The Theory of Experience Rating.

BY

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This paper traces in an informal way the general line of reasoning that was pursued in an investigation into the theory of experience rating which was made recently by the Actuarial Section of the National Reference Committee on Workmen's Compensation Insurance. This investigation resulted in the adoption by the Section of a general plan which was approved by the National Reference Committee and is now before the various Bureaus for such action as each may see fit to take.

The problem of experience rating is peculiar to workmen's compensation insurance and a few other types of insurance. The problem is not found in life insurance, except potentially in group insurance, and not at all in fire insurance so far as I know.

The problem exists only in those forms of insurance in which there is a risk-experience as distinguished from a class-experience. In the case of life insurance death occurs but once and in the case of fire insurance likewise the occurrence of a fire is so rare that the experience of the risk is of little evidential value in itself. In these cases therefore it is perforce necessary to associate the risk with other similar risks to form a class and the hazard of the risk must be identified with the hazard of the class.

In workmen's compensation insurance, some kinds of liability insurance, group insurance and possibly a few other types of insurance, the risk insured, and upon which a rate must be produced, affords an experience of its own, that is, the contingencies insured against are of sufficiently frequent occurrence so that the risk itself produces an experience having some evidential value. In such cases we have therefore both a class-experience and a risk-experience.

The problem of experience rating arises out of the necessity, from the standpoint of equity to the individual risk, of striking a balance between class-experience on the one hand and risk-experience on the other.

Here is a risk, for instance, that is clearly to be classified as a machine shop. In the absence of other information it should therefore take the machine shop rate, namely, the average rate for all risks of this class. On the other hand the risk has had an experience of its own. If the risk is large, this may be a better guide to its hazard than the class-experience.

In any event, whether the risk is large or small, both of these elements have their value as evidence, and both must be taken into account. The difficulty arises from the fact that in general the evidence is contradictory; the problem therefore is to find and apply a criterion which will give each its proper weight.

Before proceeding to make a mathematical analysis of the situation, in fact before attempting to set up a criterion for striking a balance, an enumeration may be made of the elements which will figure in the result, with an intuitive estimate of their general effect.

It is evident in the first place that the weight of the risk-experience will depend upon the risk-exposure. Other things being equal, the experience of that risk which has the larger exposure will be entitled to the larger degree of consideration. In the case of a very large risk the rate may with safety be based almost wholly upon its own experience; in the case of a small risk very little credence can be given to risk-experience and the rate must be based almost wholly upon the experience of the class.

Essentially the same relationship holds true in the case of the hazard; the larger the hazard, the larger will be the number of accidents, the exposure remaining the same, and therefore the more trustworthy the average. If, however, the varying credibility of the class-experience is taken into account, since a large hazard will affect this in approximately the same way that it affects the risk-experience, it will be difficult to say what the net effect on the balance will be.

There would be no experience-rating problem if every risk within the class were typical of the class, for in that case the diversity in the experience would be purely adventitious. The problem arises out of the necessity of assessing the degree to which the disparity between risk-experience and class-experience reflects a real divergence between the true risk-hazard and the average hazard of the class rather than mere chance. It is therefore necessary in discussing this problem to have some measure of the degree of dispersion of risks within the class, that is, the degree to which the true hazard of the various risks differs from the average hazard of the class.

Now this is strictly a matter for statistical treatment. Doubtless the risks in each classification do group themselves as to their true hazard about the average hazard of the class in some particular way that is expressible by means of some particular frequency curve. While it would be interesting in a certain number of cases to make an investigation into the actual facts, it is evident that as a practical matter for rating purposes, such a procedure for each classification would be utterly out of the question. We are therefore forced to make some assumption with regard to the law of frequency of risks of various degrees of hazard.

From a general knowledge of conditions we are safe in assuming that this law as a first approximation may be taken to be of the normal type. There will doubtless be some skewness, but since the investigation that we are to conduct is primarily for the purpose of ascertaining the proper rating structure rather than quantitative values, this assumption is under the circumstances justifiable. The standard deviation may be taken as the measure of dispersion.

Now it is evident intuitively that if the risks are concentrated within the class, that is, if the standard deviation is small, a risk-experience that departs from the average of the class can be more easily accounted for as due to chance than as due to an inherent difference in the degree of hazard. On the other hand, if the standard deviation is large, that is if the risks are diverse, it is inherently likely that a risk-experience that departs from the average is to be accounted for by a real difference in the hazard.

Another element that in theory may be taken account of is the varying credibility of the manual rate. The manual rate is established upon experience which in a majority of classifications is insufficient and which in many cases has been supplemented by judgment. It is evident that, other things being equal, the higher the credibility of the manual rate, the greater its weight in establishing the balance between class-experience and risk-experience. If, on the other hand, the manual rate is established upon insufficient experience, we shall be inclined to give greater relative credence to the risk-experience.

To summarize: the balance between class-experience and risk-experience will depend upon four elements, the exposure, the hazard, the degree of concentration within the class and the credibility of the manual rate. The larger the risk-exposure, the greater the credibility of the risk-experience, while the greater the concentration of risks within the class and the greater the credibility of the manual rate, the greater the credibility of the class-experience; an increased hazard makes both class-experience and risk-experience more trustworthy so that the net effect is not intuitively obvious.

The detailed solution of this problem depends upon the use of inverse probabilities and as the expressions involved are somewhat complicated, it will be convenient to use for this purpose symbols and an analysis adapted from the algebra of logic.

A, in a symbolic sense, may be taken to mean the happening of the event A; $A+B$ means the happening of A or B, logical addition being interpreted as "or"; AB means the happening of both A and B, logical multiplication being interpreted as "and"; AB/A means the happening of B (and therefore A) if A happens, logical division being interpreted as "if." $\bar{A}$ means not A.

$A\cdot(AB/A)=AB$, or the happening of A and the happening of B if A happens is equivalent to the happening of both A and B.

The probability of the happening of A may be denoted by $|A|$. $|A+B|=|A|+|B|$, provided A and B are completely disjunctive; in any case $|A+B|=|A|+|\bar{A}B|$ or $|A\bar{B}|+|B|$. In the expressions on the right the operation of addition is quantitative not logical, and in general the operations within the sign $|\quad|$ are logical while the operations without are quantitative. There is a relationship between the logical and quantitative operations such that in taking the probability of a logical expression, under certain restrictions, logical relations pass over into the corresponding quantitative relations.

$|A\cdot(AB/A)|=|A|\,|(AB/A)|$, from which it follows that $|AB/A)|=|AB|/|A|$. $|AB|$ is however in general not equal to $|A|\,|B|$.

Suppose the following:

P is the hazard of the class as shown by the class-experience, that is, P is the indicated hazard of the class, (known); X is the real hazard of the class, (unknown); p is the indicated hazard of the risk, (known); x is the real hazard of the risk, (unknown). As a logical symbol P will be used to mean the occurrence of an indicated class hazard equal to P* and similarly for X, p and x.

The first problem is to find $|Ppx/Pp|$, that is $|Ppx|/|Pp|$, that is, to find the probability that x is the real hazard of the risk if P is the indicated hazard of the class and p the indicated hazard of the risk.

Now $Ppx = \Sigma_X XPpx$, the sign of summation here indicating that the expression $XPpx$ is to be summed for all X's.

Therefore

$$Ppx = \sum_X X \cdot \frac{XP}{X} \cdot \frac{XPx}{XP} \cdot \frac{XPxp}{XPx}$$

and

$$|Ppx| = \sum_X |X| \left|\frac{XP}{X}\right| \left|\frac{XPx}{XP}\right| \left|\frac{XPxp}{XPx}\right|. \qquad (1)$$

These factors may be discussed seriatim:

$|X|$ is an a priori value, that is, none of the known facts, either explicit or implied, are admitted as evidence; from this point of view one value of the real hazard of the class will be as probable as another. $|X|$ may therefore be taken to be a constant c independent of the quantities P and p.

$|XP/X|$ is the probability that P will be the class-hazard indicated by experience if X is the real class-hazard. For our purposes we may suppose the contingency to be a simple one such as death. Suppose there are m persons exposed to such a hazard whose value is X. Then $|XP/X|$ may be described as the probability that, of these m persons, mP will experience the contingency in question. This probability is the $(mP+1)$th term in the expansion $[(1-X)+X]^m$ or

$$_mC_{mP}(1-X)^{mQ}X^{mP}, \quad \text{where} \quad P+Q=1.$$

* There is the possibility here of confusion, since each of these symbols is used in three senses, for instance, P is first used quantitatively, namely as the indicated value of the class-hazard, second in a logical sense as the occurrence of P as the indicated value of the class-hazard and third quantitatively in the form $|P|$ as the probability of the occurrence of P as the indicated value of the class-hazard. The context should, however, make clear which is meant.

This can be represented approximately by

$$\frac{H'}{\sqrt{\pi}} e^{-H'^2(P-X)^2} \quad \text{where} \quad H'^2 = \frac{m^*}{2X(1-X)}.$$

$|XPx/XP|$ is in reality independent of P and is therefore the same as $|Xx/X|$. This, the probability of occurrence of a risk with real hazard x within the class whose real hazard is X, is dependent upon the law of frequency of distribution of risks within the class. If we assume that this law of frequency is normal with a modulus H then

$$\left|\frac{Xx}{X}\right| = \frac{H}{\sqrt{\pi}} e^{-H^2(x-X)^2}.$$

$|XPxp/XPx|$ is independent of both X and P, that is, it is the same as $|xp/x|$, that is, it is the same as the probability of occurrence of an indicated risk-hazard p if the real risk-hazard is x. If we suppose the number of persons exposed to the hazard whose value is x is n, the value of $|xp/x|$ will be the $(np+1)$th term in the expansion $[(1-x)+x]^n$ or ${}_nC_{pn}(1-x)^{qn}x^{pn}$ where $p+q=1$. This can be written approximately $(h/\sqrt{\pi})e^{-h^2(p-x)^2}$ if we choose, where $h^2=n/[2x(1-x)]$. Collecting these factors together we have

$$|Ppx| = \sum_X c\frac{H'}{\sqrt{\pi}} e^{-H'^2(P-X)^2} \cdot \frac{H}{\sqrt{\pi}} e^{-H^2(x-X)^2} \cdot {}_nC_{pn}(1-x)^{qn}x^{pn}. \quad (2)$$

We now have to consider the quantity

$$\sum_X \frac{HH'}{\pi} e^{-[H'^2(P-X)^2+H^2(x-X)^2]}. \quad (3)$$

This can be written

$$\sum_X \frac{HH'}{\pi} e^{-(H'^2+H^2)\left[X-\frac{PH'^2+xH^2}{H'^2+H^2}\right]^2} e^{-\frac{H'^2H^2}{H'^2+H^2}(x-P)^2}. \quad (4)$$

H'^2 in reality is equal to $m/[2X(1-X)]$; it can however without serious error, since the significant values of X are in the vicinity of P, be written $m/[2P(1-P)]$. If this is done the expression above can be written

* This differs from the more familiar expression because of the fact that X and P represent *ratios* of occurrence instead of *number* of occurrences.

$$\frac{\sqrt{\frac{H^2H'^2}{H'^2+H^2}}}{\sqrt{\pi}}e^{-\frac{H'^2H^2}{H'^2+H^2}(x-P)^2}$$

$$\times\sum_X \frac{\sqrt{H'^2+H^2}}{\sqrt{\pi}}e^{-(H'^2+H^2)\left[X-\frac{PH'^2+xH^2}{H'^2+H^2}\right]^2}. \quad (5)$$

Since X is to be taken as a continuous variable, the sum in expression (5) becomes an integral, namely,

$$\int_{-\infty}^{+\infty}\frac{\sqrt{H'^2+H^2}}{\sqrt{\pi}}e^{-(H'^2+H^2)\left[X-\frac{PH'^2+xH^2}{H'^2+H^2}\right]^2}dX \quad (6)$$

and the value of this is 1.

The value of $|Ppx|$ is therefore

$$\frac{\sqrt{\frac{H^2H'^2}{H^2+H'^2}}}{\sqrt{\pi}}e^{-\frac{H^2H'^2}{H^2+H'^2}(x-P)^2}{}_nC_{pn}(1-x)^{qn}x^{pn}. \quad (7)$$

The denominator of $|Ppx/Pp|$, viz., $|Pp|$, is the same as the numerator except that it is to be summed for all values of x. It will therefore be a function of P, p, n, m, and H^2, namely a constant independent of x.

Finally therefore the value of $|Ppx/Pp|$ will be

$$Ce^{-\frac{H^2H'^2}{H^2+H'^2}(x-P)^2}(1-x)^{qn}x^{pn} \quad (8)$$

all the constants being combined into one.

In the first working out of this problem the assumption was made that the indicated class-hazard could be taken as the real class-hazard, and in the practical application of an experience rating plan this is doubtless the only feasible procedure. The process and results under this hypothesis are simpler. P can then be taken as X and no integration with regard to X is necessary.

$|Ppx/Pp|$ in that case is

$$C'e^{-H^2(x-P)^2}(1-x)^{qn}x^{pn}, \quad (9)$$

that is $H^2H'^2/(H^2+H'^2)$ reduces to H^2. This is evident directly: X and P will approach equality as the experience increases; but as m approaches ∞, H'^2 approaches ∞, and $H^2H'^2/(H^2+H'^2)$ approaches H^2 as a limit.

Mr. W. W. Greene, chairman of the Actuarial Section, proposed as an alternative treatment the assumption that all the risks in the class are homogeneous, that is, $H^2 = \infty$, and that the balance between class-experience and risk-experience be made solely on the basis of the relative credibility of class-experience and risk-experience. Under this assumption X would be the same as x, that is, the hazard of the class and the hazard of the risk would be equal. This assumption would yield the result:

$$\left|\frac{Ppx}{Pp}\right| = C''e^{-H'^2(P-x)^2}(1-x)^{qn}x^{pn}, \qquad (10)$$

that is, $H^2H'^2/(H^2+H'^2)$ would reduce to H' . This also follows directly by letting H^2 approach ∞. All three of these results are evidently of the same general form.

Let us now revert to the more general formula (8). This expresses the probability that x is the real value of the risk-hazard; this is a function of the known quantities P, p, m, n and H^2.

What criterion shall now be made use of in selecting the value of x to be used, the object namely of our investigation? The value of x that we instinctively choose is that one whose probability of occurrence is greatest, and this upon analysis means that value of x which would have made the thing which has actually occurred the most a priori probable. As Mr. A. H. Mowbray has pointed out, however, this involves a subtle repudiation of the fundamental thesis of insurance, viz., a dependence upon the law of averages.

The fundamental theory of insurance involves this, that, at the point when the effort to analyze and differentiate the hazard of various risks has been carried as far as is deemed feasible, the risks in each residuum shall be treated as of equal hazard. This means therefore that each risk shall take the average hazard of the group.

Suppose we had a large number of cases in which we knew the indicated class-hazard to be P and the indicated risk-hazard to be p. The real risk-hazard would doubtless vary from case to case, yet we should have nothing by which to distinguish one case from another and so we should be obliged to take for each the average hazard of the group. This can be done in our theoretical treatment by affecting each value of x with its corresponding frequency factor $|Ppx/Pp|$ and averaging the result; that is we should properly take the mean value of x and not the most probable value of x.

As a practical matter, however, it is expedient to use the most probable value rather than the mean. If instead of ${}_nC_{pn}(1-x)^{qn}x^{pn}$ we use the approximate function $(h/\sqrt{\pi})e^{-h^2(p-x)^2}$ and for h^2 take $n/[2P(1-P)]$, which however will be only approximately correct, we shall have for $|Ppx/Pp|$ a strictly normal and therefore symmetrical function in which the mean and the mode will agree.

In any case the discrepancy between the mean and the mode will probably be small, and not worth considering for the prime purpose of this investigation, namely, the discovery of a structure for an experience rating plan. The determination of mean values would be attended by mathematical difficulties whereas the determination of the mode is comparatively simple.

Our problem therefore is to find that value of x which will make

$$Ce^{-\frac{H^2H'^2}{H^2+H'^2}(x-P)^2}(1-x)^{qn}x^{pn}$$

a maximum. Taking the logarithm, differentiating and equating to zero, and for convenience abbreviating

$$\frac{H^2H'^2}{H^2+H'^2} \quad \text{by } J^2,$$

we have the condition for a maximum:

$$-2J^2(x-P)-\frac{qn}{1-x}+\frac{pn}{x}=0 \qquad (11)$$

which reduces to the cubic:

$$x^3-(1+P)x^2+\left(P-\frac{n}{2J^2}\right)x+\frac{n}{2J^2}p=0,$$

or by letting $n/2J^2=A$:

$$x^3-(1+P)x^2+(P-A)x+Ap=0. \qquad (12)$$

A further insight into the existence of a maximum may be had by considering the parts of (11) separately, viz., $-2J^2(x-P)$ and $n(p-x)/x(1-x)$.

When $p>P$, the first is 0 for $x=P$ and negative for $x=p$; the second is positive for $x=P$ and 0 for $x=p$. An analogous condition holds when $p<P$. Somewhere between P and p the sum of these two expressions will therefore be zero. Furthermore

it is evident from these considerations that

$$-2J^2(x-P)+\frac{n(p-x)}{x(1-x)}$$

is a decreasing function between P and p and that the solution of the cubic therefore determines a maximum. x is the adjusted value of the hazard; for our purposes, however, a more fundamental quantity will be $(x-P)/(p-P)$, namely the percentage of the deviation of the indicated risk-hazard from the indicated class-hazard which is allowed upon adjustment; let this be called z and let $p-P$ be called λ. Then $x-P=\lambda z$. Making these substitutions in (12), when thrown into the form $(x-P)(x^2-x-A)+A(p-P)=0$, we have

$$(\lambda z)^3+(2P-1)(\lambda z)^2-(A+P(1-P))\lambda z+A\lambda=0. \quad (13)$$

It is impracticable and unnecessary to consider the exact solution of this cubic; the practical problem is to find a satisfactory approximate solution.

The expression on the left of (13) which we may call y may be written

$$y=(x-P)^3+(2P-1)(x-P)^2 \\ -(A+P(1-P))(x-P)+A(p-P). \quad (14)$$

This is a cubic curve; its point of intersection with the x axis between P and p is the point in which we are interested. By dropping the first term on the right we obtain

$$y=(2P-1)(x-P)^2 \\ -(A+P(1-P))(x-P)+A(p-P). \quad (15)$$

This is the equation of a parabola osculating the cubic at the point whose x is P and therefore giving good approximate results for values of x that are in the vicinity of the indicated class-hazard. If we drop the first two terms we have

$$y=-(A+P(1-P))(x-P)+A(p-P). \quad (16)$$

This is the equation of the tangent to both the cubic and the parabola at the point whose x is P. This may be used for obtaining a first approximation to the solution of the cubic while the quadratic form may be used if a closer approximation becomes necessary.

As a matter of reference we may set down explicitly the values of z got by setting equations (15) and (16) equal to zero and solving.

From equation (15) we have:

$$z = \frac{A + P(1-P) - \sqrt{(A + P(1-P))^2 - 4A\lambda(2P-1)}}{2\lambda(2P-1)}, \qquad (15A)$$

which we may call the second approximation.

From equation (16) we have:

$$z = \frac{A}{A + P(1-P)}, \qquad (16A)$$

which we may call the first approximation.

When $p = P$, by equation (12) $x = P$; $\therefore$ z is indeterminate. z has the limiting value however from equation (13) of $A/[A + P \times (1 - P)]$. This is the same value that is given by the linear equation $-(A + P(1-P))\lambda z + A\lambda = 0$ for all values of p. That is, the first approximation to the value of z is independent of p and is the same as the value given by the cubic equation in the limiting case in which the indicated risk-hazard is the same as the indicated class-hazard.

The same result can be arrived at in another way. If instead of ${}_nC_{pn}(1-x)^{qn}x^{pn}$ we use the approximate value $(h/\sqrt{\pi})e^{-h^2(p-x)^2}$ where we take $h^2 = n/[2P(1-P)]$, equation (8) takes the form:

$$\left|\frac{Ppx}{Pp}\right| = Ce^{-[J^2(P-x)^2 + h^2(p-x)^2]}. \qquad (17)$$

Differentiating, equating to zero and solving gives

$$x = \frac{J^2P + h^2p}{J^2 + h^2}, \qquad (18)$$

or in terms of z,

$$z = \frac{h^2}{J^2 + h^2}. \qquad (19)$$

By letting $h^2 = n/[2P(1-P)]$, and $A = n/2J^2$ we have $z = A/[A + P(1-P)]$, as before.

That is, using an approximate value for ${}_nC_{pn}(1-x)^{qn}x^{pn}$ and letting $h^2 = n/[2P(1-P)]$ instead of $n/[2x(1-x)]$ gives the same result as using the first approximation by the more rigorous method.

Incidentally a further curious result may be observed. If in (18) we substitute for h^2 its more accurate value $n/[2x(1-x)]$, equation (18) reduces to our original cubic in x. That is, the effect of considering h^2 a constant in (17) when differentiating for a maximum is apparently just balanced by the error in using for ${}_nC_{pn}(1-x)^{qn}x^{pn}$ the less exact value $(h/\sqrt{\pi})e^{-h^2(x-p)^2}$. These equations have other curious mathematical properties which however it is not necessary, for the purpose in hand, to develop.

Equation (18) has an interesting dynamical interpretation. If the points p and P are weighted in the proportion of J^2 to h^2, then x, on the straight line joining P and p, is the center of gravity. We undertook in a figurative way to balance the risk-experience against class-experience; we now see in a literal way just what that balance is. It will be interesting to check it up against our intuitive estimate.

If in $z=h^2/(h^2+J^2)$ we replace J^2 with $H^2H'^2/(H^2+H'^2)$, and put $H'^2=m/[2P(1-P)]$, and $h^2=n/[2P(1-P)]$, we have

$$z=\frac{1}{1+\frac{1}{n}\left[\frac{1}{\frac{1}{m}+\frac{1}{2H^2P(1-P)}}\right]}. \tag{20}$$

From this it is evident that z increases with an increase in n, and that it decreases with an increase in m and with an increase in H^2. This agrees with our intuitive estimate. The situation as regards P is, as we surmised, complicated, particularly by the fact that H^2 itself is a function of P. Under the assumption that has been adopted by the Actuarial Section regarding the relation between H^2 and P, which will be explained later, an increase in P will produce an increase in z.

We may now turn our attention to the question of a practical method of producing a system of z's.

We have seen that if instead of ${}_nC_{pn}(1-x)^{qn}x^{pn}$ we use $(h/\sqrt{\pi})e^{-h^2(x-p)^2}$, where $h^2=n/2P(1-P)$, we obtain the relation: $z=h^2/(h^2+J^2)$ or $z=A/[A+P(1-P)]$, the same result given by the first approximation. This may be investigated in another way, namely, by throwing (17) into the form:

$$\left|\frac{Ppx}{Pp}\right| = C_1\sqrt{\frac{J^2+h^2}{\pi}}\,e^{(J^2+h^2)^{*}\left(x-\frac{J^2P+h^2p}{J^2+h^2}\right)^2}. \tag{21}$$

This is evidently a normal curve with its mode (and mean) at $x=(J^2P+h^2p)/(J^2+h^2)$. In the case, therefore, of a normal curve, in which the mode and the mean agree, the value of z will be independent of p. The fact that the second approximation, and the cubic itself, gives a value of z that is a function of p is evidently a consequence of the skewness of the frequency curve for x when ${}_nC_{pn}(1-x)^{qn}x^{pn}$ is used instead of $(h/\sqrt{\pi})e^{-h^2(x-p)^2}$.

When z is independent of p the question of the balance of the adjusted rates is not involved, as Mr. J. H. Woodward has pointed out. This may be explained in the following way: the risks belonging to a class with a given P, H and m, having a given n, may be thought of as constituting an array. But the distribution of risks in this array as to their indicated hazard will in theory be symmetrical with regard to P. Any basis for an adjustment of rates which is independent of p (or which is an even function of $p-P$) will leave the symmetry of the distribution about P undisturbed.

The second approximation, and the cubic itself, produces values of z that are greater for $p<P$ than for $p>P$, that is, it gives greater credits than debits. There are evidently curious questions involved here, depending partly upon the fact of skewness and partly upon the fact that the mode was used instead of the mean. As a practical matter it seems unnecessary to pursue these questions further because of the satisfactory character of the results produced by the first approximation, its very much greater simplicity and the fact that its use does not affect the balance.

We may therefore turn to the question of a practical treatment of the formula $z=h^2/(h^2+J^2)$ or $z=A/[A+P(1-P)]$, where $h^2=n/[2P(1-P)]$, $J^2=(H^2H'^2)/(H^2+H'^2)$, $H'^2=m/[2P(1-P)]$ and $A=n/2J^2$. $z=h^2/(h^2+J^2)$ may be written

* Incidentally attention may be called to the fact that this equation also indicates the probable error of x, namely $.67\,\frac{1}{\sqrt{2(J^2+h^2)}}$

$$z=\frac{n}{n+\dfrac{m}{1+\dfrac{m}{2H^2P(1-P)}}}\qquad\text{or}\quad z=\frac{Pn}{Pn+\dfrac{Pm}{1+\dfrac{Pm}{2H^2P^2(1-P)}}}.\tag{22}$$

Pn is the expected number of persons to suffer the contingency in question as shown by the risk-experience. Pm is the same for the class-experience; for a given classification Pm may be taken as a constant. Consider now the quantity $2H^2P^2(1-P)$. $2H^2=1/\epsilon^2$ where ϵ is the standard deviation; making the substitution we have $2H^2P^2(1-P)=P^2(1-P)/\epsilon^2$. We now come to the most difficult question of all, the determination of ϵ^2. It is obviously impossible as a practical matter to determine ϵ^2 statistically in each case. Some general assumption must therefore be made regarding its form and numerical value. In this we must be guided partly by general reasoning and partly by testing the results produced under various assumptions by an appeal to underwriting judgment. It is obvious in the first place that ϵ varies in some way with P; when the average hazard of the class is large, the variation in hazard among the risks of the class will be large, other things being equal. This is not to say that this is so in all cases but as a general proposition the statement is unquestionable. Trials were made with various laws for ϵ^2. The best results over the whole range of values of P were produced by allowing ϵ^2 to vary directly as $P^{\frac{5}{4}}$, and extensive tables were figured out on this basis. The formula is however complicated and not adapted to use without tables.

Mr. Greene made the suggestion that in equation (22) the second term of the denominator be taken as a constant. We have already remarked that Pm is constant for a given classification; there is no reason however to suppose that as the hazard increases the exposure decreases as would be the case if Pm were constant for all values of P.

This brings us to the question of whether it is desirable in actual practice to admit the varying credibility of the class-experience and hence of the manual rate. We know that the manual rates for some classifications are more reliable than for others and yet it is doubtful whether it is expedient in practice to recognize this fact except as regards the greater alterability of rates that are not fully substantiated by experience.

Mr. Greene's suggestion implies in effect that in the case of all classes we should act as though we had the same statistical resources as regards the number of accidents that have actually occurred. Another treatment which in the end would lead to a similar result upon the formula would be to assume that in all classes the statistics were ample, that is in effect, that m was infinite. Equation (22) would then reduce to

$$z = \frac{Pn}{Pn + \frac{P^2(1-P)}{\epsilon^2}}. \tag{23}$$

In either case Mr. Greene's suggestion that the second term of the denominator be taken to be a constant would imply that ϵ^2 should vary as $P^2(1-P)$. Since $1-P$ is very nearly 1, this means that ϵ varies nearly as P. As a matter of fact this does not produce satisfactory values of ϵ over the whole range of values of P. In the actual use of the experience rating plan however, the contingencies are separated on each risk into two groups and the two groups are treated independently as will be explained later, so that this offers the opportunity to select different values of K' for the two groups in the equation $K'\epsilon^2 = P^2(1-P)$. When this is done the results are very satisfactory.

The simplicity of the formula

$$z = \frac{Pn}{Pn + K} \tag{24}$$

is remarkable; not only are the operations easily performed, but another advantage arises from the fact that P and n are always associated in the form Pn, which in application involves merely earned premiums; if, for instance, it were desirable to tabulate the values of z, they could be put in the form of a one-way table instead of a two-way table which would be required if z were a function of P and n separately. The mooted question is also answered with regard to the effect of the hazard upon the balance between risk-experience and class-experience; it is apparent that the hazard plays exactly the same rôle as the exposure.

The practice of experience-rating involves the joint use of the two equations:

$$x = P + z(p - P), \tag{25}$$

and

$$z = \frac{Pn}{Pn + K}. \tag{24}$$

The application of the theory will be treated in a paper by Mr. Michelbacher. It is obvious, however, in a general way that the practical questions to be answered involve first the determination of K and second the preparation, from the experience, of the quantities Pn and p, the determination of p being the main practical problem. A few general observations with regard to these matters may be made, without exceeding the proper confines of this paper.

In the preceding discussion P, p and x are hazards, that is probabilities. These quantities are connected with the corresponding rates $[P]$, $[p]$ and $[x]$ by a relation of the general form:

$$\text{Rate} = \frac{\begin{pmatrix}\text{Number of workers}\\ \text{exposed}\end{pmatrix}\begin{pmatrix}\text{probability of}\\ \text{accident}\end{pmatrix}\begin{pmatrix}\text{average loss}\\ \text{per accident}\end{pmatrix}}{\begin{pmatrix}\text{number of workers}\\ \text{exposed}\end{pmatrix}\begin{pmatrix}\text{average annual payroll}\\ \text{per worker}\end{pmatrix}}$$

The quantity expressing the number of workers exposed cancels out of both numerator and denominator; the average loss per accident and the average annual payroll per worker are assumed constant for a given classification and a given contingency, so that P, p and x are equal respectively to $[P]$, $[p]$ and $[x]$, each multiplied by the same constant, which may be called a. In equation (24) if for P, p and x are substituted $a[P]$, $a[p]$ and $a[x]$, respectively we have

$$z = \frac{[x] - [P]}{[p] - [P]} = [z] = \frac{[P]n}{[P]n + \dfrac{K}{a}},$$

From this it appears that $[z]$, the percentage of the difference between the manual *rate* and the indicated *rate* which is allowed upon adjustment, is given by an expression of the same form as equation (24); equation (24) may therefore be interpreted in terms of rate as well as in terms of hazard, the only difference being with regard to the value of the constant K.

In practice K must be determined by judgment. This will be treated by Mr. Michelbacher. If in equation (24) P is treated as a rate and n as the number of year-workers exposed Pn will be earned premiums. P is obtained by the application of manual or manual and schedule; n, the number exposed, is obtainable from the payroll exposure.

It will be observed that there are no artificial stops such as neutral zones or maximum allowances in connection with this theory. Complete control is found in the formula itself. The only artificial stop that is necessary is a minimum to exclude risks so small that the cost of rating would be out of proportion to the results produced.

The theory developed in this paper contemplates independent occurrences of a simple contingency such as death. Catastrophes are by the nature of the hypothesis excluded. It is obvious as a matter of practice that some concessions must be made to practical conditions on both these points. We cannot insist that the accidents shall be entirely independent and in practice we are not dealing with simple contingencies.

It was found feasible to split the contingencies into two groups, in the first death and permanent total disability, in the second all other losses. Each of these groups is treated separately and the final rate is secured by addition of the two adjusted rates. Similarly it was found satisfactory to exclude the excess of catastrophic losses above a certain point.

A word should be added with regard to the relationship between experience rating and schedule rating. There has never in the past been any conscious and well-considered effort to combine manual rating, schedule rating and experience rating into a single consistent system; in fact it has been generally, although reluctantly, recognized that schedule rating and experience rating were to a considerable extent different ways of doing the same thing, and in effect they have doubtless overlapped; experience rating approaching the problem from the retrospective point of view, schedule rating from the prospective point of view. There has been a certain fiction that, as the proper field of schedule rating was physical condition as revealed by inspection, so correlatively experience rating should cover the field of the moral hazard which could not be reached by the schedule. Unfortunately for this theory experience does as a matter of fact reflect both moral and physical conditions, so that instead of having one system covering physical condition and one covering moral condition, we have in fact one system covering physical condition alone and one system covering both physical and moral condition.

Each however has its peculiar value. Except in the case of small risks experience rating is doubtless in general the better guide to the hazard. In the case of small risks, however, schedule rating is the only system that produces substantial variations; it is the only system furthermore whose effect is immediately felt in the rate when a plant is brought into good condition.

The ultimate place of schedule rating depends however not so much upon its primary value in measuring the hazard as upon its secondary value as a basis for the prevention of accidents. It is altogether desirable from the standpoint of public policy that there should be some immediate and perspicuous correlation between physical condition and the cost of accidents, and while schedule rating should be developed so far as possible as an exact measure of the hazard, and for this purpose statistical sources must be drawn upon far more than in the past, nevertheless the development of the schedule must be largely guided by a consideration for its place in public economy.

I believe the time has now come when there can be and there should be a complete reconsideration and readjustment of the manual system, the schedule system and the experience system in the effort to develop one thoroughly concatenated and consistent rating system. This involves the necessity for a thoroughgoing analysis of the logic and philosophy of rating.

An illuminating suggestion was made by Mr. Woodward during the work of the Actuarial Section to the effect that the schedule should be viewed as a* refinement of the manual system of classification.

The manual proceeds by simple enumeration of classes. It is impracticable, however, to follow this method beyond a certain point; the future development of the manual should probably be toward simplification rather than amplification. The schedule by analysis and combination provides a method for carrying the process of classification further. Suppose for instance the schedule recognized three characteristics, each having a bearing upon the hazard and suppose that each of these characteristics had five different quantitative values that it might assume, then the possible variations produced by the schedule and superimposable upon the manual would be 5^3 or 125. The place of experience rating in this theory now appears. Manual and schedule together may be considered still to deal with classes, although classes that are greatly

* This point of view had also been held by Mr. Greene; see p. 72 of Vol. III of the Proceedings of the Casualty Actuarial and Statistical Society.

refined. The experience rating plan, however, deals with the particular risk. We have, therefore, in accordance with the theory developed in this paper, the problem of balancing the class effect against the risk effect. This general point of view seems to indicate one possible basis for a thoroughgoing rating theory; in fact the National Reference Committee has already adopted this point of view by providing that the basic rate for experience rating shall be the manual rate as affected by schedule rating. There are, however, also other points of view.

I hope that the future may see important work done along these lines and that an actuarial theory for workmen's compensation insurance rating may be developed as consistent and well-balanced as that of life insurance and going beyond it in its nicety of measurement.