a suitable text for purposes of statistical instruction in our American schools and colleges.

Throughout previous discussions of Mr. Fisher's articles there has been an unfortunate characterization of statistics as "merely practical" on the one hand, and as "merely theoretical" on the other. It is about time we ceased to conveniently, but falsely, distinguish between these two alleged natural classes of statistical subject matter. We have recently had an illustrious example of the intense practicability of the "merely theoretical" in surgical statistics, a branch of that larger subject of general medical statistics in which casualty statisticians should strive to become proficient.

The eminent French surgeon, Dr. Alexis Carrel, in association with Dr. Alice Hartmann, proceeded by the ordinary "practical" methods of direct observation and description to collect the necessary surgical facts for a statement of the "normal law of cicatri-

* Quarterly Publications of American Statistical Assn., December, 1916, p. 468.

zation." They concluded, among other things, that the rate of normal healing of a wound is greater at the beginning than at the end of the period of repair; that the rate of healing depends upon the area more than upon the age of the wound; and that there appeared to be a constant relation between the size of the wound and the rate of cicatrization. The method of simple description and observation is not in itself sufficient to establish a basis for the numerical expression of a "normal law of cicatrization." In other words, description alone (as far as the "practical" statisticians are in the habit of going) did not conclusively state in numerical terms the relation between the area of a wound, the rate of cicatrization and a constant characteristic of the case, say, the age of the patient.

Carrel turned over the *descriptive* data to the French mathematician and physicist, Pierre Lecomte Du Noüy, for *analytic* study and for the formulation of the "normal law of cicatrization." Du Noüy devised a simple extrapolation formula, which can be found fully explained in the *Journal of Experimental Medicine* for November, 1916. The French physicist maintained that the cicatrization of sterile wounds could be studied in precisely the same way as other physicochemical phenomena. The formula developed by Du Noüy calls for the calculation of an "index" for each case, from plotted curves representing the observed relation between the age of patients and the area of the wound in a test series, and the employment of this "index" in the computation of the theoretical area of the wound after a given lapse of time from the date of first observation. Thus, given an "index" of 0.0416 for a certain patient at a stated age, with a sterile wound of an area of 9 sq. cm., we can estimate that four days later the area of the wound will be 6.45 cm., and three days after that, 4.84 cm., etc.

The plotting of the ideal curve for the individual, suited to the age of the patient and the area of the wound, enables this mathematician at Hospital No. 21, Compiègne, France, to predict the probable date of discharge of a surgical patient. The "practical" surgeon, in arranging for the efficient movement of the sick population under his care, makes use, first, of *description* of a series of representative cases, and second, of *analytic* study of the cases and the statement of a general law covering the surgical phenomenon of normal healing of wounds.

As "practical" men, therefore, we ought to take a page out of the book of the French surgeons' experience and abolish the useless emphasis upon and distinction between the two older concepts of statistics as (*a*) theoretical and (*b*) practical. We ought to earnestly desire union between the two schools and establish a tentative working definition of statistics in the service of American casualty practice as: (1) *description* of masses or groups of persons, things and events, in terms of the mass or group, (2) mathematical and other *analysis* of statistical description, especially in

testing the reliability of conclusion based upon statistical results.

MR. ALBERT H. MOWBRAY: If many of our members have had the same experience I have, they should certainly be grateful to Mr. Fisher for presenting this paper, because I remember in my preparatory work for my examinations studying through inverse probabilities, as we call them, and finding them an extremely hard subject and one the practical value of which I found it very hard to estimate or gauge. I am therefore very glad to see some one put forward a practical application of some of the theories of inverse probabilities and one which does seem to have a practical value. With regard to the example Mr. Fisher cites, it seems to me that there is one considerable difficulty, in that he takes as his criterion, as I recall it, a loss of a certain size, without regard to what might be the cause or makeup of that particular loss. I assume that that is not subject to such very great criticism, because, as I understand it, this note is presented rather as an illustration of the theory than as an attempt to solve the particular problem that is used as the illustration. I am the more moved to make these remarks here from seeing a letter from a rather—well, I don't know just what I want to say, but at least from a safety engineer, who very evidently has not at all carefully read the paper and who referred to some men appearing in the Casualty Actuarial and Statistical Society and attempting to substitute an integral sign for a safeguard and some mathematical formulae for a safety engineer. I think the entire spirit of the paper has been misinterpreted by the engineers, and I think that if the engineers and some of us who are working on safety ratings would study some of the things that this paper has intended to bring to our minds, we might all profit considerably.

MR. JAMES D. CRAIG: The first point that appealed to me was the lack of any definition of Bayes's Rule. As Mr. Mowbray says, it is what we commonly call inverse probabilities. Mr. Fisher in his book, which I happen to have with me, states: "English writers have lost sight of the true Bayes's rule and substitute a false, or, to be more accurate, a special case of the exact rule in the different algebra books under the discussion of the so-called inverse probability." It seems, therefore, that what we commonly know as inverse probabilities is really a special case of Bayes's Rule.

Mr. Fisher might help us out more in his papers if he would give a little more detail; lead us along slower as it were. For instance, on the top of page 44 there are some integrations, but the functions used are not defined. Evidently (y) is the probability of an event's happening, and $(1-y)$ the probability of its not happening. The integration is between 0 and 1, for the reason that y being a probability cannot have a value exceeding 1. Again from the expression given, Mr. Fisher deduces the factorials on the top of the next page. These are a little hard to see, without some study, although they are not hard to deduce. Integrating

by parts, the second part will be an expression with the power of y decreased by 1 and of $(1-y)$ increased by 1, while the first part is a quantity which reduces to 0, when the limits are substituted. Continue integrating by parts, and each time the first part vanishes when the limits are applied; while in the second part the exponent of y decreases, and that of $(1-y)$ increases by unity each time. The coefficient of the second term receives an additional factor in both numerator and denominator, so that after m such integrations the y term becomes unity in the last integral. Before multiplying by the constant $\frac{\lfloor t}{\lfloor n \lfloor t-n}$ the numerator for the expression $P_{(t,\,n)}$ is therefore $\frac{\lfloor m+n \lfloor S-m+t-n}{\lfloor S+t+1}$. The denominator is $\frac{\lfloor n \lfloor S-m}{\lfloor S+1}$ giving a value for $P_{(t,\,n)}$ of

$$\frac{\lfloor t}{\lfloor n \lfloor t-n}\left[\frac{\lfloor m+n \quad \lfloor S-m+t-n \lfloor S+1}{\lfloor S+t+1 \lfloor m \lfloor S-m}\right].$$

Mr. Fisher evaluates this expression by Stirling's Law as given on page 101 of his book.

In the second example, on page 45, it might be of some advantage to express that probability and its answer in terms with which we are more familiar. The probability is: having 80 events occur out of a possible 100, what is the probability that exactly 3 events will occur out of 25? In this form we would probably recognize it, ascertain the value by extending the binomial $(\frac{8}{100}+\frac{92}{100})^{25}$, to where the exponent of $\frac{92}{100}$ is 3, the value of which term represents the answer. It works out to .185020. When the answer is comprehended, it is simple to realize that the table on page 46 is a list of the probabilities that the event y will occur exactly 0, one, two or three, etc., times.

This is not meant as any general discussion of the paper, but simply as a possible help to those who might wish to study the expressions.

MR. ARNE FISHER:

(AUTHOR'S REVIEW OF DISCUSSIONS.)

Mr. Craig is justified in saying that I have not given the detailed steps in the derivation of the various formulae. However, I referred to the discussion on page 101 in my work on probabilities where the detailed work is shown and did not consider it worth while to reprint this in the paper.

The variations in the second example may all be explained as due to sampling. This, however, does not eliminate a loss con-

siderably higher. You might indeed have a loss, due to a catastrophe and which might exceed the total payroll several times. In order to treat all those variations one had to go to a detailed study of the frequency curves of such losses. What I wanted to emphasize was that many of the losses might simply be due to pure sampling and not to a special cause or condition in a particular factory.

Bayes's Rule, dating from 1763, has been forgotten and only used in examination questions, which often plainly show that the examiners have no idea of the importance of the theorem. I have recently had some correspondence with Major Greenwood, of the Lister Institute of Preventive Medicine of London, wherein he informs me that he and Mr. Yule, whose admirable text-book on statistics you probably all know, have made some researches on the Theorem of Bayes, but due to the war stress, it would probably be some years before they would be able to publish the final work.

In regard to Mr. Kopf's remarks I feel thankful for what probably is the first encouragement I have had in my work on mathematical statistics in this country. Mathematical statistics is by many regarded as a brand new science. The fact is, however, that it by no means is a stripling baby, but a science with old and renowned ancestors. Its family tree dates back to a time when the word statistics was unknown, because no statistical data were collected. The "Ars Conjectandi" by James Bernoulli and the "Doctrine of Chances" by de Moivre were written before the American Revolution. These earliest works on probability can still be read to advantage. The greatest work on the subject is, however, Laplace's "Theorie des Probabilités," published at the beginning of the nineteenth century. You will therefore see that you have possessed a first class technique, as fine a set of tools as you could wish, for more than 100 years. Why have you not used the tools? The subject of frequency curves is treated in Laplace's book, and his treatment is, as Charlier has shown, the most general we possess. Correlation, of which we hear so much of late, was introduced by the Belgian astronomer, Bravais, in 1846, but it was never used until Sir Francis Galton applied it to statistical measurements in the early eighties.

After that the biometric school of England grew up under the very able leadership of Pearson. Pearson and his pupils do excellent work, but their attitude is often one-sided, and many things they consider as new discoveries are really old things treated in a different way by the old French classics. We Scandinavian statisticians have tried to bring the old French masters and the modern Englishmen together and view their researches from a common point of view. How far we have succeeded I leave to others to judge.

The reason that the English speaking world has lost sight of the wonderful mine of information contained in Laplace's book on

probabilities I think is due to the fact that it was written during the Napoleonic wars, in fact, the first volume was dedicated to Napoleon, and of course you know that the English did not have any great love for anything connected with Napoleon round 1800–1814, and consequently they did not read much of Laplace. I hope earnestly that one result of the present war will be that English speaking statisticians, both in England and America, will turn their eyes toward that old volume of Laplace and will study it, and I can assure you that you will derive much benefit from it.