Abstract of the Discussion of the Papers Read at the Previous Meeting.

UPON COMBINING COMPENSATION EXPERIENCE FROM SEVERAL STATES—WINFIELD W. GREENE.

VOL. VI, PAGE 10.

WRITTEN DISCUSSION.

MR. E. H. DOWNEY:

Mr. Greene's paper deals with the method of combining compensation pure premium experience had under different scales of benefits. As a contribution to the technique of experience differentials the paper is wholly admirable. Mr. Mowbray and others have discussed the technical aspects of this subject. What follows will relate to the fundamental fallacies of any method of experience law differentials.

The underlying thought of the experience differential is that the experience of a single jurisdiction of any considerable size is sufficient to indicate the *general rate level* for that jurisdiction, but that the experience of many states is necessary to establish the proper relationship between industry classification rates. The general rate level is a matter of total premium income; classification rates are a matter of premium distribution by industries. In the full-blown application of this method, the experience of Pennsylvania would determine that the aggregate rates for Pennsylvania shall be to the aggregate rates for New York (assuming the same industry distribution of insured payrolls) as .65 to 1.00; whereas the absolute Pennsylvania rate for any given classification will be determined, not by Pennsylvania experience, but by the combined experience of many states. Loss experience for this purpose is combined by means of conversion multipliers obtained by the now familiar methods of pure premium comparison. These conversion multipliers represent, not the actually experienced pure premium ratios of particular classification, but the weighted average (or aggregate) ratio for a large number of classifications.

The fundamental fallacy of this procedure resides in its underlying assumptions. The procedure will be valid only if, and insofar as, a uniform relationship between pure premiums can be predicated of the several classifications which are so treated as a unity for the purpose of deriving the conversion multiplier. It so happens, however, that the pure premium ratios as between any two jurisdictions are not the same for any two industries. Nor are these divergent ratios a matter of accident: they depend upon inherent causes which no mathematical jugglery can overcome. No two scales of benefits stand in constant relationship for the several degrees of injury and no two industries present the same severity distribution of injuries. As between New York and Pennsylvania, e.g., taking benefits and wages at the level of 1917, the conversion multipliers should be approximately as follows: For deaths 1.40, for permanent totals 4.00, for loss of arm 2.30, for loss of hand 2.20, for loss of eye 1.64, for loss of fingers 2.50, for temporary dis-

TABLE I.

FREQUENCY DISTRIBUTION OF INJURIES.

Pennsylvania Schedule Z, Policy Years 1916 and 1917.

Industry.	Payroll (000 omitted).	Pure Premium.	No. of Comp. Accidents	Number of Specified Injuries per 1,000 Comp. Accidents.				
				Death.	Maj. Perm.	Loss of		
						Hand.	Foot.	Eye.
1	2	3	4	5	6	7	8	9
All industries	$2,724,709	$ .73	88,234	30	23	7	3	12
Anthracite mining	49,661	2.76	3,931	67	24	5	6	10
Bituminous mining	263,689	1.83	17,778	41	23	6	6	10
Quarrying	25,249	2.00	1,663	41	39	6	10	23
All manufacturing	1,163,432	.60	38,840	16	24	10	2	12
Baking	14,980	.71	537	16	40	33	—	3
Blast furnaces	10,410	1.64	612	60	15	5	5	5
Rolling mills	77,561	.83	2,991	20	18	6	3	9
Steel foundries	20,335	1.20	1,047	30	24	6	4	13
Iron foundries	34,690	.89	1,906	15	16	3	2	9
Machine shops	95,751	.67	3,507	11	27	8	1	17
Planing mills	12,333	1.00	679	15	27	20	1	6
Glass ware mfg	24,123	.28	428	2	25	7	2	16
Carpentry—N.O.C.	15,854	1.62	1,013	19	23	8	6	9
Concrete construction	18,488	1.50	1,034	37	25	6	6	12
Iron erection	6,768	3.30	640	70	35	14	7	12
Drivers	44,159	.80	2,139	24	12	4	2	4

abilities 1.50. Even these ratios vary with wage levels and consequently are not the same for the building trades as for candy making, nor are they the same for 1919 as for 1917. What makes these differences important from the present standpoint is the extremely variable severity distribution of injuries. Deaths range from one in fifteen compensable accidents for iron erection to one in four hundred for glassware manufacturing; major permanents are one-third the number of deaths in anthracite mining and five times

the number of deaths in laundries; lost eyes are sixty per cent. of all major permanents in quarrying, and fifty to sixty per cent. in foundries and machine shops whereas lost hands are sixty to seventy per cent. of all major permanents in bakeries, laundries and printing establishments (See Table I). Minor permanents are an insignificant fraction of pure premium in blast furnaces, rolling mills, foundaries and machine shops but they play an important rôle in the losses of planing mills, paper box factories and sheet metal works.

The severity distribution of injuries being thus dissimilar, and the several scales of compensation disparite, it follows that no conversion multiplier which is true for planing mills can be true for machine shops and that no average conversion multiplier will hold good for any specific industry. The use of conversion multipliers, however derived, distorts the classification pure premiums for every state (See Table II). The error may be reduced by refined analysis but no refinement upon a wrong method can reach the fundamental source of error. If pure premiums be broken up into death and permanent total, major permanent, minor permanent, temporary and medical, the several fractional conversion multipliers will distort the classification experience less than the total multiplier, yet considerable distortion will remain, particularly in respect to death and permanent total disability benefits, major permanent disability benefits and medical benefits. The distortion is further reduced by computing separate conversion multipliers for broad industry divisions, but only at the cost of narrowing the basis of comparison and magnifying the influence of chance deviations. Obviously an experience differential derived from a very limited exposure is utterly unreliable. But the calculators of experience differentials must steer between Scylla and Charybdas: either their industry groups are a heterogeneous hodge-podge—like the "outdoor," "in-door" and "loft" industries of the National Council in the 1920 rate revision—or the statistical basis is too narrow to produce a dependable average. Really homogeneous industry groups are scarcely broader than individual classifications and a separate experience differential for each classification is a *reductio ad absurdum*. The attempt to enlarge the experience of a single state without altering its quality is analogous to the famous feat of the gentleman who lifted himself over the fence by his own bootstraps.

Mr. Greene's formula does not touch this fundamental defect of experience differentials. It is a method, simply, of correcting the aggregate, or average coversion multiplier for the state's experience as a whole, or for that segment of the state's experience which is treated as an entity for the purpose in hand. As such, the formula is equally applicable to total or fractional multipliers; it will reproduce the losses of the given state with equal fidelity whether taken *in toto,* or divided into partial pure premiums by severity of injury. What the method overlooks is that rates are made by classifications, and not for industry as a whole. It is not enough that the general level of rates shall correspond to a reasonable loss ratio; the rate for each industry must be, so far as possible, both reasonable and adequate. No method which produces unreasonable results for specific industries, however well balanced in the

TABLE II.

COMPARISON OF PENNSYLVANIA PURE PREMIUMS, SCHEDULE Z, POLICY YEARS 1916 AND 1917, WITH NATIONAL COUNCIL PURE PREMIUMS CONVERTED TO PENNSYLVANIA LEVEL OF BENEFITS.

Industry.	Payroll.		"All other" Pure Premium.		Deviation in Per Cent. of Penna. P. P.
	Penna. (000 omitted).	Natl. Council (000 omitted).	Penna.	N. C. Converted to Penna.	
1	2	3	4	5	6
Quarries—N.O.C.	$ 8,470	$ 25,674	$.70	$.62	−10
Baking	14,980	72,284	.29	.25	−17
Wool spinning and weaving	35,273	191,078	.13	.11	−15
Fur hat mfg.	5,015	31,857	.12	.06	−50
Laundries—N.O.C.	8,303	49,007	.35	.20	−43
Planing mills	8,403	56,084	.41	.43	+05
Paper mfg.—N.O.C.	5,683	38,104	.20	.28	+40
Printing	28,386	134,325	.14	.12	−14
Blast furnaces	10,410	13,805	.32	.43	+33
Steel foundries	14,547	25,410	.35	.37	+06
Iron foundries	34,690	94,401	.29	.28	−03
Machine shops	75,441	314,545	.22	.23	+05
Cement mfg.	10,705	19,262	.27	.31	+15
Masonry—N.O.C.	10,715	50,227	.38	.48	+26
Drivers	44,159	213,022	.27	.26	+04
Garages	20,491	146,506	.20	.17	−15

aggregate, will suffice for sound rate making. It is at precisely this point that the experience differential fails. The whole attempt to distinguish between general rate level and classification rates is fundamentally fallacious. The general rate level is produced by the classification rates and will be accurate just in so far as the classification rates are accurate.

The need of combining classification experience from different states arises only in respect to those classifications for which the experience of the given state is insufficient to afford a dependable rate indication. Such combination can proceed only upon the assumption that the industry covered by the classification is substantially homegeneous in the several states whose experience is to be combined—i. e., that the number and severity of injuries per million of payroll would be the same in all these states if the exposure were broad enough to produce a dependable average. If this assumption be invalid, the combination is wholly illicit and no sleight of hand, whether by experience differentials or otherwise, will remedy the evil. Given, from Pennsylvania Schedule Z, $10,000,000 of payroll in Paper Mfg., with 21 deaths, 4 major permanents, and 360 temporary disabilities, the problem to be solved in rate projection is whether this number and this distribution of injuries per unit of exposure are normal to the industry. A convincing answer to this question can be obtained only by comparing the experience reported for Pennsylvania with the experience of the same industry in other states. Massachusetts experience is not appropriate for this purpose because paper making in Massachusetts and in Pennsylvania are quite dissimilar. The experience of New York and Wisconsin, on the other hand, is known to be fairly comparable with that of Pennsylvania. If, then, the combined experience of these three states for the policy years 1916 and 1917 shows $40,000,000 of payroll, with 60 death, 50 major permanents and 1500 temporary compensatable disabilities, we have a reasonably adequate basis for the determination of the accident severity rate of that industry per million of payroll at the then wage level.

The next step is to pass from accident severity rates to pure premiums. Here nothing short of individual valuation of accidents will serve the turn. An average value of death benefits will not answer, because the number of deaths in the Pennsylvania experience (21) is not sufficient to give a reliable average. To assume that the average cost of deaths in paper making is the same as for manufacturing industries as a whole, or is identical with the average for any other industry group, is to beg the very question at issue. It is only because Pennsylvania experience is deficient that it required to be supplemented by the experience of other states; this deficiency is not confined to fatality rates but extends as well, and even more emphatically, to the wage and dependency distribution. Whence it follows that a reasonable estimate of death pure premium can be obtained only by applying the Pennsylvania death benefits to individual death reports. Still less can permanent total and major permanent disability pure premiums be converted from one scale of benefits to another by any species of multiplier applied to monetary losses. Here again the very question to be solved is the frequency distribution of major permanents by part of body affected, and here also the only method of valuation which is even approximately accurate is to apply Pennsylvania benefits to the individual accidents as reported for each state. Minor permanents and temporaries can be valued *en masse,* but only when distributed into wage and duration groups for the specific industry, which also can be conveniently done only from cards punched from individual accident reports. Medical benefits, lastly, as between different jurisdictions, are affected by varying time and monetary limits which impinge unequally upon different industries. As between New York and Pennsylvania, e.g., the medical multiplier is not the same in the building trades as for clothing manufacturing, nor is it the same for house carpentry and iron erection. The low limits of the Pennsylvania law affect most markedly the severe injuries and the ratio of Pennsylvania to New York benefits is inversely proportionate to the frequency of these injuries. For this reason a dependable translation of medical benefits necessitates a distribution of medical aid cases by monetary amounts, which in turn requires an individual card analysis. It is often the case, of course, as in the instance cited of Paper Mfg. in Pennsylvania, that the experience of the given state in the given industry is ample in respect to medical and temporary disability benefits and requires to be supplemented only in respect to death and permanent disability benefits. In such cases combined experience need be used only for the death and permanent disability pure premiums, which course will notably lessen the labor of conversion. But wherever loss experience is to be combined at all, only individual valuation of claims will yield defensible results. If it be objured that the procedure will entail much labor, the answer is that Providence has provided no royal road to statistics.

If the foregoing contentions be sound, the experience differential is in the nature of a makeshift: an attempt to evade the necessity of statistical analysis and to substitute mathematical hocus-pocus for knowledge of facts. All this is not to be-little the efforts of those who have labored to perfect the mechanism of pure premium conversion. Much progress has been made since the first crude application of experience differentials by the Pennsylvania Bureau in 1918 and present methods represent an immence advance over the flat theoretical law differential of unhallowed memory. The time has come, however, after nine years' experience with compensation insurance, to develop an actual statistical basis for rate making.

MR. H. C. CARVER:

In a valuable contribution to our Society, Mr. Greene has called attention to the fact that conversion factors may quite readily be calculated by imposing the condition that the expected and actual losses, for representative classifications of the basis state must be equal.

Representing in general terms the payrolls and actual losses for the basis state by P_b and L_b, and by P_a and L_a, the corresponding items for the additional state, the above condition requires that

$$\Sigma P_b \frac{L_b + EL_a}{P_a + P_b} = \Sigma L_b, \tag{1}$$

which can be reduced to

$$\Sigma \frac{EL_a P_b}{P_a + P_b} = \Sigma \frac{L_b P_a}{P_a + P_b}, \tag{2}$$

and if E can be considered a constant for the group of classifications to which Σ refers, we have

$$E = \frac{\Sigma \frac{L_b P_a}{P_a + P_b}}{\Sigma \frac{L_a P_b}{P_a + P_b}}. \tag{3}$$

Although a comparison of the actual and expected losses obtained by using Mr. Greene's method of approximating the value of E defined by equation (3) has appeared slightly unsatisfactory to Mr. Downey, I cannot but help believing that Mr. Greene's formula (2) can produce wonderfully satisfactory results if we do not impose the condition that E is a constant.

The results of Table I indicate that E is a function of the class hazard rather than a constant which, for a specified kind of injury, is common to any one broad group of industrial classifications.

TABLE I.

"ALL OTHER LOSSES."

Pure Premium Group.	Classifications in Groups.	Indicated N. Y. p.p. Π_b.	E N.Y.—Basic State Penna.—Additional State.
1	8028, 8000, 8016, 9071	.00– .25	4.09
2	9050, 2623, 3808, 8003	.25– .50	3.20
3	7205, 5380, 8380, 2121	.50– .75	2.40
4	3632, 2000, 5183, 5602	.75–1.00	3.34
5	8222, 6042, 3864, 2803	1.00–1.25	3.52
6	2730, 6220, 5500, 7219	1.25–1.50	3.63
7	5022, 5160, 2760, 4234	1.50–2.00	4.69
8	5401, 2702, 5204, 5545, 5474	over 2.00	4.50

The values of E were calculated by using formula (3).

It appears to me that the high values of E for the low premium classifications can be accounted for by the fact that Pennsylvania does not provide for minor dismemberments in 306 C of the act.

Although we might assume that $E = f(\Pi_b)$ for a limited region could be represented approximately by a parabola, better results can be arrived at, and more easily, by considering that the relation is linear for each of the regions $\Pi_b = 0$ to .50, and .50 to 2.00, and therafter constant. Placing, therefore,

$$E = a + b\Pi_b,$$

equation (2) becomes

$$a\Sigma \frac{L_a P_b}{P_a + P_b} + b\Sigma \frac{L_a L_b}{P_a + P_b} = \Sigma \frac{L_b P_a}{P_a + P_b}. \tag{4}$$

Using the method of averages for the zone $\Pi_b = 0$ to .50 we obtain from groups 1 and 2 of Table I the equations,

$$23463a + 5009b = 95855,$$

$$27780a + 104811b = 88808,$$

and from the groups

7205, 3632, 2000, 6042, and

2730, 5022, 4234, 5401

the equations,

$$385280a + 100347b = 124101,$$

$$251103a + 115535b = 53483.$$

From these we may deduce

TABLE II.

Π_b.	E.	$\Pi_a = \frac{\Pi_b}{E}$.	Π_b.	E.	$\Pi_a = \frac{\Pi_b}{E}$.
0	5.2	.00	.50	2.7	.18
5	5.0	.01	.60	2.9	.21
10	4.7	.02	.70	3.0	.24
15	4.4	.03	.80	3.1	.26
20	4.2	.05	.90	3.2	.28
25	3.9	.06	1.00	3.3	.30
30	3.6	.08	1.25	3.6	.34
35	3.3	.10	1.50	3.9	.38
40	3.1	.13	1.75	4.2	.42
45	2.8	.16	2.00 and over	4.5	.44 and over

This table, based upon the experience of only a few classifications will enable one to read off the "expected" reduction factors from either Π_b or Π_a.

Example I.

From classification 8028, of Group III, we have,

$$P_b = 323540, \quad P_a = 575140,$$

$$L_b = 64773, \quad L_a = 28941,$$

$$\Pi_b = .20, \quad \Pi_a = .50,$$

and from Table II, we see that taking either Π_b or Π_a as a criterion

$$E = 4.2.$$

Example II.

Classification 5022, Group I.

$$P_b = 50770, \quad P_a = 107150,$$

$$L_b = 98900, \quad L_a = 40937,$$

$$\Pi_b = 1.95, \quad \Pi_a = .38,$$

Using Π_b and Π_a, in order, we have the two indicated values for E, 4.5 and 3.9 respectively. Weighting these according to payroll, i.e., 2 to 1, we have

$$E = 4.3.$$

Example III.

Classification 3631, Group II.

$$P_b = 14930, \quad P_a = 143410,$$

$$L_b = 14482, \quad L_a = 77579,$$

$$\Pi_b = .97, \quad \Pi_a = .54;$$

$\therefore E$ is weighted between 3.3 and 4.5, i.e., 4.4.

Although a test over a large number of classifications shows that this modification of Mr. Greene's formula produces in the majority of cases smaller differences between the actual and expected losses, I believe its greatest value lies in its ability to project pure premiums.

Of course, discrepancies between values of E, as determined by Π_b and Π_a are bound to occur, due to a limited exposure, but by the use of weights whenever necessary, the total reduced losses from all states and therefore the basic pure premiums, ought to be very dependable for a large number of our classifications. Then, since values of

$$E = f(\Pi_b)$$

can be calculated from the classifications having the largest exposure, we may project the basic pure premiums, retaining a maximum amount of accuracy and dependability. Of course, in projection, values of E should be determined from a consideration of the basic pure premium only.

It seems to me that Mr. Greene's method of attack is exceedingly direct and logical, but that by giving E a little more freedom, better results can be obtained.

ORAL DISCUSSION.

MR. H. E. RYAN:

Anything from the pen of Dr. Downey must be read with more than ordinary interest. Dr. Downey generally has something to say and he says it in such a manner that it can rarely be misunderstood. It has occurred to me on several occasions, in noting the apparent intolerance on Dr. Downey's part of methods which introduce the higher mathematics, that perhaps after all, the major difference between himself and those with whom he finds it difficult to be at peace may be due in large measure to the fact that Dr. Downey views things from the statistical rather than from the actuarial standpoint.

I must say that I am not inclined to be amused at Dr. Downey's newly-coined expression "mathematical hocus-pocus." It gives me very much the same feeling that the rather too frequent use of the word "conjectural" gives, when applied to estimates according to mathematical methods and reasoning of qualities which by their very nature must be estimated when the statistical material necessary for exact measurement is lacking. I equally condemn pure conjecture. When the actuary calculates the net premium for a joint and survivorship annuity, he might be accused of embarking upon the field of conjecture, and if the actual experience should not coincide with the assumptions underlying the calculation, I suppose he might even be accused of having employed "mathematical hocus-pocus."

It seems a little unfair to criticize in such terms a genuine endeavor to reach, by mathematical reasoning, a solution to a problem for which there is no precise answer. Dr. Downey seems to find fault with us because we do not accomplish all of our results by the laborious processes of long-hand arithmetic. By the same token, it may be that he would have little faith in the calculated value of the net premium for say, a whole life policy, unless perchance it were arrived at by discounting, with mortality and interest, the successive annual claim payments and the corresponding premium contributions. This was the method in vogue before the invention of commutation columns, but now fortunately, by means of that very useful artifice, it is possible to produce the desired result with the minimum of mathematical labor.

There are, it is true, times when the use of actuarial short cuts tends to obscure the successive stages involved in obtaining a given result. For this reason, it is highly desirable that results reached by such means should be unscrambled and thoroughly explained. Is it fair, however, to intimate that actuarial short cuts obscure the results themselves? I am sure that to this proposition actuaries will not agree but they may have some difficulty in winning over to their point of view statisticians trained along traditional lines whose acquaintance with the actuarial method of analysis may not happen to be particularly close. Conjecture and mathematical hocus-pocus are not of the essence of actuarial science.

MR. S. B. PERKINS:

With regard to the criticism of Mr. Greene's paper, which has just been submitted (Discussion by Mr. Downey), it occurs to me that there is one fact which should be emphasized at this time. The point has been raised, and has been exceptionally well taken, that the distribution of accidents of a particular industry in a given state should be the basis of rate making for that classification and state.

The intimation is, however, that a single distribution of accidents is being used for all classifications in all states by the National Council. The fact is that, wherever a classification in any state has a sufficient amount of experience to be indicative, it is being rated on its own experience wherever that experience differs from the combined experience, and in this manner the accident distribution for that classification in the state in question is being given primary consideration. As a matter of fact, Mr. Greene's formula and method were originally devised, in my estimation, as a means of combining the experience for all states for those classifications which did not produce enough experience in any individual state to be indicative and has proved to be an admirable solution to that particular problem.

To carry out the idea of individual classification accident distribution to its logical conclusion, an accident distribution would have to be obtained from the experience of each classification for each state. It is obvious that, if for many classifications there has been too little experience accumulated to even be indicative in itself as a rate measure, it would be ridiculous to attempt to use it as a basis of an accident distribution. It occurs to me, therefore, that rates for the classifications and states for which the National Council has used accumulated experience on the assumption that one accident distribution holds for the whole country are more defensible and less fallacious than would be rates based upon accident distributions obtained from the inadequate experience of individual classifications and states.

AUTHOR'S REVIEW OF DISCUSSIONS.

MR. WINFIELD W. GREEENE:

Dr. Downey's criticism may be summarized as follows:

1. There is one proper way in which the compensation experience of several states for a given industry may be combined, and that is by actually revaluing each accident in terms of the benefit schedule of the basic act.

2. The method of experience differentials produces incorrect results for the following reasons:

(*a*) The frequency distribution of accidents according to their severity varies from classification to classification.

(*b*) The ratio between the respective costs of the benefit schedules of any two states varies according to the nature of the injury.

(*c*) Wage scales differ, not only by classification, but also by state, for the same classification.

(*d*) The combined effect of (*a*), (*b*) and (*c*) is to make a conversion factor determined for the entire manual, or for a group of related classifications, inapplicable to each of the individual classifications involved.

(*e*) The difficulty outlined in (*d*) cannot be overcome by computing many conversion factors for each of a considerable number of classification groups, because this latter course would cut down the statistical basis of the computation to a point where the result would be undependable.

Dr. Downey will admit, I believe, that a revaluation of the original accidents in terms of the basic act involves substantially more labor than that required by the experience differential method. He says, "if it be objured that this procedure will entail much labor, the answer is that Providence has provided no royal road for statistics."

If it is admitted that it is necessary to combine experience from several states for any purpose the amount of labor involved is a consideration. Those familiar with the recent work of the National Council must agree that even the method employed (experience differentials for "all other" and "medical" and combination of "D. & P. T. D." cases by number and average cost) requires quite as much of a burden of clerical labor as is tolerable. The revaluation method would require the determination of what might be termed accident tables, not a single standard table for all classifications, but one for each classification, or, at least, for each group of related classifications. In order to determine pure premiums upon the basis of the combined experience for each of, let us say, thirty states we would have to make 30,000 distinct applications of such an accident table to a given benefit schedule, assuming that a separate combination would be made for each of 1,000 classifications. If the number of accident tables could be cut down to 400, 12,000 calculations of this kind would be required.

Considering the preliminary labor of preparing the accident tables, which might be greater than that involved in their application, it would seem that Dr. Downey proposes a remedy hardly preferable to our present ills.

Now if the individual revaluation method is as laborious as I believe it to be, we should not consider the adoption of it for one moment if the present method produces results which are approximately correct. I say "approximately" because I believe absolute accuracy to be unattainable in the matter of compensation insurance rates. There are many things which may happen,—and some of them generally do,—which will result in disturbing that "relativity" between classification pure premiums which existed while the experience was being accumulated. Familiar past examples are the world conflict and entry of the United States therein.

Dr. Downey has enumerated several conditions which I admit tend toward inaccuracy in the experience differential method. Do these conditions tend to balance each other? Is the experience differential method reasonably accurate in spite of them?

My critic thinks not, and in support of his view submits his Table II entitled "A Comparison of Pennsylvania Pure Premiums —Schedule 'Z'—Policy Years 1916 and 1917—with National Council Pure Premiums Concerted to Pennsylvania Level of Benefits." The table in question shows for sixteen classifications the payroll for Pennsylvania and for the National Council experience, also the "all other" pure premium for Pennsylvania and "National Council Converted to Pennsylvania." The last column shows the deviation in percentage of the National Council converted pure premium from the Pennsylvania pure premium. These deviations run from 3 per cent. to 50 per cent. There are nine negative deviations and seven positive deviations.

It is not permissible to have as much as a 50 per cent. error in the "all other" pure premium, if it is really an error. However, before making final judgment as to the accuracy of the experience differential method, I would like to submit a modification of this same table (Table A.) In this I have shown, instead of payrolls, losses upon the Pennsylvania basis, both for Pennsylvania and the National Council. Further, I have rearranged the table to show the data for the classifications in descending order of Pennsylvania losses.

What a different light shines upon the subject as a result of the rearrangement, and the exhibition of losses instead of payrolls! The most striking thing is the consistency with which the deviation increases as the Pennsylvania losses decrease. For example, in the first five classifications, those having the largest Pennsylvania losses, the greatest deviation is 11 per cent. and the average deviation is 6 per cent. The average for the first three classifications, all having over $100,000 Pennsylvania losses, is only 4 per cent.

In the next five classifications (having losses between $39,000 and $46,000) the average deviation is 17 per cent., while in the last group of six classifications (the losses vary from $6,000 to $34,000) the average deviation is 31 per cent.

A simple graduation of the last column using a moving average of three results as follows: 4, 6, 7, 11, 12, 14, 18, 19, 15, 17, 27, 30, 33, 38; with only one break in the curve and that a slight one. In other words, the deviation varies inversely with the volume of Pennsylvania losses.

This rearrangement of Dr. Downey's table can hardly be accepted as conclusive proof of the soundness of the National Council combination of experience—but it is highly favorable to the very method which the Doctor indicts; for where the Pennsylvania experience is substantial, it confirms the national experience. It would seem to be a corollary that where the national experience is substantial and the Pennsylvania experience is insignificant the national experience is a more reliable approximation to the Pennsylvania pure premium than is the actual Pennsylvania experience.

TABLE A.

COMPARISON OF PENNSYLVANIA PURE PREMIUMS WITH NATIONAL COUNCIL PURE PREMIUMS (THE LATTER CONVERTED TO PENNSYLVANIA LEVEL OF BENEFITS).

Classification.	Losses[1] Penna. Basis.		All Other Pure Premium.		% Deviation.	
	Penna.	National.	Penna.	National.		
Machine shops....	165,970	723,454	.22	.23	+05	Average =6%, Maximum =11%
Drivers...........	119,229	553,857	.27	.26	−04	
Iron foundries.....	100,601	264,323	.29	.28	−03	
Quarries (N.O.C.) .	59,290	159,179	.70	.62	−11	
Steel foundries....	50,915	94,017	.35	.37	+06	
Wool spinning and weaving........	45,855	210,186	.13	.11	−15	Average =17%, Maximum =26%,
Baking...........	43,442	180,710	.29	.25	−14	
Garages...........	40,982	249,060	.20	.17	−15	
Masonry (N.O.C.).	40,717	241,090	.38	.48	+26	
Printing..........	39,740	161,190	.14	.12	−14	
Planing mills......	34,452	241,161	.41	.43	+05	Average =31%, Maximum =50%
Blast furnaces.....	33,312	59,362	.32	.43	+33	
Laundries (N.O.C.)	29,061	98,014	.35	.20	−43	
Cement mfg.......	28,994	59,712	.27	.31	+15	
Paper mfg. (N.O.C.)	11,366	106,691	.20	.28	+40	
Fur hat mfg.......	6,018	19,114	.12	.06	−50	

The above conclusion is confirmed by a comparison of New Jersey and national pure premiums for all classifications, which in the experience before the Council exhibited New Jersey losses of more than $22,000 (New York basis). The results of this test appear in Table B. It should be borne in mind that the pure premiums and losses in Table B are upon the New York basis (i.e., converted to New York level). This does not affect the significant feature of the table, which is the percentage of deviation of the national pure premium from the New Jersey pure premium (column 5).

As in the Pennsylvania classifications selected by Dr. Downey there is a clearly defined tendency for the deviation to increase as

[1] Derived from Dr. Downey's Table II. by applying pure premiums to payrolls.

TABLE B.

New Jersey.

Comparison of National and State "All Other" Pure Premiums.

All Classes with More Than $22,000 New Jersey Losses (N. Y. Basis).
Note: Losses and P. P. are shown upon New York Basis.

Code No.	Classification.	New Jersey. Losses (in thousands). (1)	New Jersey. Pure Prem. (2)	National. Losses (in thousands). (3)	National. Pure Premium. (4)	% Deviation of Natl. from N. J. Pure Prem. (5)	
3632	Machine shop—no fdy.	212	.70	2,414	.77	+10	
7310	Stevedoring (N.O.C.)	152	3.57	1,597	4.19	+18	
5401	Carpentry (N.O.C.)	101	2.25	1,738	2.75	+22	Average = 14%,
4524	Chemical mfg. (N.O.C.)	91	1.26	314	1.16	−08	
4420	Rubber tire mfg.	89	1.04	241	.79	−24	
3643	Electrical apparatus	87	.66	452	.57	−14	Maximum = 24%
7205	Drivers	82	.67	1,705	.80	+19	
3631	Machine shop—fdy.	69	.94	803	1.08	+15	
5022	Masonry (N.O.C.)	66	1.92	890	1.77	−08	
6003	Pile driving	63	2.96	303	2.84	−04	
6861	Shipwrights	50	.85	163	1.00	+18	
5643	Carpentry—private	49	1.06	900	.98	−07	
2000	Bakeries	49	1.05	600	.83	−21	
4410	Rubber goods (N.O.C.)	48	.81	179	1.12	+38	Average = 23%,
3633	Projectile shell or case	46	1,64	310	1.00	−39	
6042	Road making	46	1.46	633	1.08	−26	Maximum = 39%
2413	Textiles—dyeing	45	.44	206	.51	+16	
7380	Chauffeurs	43	.54	735	.61	+13	
5474	Painting—interior and exterior	43	2.34	480	2.92	+25	
3724	Millwrights	43	1.53	382	1.08	−29	
2731	Planing mills	41	1.99	811	1.45	−27	
3607	Engine mfg.	38	1.72	165	.98	−43	
5204	Concrete work	38	2.69	575	2.03	−24	Average = 28%,
7219	Truckmen	36	1.32	892	1.73	+31	
3116	Tool mfg.	36	1.10	348	.59	−46	
3808	Automobile mfg.	35	.67	569	.63	−06	
5040	Iron work—erection	35	5.00	544	5.82	+16	Maximum = 46%
3100	Forging works	34	3.08	253	1.83	−41	
1622	Quarries	34	2.75	555	2.16	−21	
4510	Acid mfg. (N.O.C.)	33	1.96	126	1.50	−24	
3081	Foundries—iron	32	.68	818	.87	+28	Average = 24%,
2623	Tanning	31	.65	343	.62	−05	
3030	Steel works	31	2.44	220	1.63	−34	
3241	Wire drawing	30	1.07	179	.87	−19	
5642	Masonry—private	28	1.14	587	1.40	+23	Maximum = 52%
3881	Car mfg.—R.R.	28	3.22	168	1.53	−52	
3089	Pipe mfg.—cast iron	27	1.08	60	1.16	+07	

TABLE B—*Continued.*

Code No.	Classification.	New Jersey. Losses (in Thousands). (1)	New Jersey. Pure Prem. (2)	National. Losses (in Thousands). (3)	National. Pure Prem. (4)	% Deviation of Natl. from N. J. Pure Prem. (5)	
2121	Breweries..........	27	.91	415	.84	−08	Average =24%
5602	Additions, alterations, etc.........	27	.52	356	.73	+40	Maximum =25%
4580	Fertilizer mfg.......	26	.93	109	1.19	+28	
5500	Paving (N.O.C.)....	26	.83	332	.87	+05	Average =31%,
3082	Foundries—steel....	26	1.25	291	1.15	−08	
3620	Boiler making......	25	1.46	376	1.55	+06	
2303	Silk mfg...........	24	.07	123	.12	+71	
2803	Carpentry—shop...	24	1.79	208	1.28	−29	
3548	Printing machinery.	24	1.29	93	.70	−46	
3400	Metal goods (N.O.C.)........	24	1.14	468	1.78	+56	Maximum =74%
5437	Carpentry—interior	24	.39	260	.68	+74	
6106	R.R. Construction..	23	1.76	266	1.21	−32	
3220	Can mfg...........	22	1.76	137	1.63	−08	
4233	Paper board mfg....	22	1.19	119	1.08	−09	

the New Jersey loss exposure decreases. The following brings out this tendency:

Classifications.	Deviation of National from New Jersey Pure Premium. Average.	Maximum.
First 10	14%	24%
Second 10	23%	39%
Third 10	28%	46%
Fourth 10	24%	52%
Last 11	31%	74%

The extent to which in practice it is found necessary to establish "state exceptions" is a rough indication of the degree to which the national pure premiums are satisfactory as a measure of relativity for classifications having a significant exposure within a single state. The facts upon this important point will soon be common knowledge.

The paper under discussion does not involve higher mathematics. The several formulæ presented were intended merely as practicable solutions of a practical problem. Mr. Mowbray's comparison of the several experience differential formulæ, which I understand he will embody in a paper to this Society, brings out the fact that the experience differential determined according to the theory laid down in my paper would be the ratio of the aggregate pure premium for the basic state to that of the other state were it not for differences between states in the distribution of payrolls according to classification.

If I have correctly interpreted Mr. Carver's suggestion it is to make E a function of the pure premium; to determine E directly from the experience in case of a few classifications having a dependably broad exposure; and to derive E for the remaining classifications by interpolation. This interpolation could rest upon a variety of assumptions. The mathematical form of the assumption could be selected with reference to (*a*) conformity to the facts (*b*) convenience or operation.

Mr. Carver will I am sure agree that this procedure should not be adopted for practical use without further tests. If these tests yield satisfactory results the only objection I can forecast at this time is that we will not, as I see it now, be cutting down the labor incident to the combination of experience, and I think we do need to cut down on the work if it can be done without undue sacrifice of accuracy.

If Mr. Carver's method were followed, we might eliminate some of the labor by making only two subdivisions in the pure premium, death and permanent total (or major permanent) and "all other." It is possible that in combining "all other" and medical we would not materially alter the basic pure premiums.

It is to be hoped that Mr. Carver will utilize the National Council experience for the purpose of trying out his suggestion. He may find it necessary to adopt more than one mathematical law for the value of E in terms of pure premiums.

The writer would be glad if Mr. Carver or some other member of the Society would test the practicability of combining experience by simple addition without resort to factors of any kind, leaving all multiplicative operations to the second phase of rate-making, namely, projection.

Obviously if we do add the losses and payrolls respectively for all states, our results will be unintelligible owing to variability in the distribution of payrolls between states. It is, however, just as easy to add the number of accidents as it is to add losses. The figures handled are smaller and the computation and application of reduction factors would be avoided.

If we add for all states for a given classification the payroll, the number of deaths and major permanent disabilities, and the number of "all other" disabilities we could obtain pure premiums for any state by applying appropriate average values. (The average value for "all other" disabilities would include medical.)

Of course, accidents resulting in no payment or in medical only would be disregarded for purposes of the addition.

Great care would have to be exercised in determining the average values for death and major permanent, and "all other," injuries respectively. I am inclined to think that this problem could be met without serious difficulty by one of two expedients.

(*a*) Securing a dependable average by combining related classifications.

(*b*) Determining averages for classifications having a large exposure direct from their experience and deriving averages for other classifications by reference to the rate of accident per unit of payroll.

In any event we would be no worse off on the "death and major permanent" than we have been upon the "death and permanent total" heretofore. As for the "all other" average value, I believe the possibility that this average is a function of the accident rate is worthy of serious investigation.

I have phrased the above upon the assumption that before another national revision the available experience will segregate major permanent disabilities. If this turns out not to be the case the foregoing still applies *mutatis mutandis.*

It may be that refinements in compensation insurance statistics will have outlived their usefulness as soon as we have discovered the mathematical laws underlying the pure premium. If we can express the average cost per accident and hence the pure premium in terms of the accident rate per unit of payroll, we shall have but little interest in any statistics except payrolls, number of compensated accidents and aggregate losses. We should not forget that recently we decided that the time-honored division between paid and outstanding losses could after all be dispensed with.

Having determined your pure premiums from a consideration of the accident rate, and the relation of average accident cost thereto, you would quite likely find that your expected losses would not be identical with your actual state losses. I believe that the deviation between actual and expected losses for an entire state according to the method I have outlined would not be great and would be due either to variation between states in wages and in the true accident rate (man hour basis). This "slack" could be taken up as a flat percentage for the entire state, or by schedules, or groups of schedules.

AIRCRAFT INSURANCE—WALTER G. COWLES.

VOL. VI, PAGE 31.

WRITTEN DISCUSSION.

MR. A. MCDOUGALD:

I have read Mr. Cowles' paper with considerable interest, and particularly his pictures of the invention and development in America, as a means of transport, of the railroad train, trolley and automobile, the forerunners of aircraft. It is interesting, too, to note that the trolley, like the railroad, not being adaptable to purposes of sport, a development of these vehicles in the commerial sphere was not delayed by misapplication. Whilst as a matter of history this cannot be said of the automobile, there appears to be small chance (in England at least) of the hindrance of sport to the commercial advancement of aircraft. For this prospect we are probably indebted to the war as having ushered in a period of earnest striving and endeavor in the realm of recuperation, and to the post-bellum call for every available and effective means of transport in aid of recovery from the recent years of horror and deprivation.

Aircraft risks as the subject of insurance are new, and it must necessarily be some time before any dependable data can be collected on which to base equitable premium rates. In the meantime the arbitrary rates will be governed by considerations of analogy and argument and influenced possibly to some extent by competition. Repeated trial rates may be expected to approximate with increasing closeness to the ideal which can alone emerge as the passing years accumulate their experience data.

The author of the paper has performed his task so well as to leave small room for criticism, though superabundant space for discussion: these features I take to be characteristic of the author and of his subject respectively.

Viewing through the author's eyes the prospects of American aviation insurance, the impression conveyed by a perusal of the paper is that the author is well acquainted with the practical conditions of the problem, and that in the main, if not entirely, his forecast of the development of the insurance demand and supply may be relied upon. He writes: "Nor do we see in the near future the promise of a development likely to result in the early establishment of the field for casualty insurance presenting a sufficient volume to permit the application of fundamental insurance principles." If this be true for America, it strikes me as being somewhat pessimistic of application elsewhere. This brings me to a sudden halt at the standpoint of the author. He has attacked his subject with such vigor and success that I can but feebly attack him on one point, namely, his caption, "Aircraft Insurance," whereas he writes chiefly and so entertainingly on "Aircraft Insurance *in America.*" If, however, I extend the title of his paper, I am poorly qualified to question his practical conclusion, that, of aircraft insurance in America, there is none—yet.

In these circumstances, and allowing the caption to stand, it occurs to me that it may be of passing interest to the members if I endeavor, as briefly as may be, to record something of the position of aircraft insurance on this side of the water.

Dealing first with the present position in the United Kingdom: in February of last year the British Government set up a department of Civil Aviation, and on the 1st May following, the official date of the opening of civil flying in England, a small government staff began to attack the multiplicity of problems and difficulties attending the transition period from war to peace, this work including (*a*) the framing of Air Navigation Regulations for the control of civil flying at home, as distinct from the International Air Convention governing the regulations for international flying, and (*b*) the fixing of air traffic routes, etc. The International air Convention was based in the main on the Air Navigation Regulations, and was signed in October last by eleven out of the thirteen nations which were parties to it, the United States and Japan not then being in a position to sign. It is expected that very shortly one code of rules for the air will obtain throughout the whole of the civilized portions of Europe. In the meantime, British commercial aircraft have already visited the principal cities of eight of the European Countries, namely, Amsterdam, Brussels, Christiania, Copenhagen, Lausanne, Madrid, Paris and Stockholm. Considerable progress, too, has been made with the reconnaissance and development of imperial air routes, including Cairo to Karachi, Cairo to the Cape and India to Australia.

A return shows that during the first six months of civil aviation the greater number of accidents occurred in getting off and landing. It would, therefore, appear that the proportion of accidents to flights made may be a truer guide to the underwriter than that for hours flown. A Communiqué issued by the British Air Ministry announces that returns voluntarily supplied by civil aerial transport firms in the United Kingdom for the eight months ending the 31st December last show that for Great Britain and on the Continental route 403 machines were in use, 35,330 flights were made and 8,368 machine hours were flown, the mileage totalling approximately 593,000. The number of passengers carried was 64,416 and the weight of goods carried amounted to 67,143 pounds.

The following figures show the proportion of accidents over the whole period:—

No. of accidents resulting in death of one or more occupants of machines	4
No. of non-fatal accidents resulting in injury to occupants of machine	8
No. of accidents resulting in death of third party (occupants of machine uninjured)	1
No. of accidents in which no one was killed or injured	5
Total accidents reported	18
Approximate number of machine miles flown per accident	32,900
Approximate number of machine flights per accident	1,960
Approximate number of machine hours flown per accident	465

The most common cause of accident was engine failure, of which there were six cases.

Experience has proved the necessity for the punctual report and investigation of accidents, as only by this means can weak points in administration, personnel and material be eliminated and the safety of the public proportionately increased.

A special branch of the Department of Civil Aviation is concerned with the building, inspection and licensing of aerodromes, and the licensing of pilots competent to control the machines, and of aerodrome officials qualified to pass machines as fit for flying. Every machine has to be registered and numbered in the same way as a motor car, and if flying for hire, must in addition be certified as "airworthy." Certified customs aerodromes have been constituted for dealing with regular traffic arriving by air from other countries.

One of the lessons learned during the war was the supreme necessity for rapid ground communication. A special branch of the Air Ministry has charge of this particular work and covers the requirements of both the civil and service sides. An important part of the work is in connection with wireless telegraphy and telephony and the development of aerial navigation by means of directional wireless. On the London-Paris air route a wireless liaison was early established, and machine and weather reports are successfully exchanged by numerous messages per diem. Meteorological reports are now transmitted almost entirely by wireless, and weather forecasts are distributed three times a day from the Air Ministry. A complete scheme of wireless organizations for the entire meterological service of England has now been prepared. This branch is also responsible for the supervision of the training and examination of the wireless personnel.

On the navigational side of the work experimental strip maps of certain routes have been prepared for all aerial purposes, and "flying directions" compiled containing information as to landing grounds, aerodromes, wireless and meterological data—the first of a comprehensive library of air maps and flying directions which it is hoped to produce for all the main routes. Charts and maps are also prepared and issued for all intended flights on new routes. Experimental lighthouses have been erected on land on the London-Paris route to assist such services as necessitate commercial night flying. A research branch of the Air Ministry has been established, and the government meterological office has been transferred to the Ministry. Endeavors are being made further to assist scientific research, and in addition to the ordinary routine work of the meterological office, special attention is being devoted to the requirements peculiar to flying. A system of weather maps is being produced at six-hour intervals from information supplied by a network of meterological stations, and reports and forecasts covering various aerial routes have been prepared and issued, together with maps showing the speed and direction of the upper wind over each.

The rigid airship has not been exploited in England to the same extent as the aeroplane, but the commercial possibilities of this type of aircraft in connection with long-distance flights are fully realized.

Some figures have been given showing the number of flights during the eight months ending December last, and the passengers carried. During that period four pilots were killed and six injured, and one passenger was killed and ten were injured.

The percentage of casualties was as follows:

Pilots killed per thousand flights made by pilots	.11
Pilots injured per ditto	.17
Pilots killed per 1,000 hours flown by pilots	.48
Pilots injured per ditto	.72
Pilots killed per 1,000 passengers carried	.016
Passengers injured per 1,000 passengers carried	.16
Passengers killed per 1,000 hours flown by passengers	.06
Passengers injured per 1,000 hours flown by passengers	.61

For these figures and a good deal of the foregoing information I am indebted to the courtesy of the Actuary of the British Air Ministry. In aircraft development we are probably in England about as far advanced relatively as was America twenty years ago in regard to the automobile. We have, however, travelled far enough along the road of progress to demonstrate that the possibilities of the future development of aircraft as a commercial proposition are very great. Once the postulates of reliability, safety, comfort and economy have been met, as they undoubtedly will be, civil aviation must play an increasingly important part in the development of civilization. These demands may and will doubtless be met in the comparatively near future by the combined forces of science and insurance. As these lines are being written, announcement is made of an invention involving a material saving of horse power, weight and wing dimensions, whereby, without diminution of commercial capacity and usefulness, the dangers at present incident to getting off and landing may be considerably lessened, and insurance made *pro tanto* more attractive. Another invention of quite recent date enables a pilot to steer his chaft in a fog as unerringly as he might do in a clear atmosphere.

Under the stress of war the aeroplane was scientifically developed with great rapidity. The future relationship of the commercial to the service aircraft may develop into that of the mercantile marine of a country to its navy, and furnish a potential reserve of material value in any future crisis. If such a time of crisis should ever again loom up, the joint service and civil aviation forces of America and of the British Empire should surely suffice to settle the dispute in short order.

A particular word as to aviation insurance in England. Soon after the armistice, the leading insurance companies combined to form a pool to take care of aviation risks. A committee was formed, and with the advice and assistance on technical points of an aviation engineer, this committee fixed premium rates to the best of its judgment for (*a*) accident to the passenger, and (*b*) a comprehensive insurance, i.e., a policy covering practically all risks incident to the ownership and control of the machine. Each company issued its own policy in agreed common form, crediting the pool with the premium and charging there against all claims. The pool committee was in control of all claim settlements. Policies were issued up to specified limits and the balances between premiums and claims were proportioned periodically amongst the pool members according to their shares therein. Under this scheme policies could be issued by individual companies up to a limit of, approximately, £100,000 against liabilities to the public. The experience of the pool has been that the aggregate claims during the eight months ending December last were nearly 250 per cent. of the net earned premiums pooled. The total number of insured passengers carried was 3,600. These were all carried without any accident. There were only four accidents in all. In each instance the machine was wrecked. One case was that of a forced landing; another was the direct result of fog, and two were instances of engine trouble. In none of these four cases were any insured passengers aboard. The volume of insurance was, fortunately for the insurance companies, only of modest dimensions, so that each company in the result secured a valuable experience at a reasonably small cost. Hitherto premiums have generally been based on the intended number of flying hours, but this system has not proved satisfactory and rates are now being charged per flight, with differentiations for intended duration and distance.

Some demand for insurance has sprung up in Australia and is being met, and there is a strong probability of such an extension of commercial flying within the coming years between the far-flung countries of the British Empire as may exercise the mind and genius alike of the scientist and of the underwriter to the full.

The air lines of commerce between England and foreign countries are being laid down, and the insurance demand is imminent and must be met. Only to-day a 3-day-per-week service to and from Holland is announced, leaving London at 10 a.m., arriving at Amsterdam just after mid-day. Other routes will quickly follow. It may not be so very long before the airship commercial service between England and America is in full swing, and its insurance requirements being met by a pool membership of American and British companies with the *Casualty Actuarial and Statistical Society of America* in charge of the accident statistics on the American side. Such a vision requires much less imagination than would have been necessary even three years ago to visualize to-day's status of commercial aviation. Mr. Cowles may yet be called upon for a further edition of his thought-provoking paper .

MR. H. E. FEER:*

When England issued the first policy on an airplane seven years ago, before the war, she had the privilege of being the first country to write this line of insurance. At that time the risks covered were principally so-called "school risks"; that is, insurance given against the accidental damage to which a machine is exposed if used by a pupil during his instruction. Later on, at the time just before the war, and during its first months, the insurance cover was extended to comprise also delivery flights; that is the flights of a newly bought machine from the seller to the purchaser. The premium for this latter insurance was fixed according to the miles flown, but most probably also varied according to the different routes. Yet, after a short time this development was entirely interrupted by the war, and it was only after the armistice that a resumption took place. Aviation insurance then began to make rapid progress, and there is no doubt that England possesses today the best developed market for this line. It is now possible to cover at London every kind of risk, pertaining to the crew as well as to the ship, and for both heavier and lighter than air machines. The competition in this business is very keen, and in an article of April 24th, 1920, the English magazine "The Policy" quotes: "Taking a general view of aviation insurance during the past year, one is struck with the extraordinary competition for so little an amount of business."

I do not wish to enter into a discussion of the situation in other European countries, such as France, Italy, Switzerland, etc., in all of which aviation insurance is engaged in only to a limited extent,

* Mr. Feer, who was in this country at the time of the meeting representing a Scandinavian insurance pool, was invited by the President to present this statement to the Society.

and I turn towards Scandinavia, which, for the purpose of my remarks, comprises Finland also. During the war Scandinavia had very little commercial flying. In 1919 this situation was somewhat changed, but I still doubt whether aviation in the northern countries is as much developed as in England, where today a widespread net of air lines is in operation. The Scandinavian insurance companies, however, realizing future possibilites, decided to carry aviation insurance from the very beginning along sound lines, and they therefore organized during 1919 a pool, the Scandinavian Pool for Aircraft Insurance, which is represented in the United States through my office. The pool comprises today about ninety companies, to which it prescribes a strict underwriting policy, in fixing forms, rates, rules for adjustment, etc. I believe I am safe in saying that probably no better organization of this kind exists today in the whole world.

Besides this underwriting organization, the pool provides also for a proper statistical institute, the "Aircraft Registry Inc." The purpose of this second organization consists of gathering and keeping in a proper way all statistics pertaining to aviation insurance and in making them available not only for the pool, but also for any other insurance company in the world.

As it is the particular object of your Society to analyze and study the making up of premiums and tariffs, I may now turn towards a short discussion of the tariffs of the Scandinavian pool. Of course it has to be considered from the very beginning that these tariffs are drawn up to fit entirely different conditions, and that American underwriters, in fixing their own rates, can in no way consider them as a precedent. However, they give a good insight into the lines along which the pool carries on its business and into the opinion of the Scandinavian companies about aviation insurance in general.

Actually there are two tariffs available, one of which applies to machines heavier than air; the second of which applies to machines lighter than air. I might say that the basis on which they were drawn up, was most probably the same as that of the American tariffs, namely, a certain number of governmental figures pertaining to military and naval flying and a great amount of guess work. In fact the pool itself does not consider them as tariffs in the usual sense of the word, but merely as a set of instructions, to be used by the central rating office in a general way, when fixing the rate for a special risk. I may here quote from the "General Conditions of the Pool":

"As aerial voyage risks are singularly unequal regarding the machine, the pilot, the rates, etc., and as the corresponding premium has to be changed very promptly, it has not been possible to work out any premium tariffs in the usual sense which could be used by the company without further reference. Until further notice the premium will be fixed for every single case by the department managements, for the guidance of which the pool management has fixed certain minimum premium rates."

The order of the two tariffs is as follows:

Tariff number one covers: accident, liability and so-called "transport" insurance; that is, insurance against all risks pertaining to the ship itself or to its cargo.

Tariff number two is less developed and provides, so far, for accident and liability insurance only.

Accident Insurance.—Accident insurance is issued either as a trip or as a period policy, and is given as a complimentary cover to a usual accident policy which has already been, or is still to be taken out. The applicant is subject to a medical examination. The highest amount of insurance granted is Kr. 100,000 and the minimum premium is fixed at Kr. 20. Beyond that amount the rates are graded according to the relation between the assured and the use made of the machine; they are different for private owners operating their own machines, for pupils and teachers of a flying school, for employees and workmen of an aircraft factory, and for third parties attending a flight as passengers. The rates are generally fixed for one, three, six or twelve months, except for passengers who pay the premium of their trip tickets according to the route then intended to fly; for instance, Stockholm-Copenhagen, Stockholm-Christiania, or Christiania-Goteborg, etc.

Liability Insurance.—The cover given under this line amounts to Kr. 50,000 for one person and Kr. 150,000, for one accident, together with Kr. 20,000, for property damage. The rate is different for private flying machines, for flying schools, for industrial demonstrating and delivery flying, and finally for commercial flying in general. All premiums are computed on a one year basis and the first hundred kroner of every plane are charged against the assured. This is a kind of exclusion of small losses, as it is practised by the American aircraft underwriter in the aircraft collision line. Other important features of the Scandinavian policy are the exclusion of all flights between one hour after sunset and one hour before sunrise; furthermore, the right of the company to cancel the policy within four days after any loss. The question of the qualification of the pilot bears a certain, although very limited, importance for this line, inasmuch as flights executed by non-certificated pilots are not covered by the insurance.

In this connection I may mention that the Scandinavian tariffs do not appear to provide for that line of business, which is probably the most interesting one for the present meeting, namely, compensation insurance. As you will have remarked, this cover is contained to a certain extent under the heading "Liability Insurance," but nowhere is it dealt with as a separate line of business, and I am at a loss to say exactly how far it comes under the scope of the activities of the pool. The reasons for such an apparent lack of an important line of the business may be different in each of the four Scandinavian countries, but most probably it has to be sought either in the absence of compensation laws at all or if there are any, in their limited application.

Transport Insurance.—As I have mentioned before, the Scandinavian pool includes under this heading all damage sustained by the ship itself or by the merchandise and other kinds of cargo transported in it. Consequently transport insurance covers the risks of landing, falling, striking against fast moving objects, collision, fire, explosion, lightning, and theft. Variation of the rate is made according to whether the machine is used for pleasure, for instruction, for demonstration, or for general business purposes.

I have intentionally abstained so far from reading any figures, either concerning rates of premiums. Of course such figures are contained in the tariffs, but as the amounts, terms, the currency and all other general items that they refer to, are entirely different from the American ones, I feel that their indication would be of very little interest. The knowledge of European rates is only of value for an American underwriter if he can compare them with his own, that means, if all pertaining points are brought on an identical basis. In our case, however, such a procedure is extremely difficult, and if I try to do it for a few rates, in spite of all obstacles, I am aware myself, more than anybody else, how limited the correctness of my figures is.

The first comparison I have made covers rates for personal accident. A private owner piloting his own plane pays under the tariff of one of the leading American companies $15 on a principal sum of $100. The same man while flying in Scandinavia would pay Kr. 12 on Kr. 1,000. If an owner is not piloting, but traveling in his own plane as a passenger, he pays in the United States $8 on $100, while in Scandinavia the rate is the same as mentioned above, namely, Kr. 12 on Kr. 1,000.

In property damage and liability insurance a comparison shows the following:

A *sea plane* used for pleasure pays for property damage the following in the U. S. and Scandinavia, respectively:

$	5.00	on a cover of $	100.00
Kr.	3.75	on a cover of Kr.	1,000.00

The liability rate on the same plane would be in the United States and Scandinavia, respectively:

$	1.00	on a cover of $	100.00
Kr.	1.50	on a cover of Kr.	1,000.00

If the same plane is used for commercial purposes, the property damage rate is in the United States and Scandinavia, respectively:

$	5.00	on a cover of $	100.00
Kr.	9.375	on a cover of Kr.	1,000.00

On liability insurance, the rates are respectively for the United States and Scandinavia:

$	1.00	on a cover of $	100.00
Kr.	3.75	on a cover of Kr.	1,000.00

If a *land plane* is used for pleasure, it pays for property damage in the United States and Scandinavia, respectively:

$	5.00	on a cover of $	100.00
Kr.	7.50	on a cover of Kr.	1,000.00

For liability insurance:

$	1.50	on a cover of $	100.00
Kr.	4.50	on a cover of Kr.	1,000.00

Finally, the rates for a *land plane* used for commercial purposes, for property damage, for the United States and Scandinavia, are respectively:

$	5.00	on a cover of $	100.00
Kr.	15.00	on a cover of Kr.	1,000.00

and for liability insurance:

$	1.50	on a cover of $	100.00
Kr.	9.00	on a cover of Kr.	1,000.00

The most striking conclusion to be drawn from this comparison is doubtless the extremely low level of the European rates. The limited time does not allow me to discuss the reason for this fact, but for general information I may mention that this difference between European and American rates shows not only here but, as far as I know, in most lines of insurance. I may refer here to the general fire rates, which in Europe are always at least one fifth of those in the United States.

All of the foregoing remarks are based exclusively on information and documents, and not on personal investigation. Their value is, therefore, very limited and I myself am eager to amplify my knowledge of European aviation insurance by personal study. I hope to visit Europe this summer and to spend a good deal of my time on aviation insurance, and I trust, after my return in September, to be able to report more completely and more accurately on aviation insurance in Europe than I can at the present time.

CONTRIBUTED BY INFORMATION DEPARTMENT, MANUFACTURERS AIRCRAFT ASSOCIATION.*

1. *Mr. Cowles points out that the field for aircraft insurance is limited.* The uses of aircraft from the underwriters' point of view may be placed into two classes, governmental and civilian. It is

* This memorandum was prepared and presented to the Society upon the invitation of the President.

with the civilian use of flying machines that we are most concerned, because federal agencies will either take care of their own insurance or be guided by the rules governing commercial aviation and its insurance. Though the development of commercial aviation dates only since the signing of the armistice, it is rapidly progressing past the point where the public considered the airplane or dirigible only a dangerous toy. People now take aviation seriously. They are being impressed daily with the idea that one day everybody will have an opportunity to fly. They naturally look to the insurance companies for protection. This is where the actuary enters. This is where the actuary looks to those engaged in commercial aviation for correct figures and statistics.

Commercial aviation is slowly but surely becoming an established fact. Transportation companies are being organized in every state in the Union. These companies, many of them at present only entering upon the capital-seeking stage, propose to carry passengers, freight, mail and express. They have charted routes between large cities. They have interested all the trade bodies along their aerial right of way. These chambers of commerce or boards of trade are in turn spurring on their municipal governments toward the establishment of municipal landing fields, air terminals whereon the planes may alight for passengers and freight and for fuel and supply.

A recent survey made by the Manufacturers Aircraft Association discloses that at least fifty aerial transportation companies are in process of organization. Some of them will never get beyond that point, it is true, but the others, if properly officered and financed, have wonderful opportunities for expansion.

Then there are industrial and engineering concerns which have been compelled to adopt airplanes to facilitate transportation on certain occasions. Trial of this new vehicle has frequently resulted in keeping a certain number of planes for the important duty of carrying company officials or papers, often supplies and payrolls. It would be impracticable at this writing to check up the number of private industrial organizations actually employing the airplane daily. There are, perhaps, a hundred. More important for our purpose, however, is the fact that the number is gradually increasing.

The companies using aircraft and trying to place it on a paying basis are increasing. There is a real tendency on the part of all manufacturing and operating companies, such as the passenger and freight carrying services, to practise straight flying, eliminating the element of risk attached to exhibition, stunt or trick performance.

Aircraft are being used for pleasure travel. While the newest commercial machines, produced by representative manufacturers of America, have not yet been placed in quantity production, there is every indication that, given the proper encouragement, quantity production will be possible. Mr. Cowles has ably discussed the handicaps that aviation must overcome before we reach this state. Chief among these are the expense, both initial and overhead, and the risk. Quantity production will reduce the expense. Aerial laws and increased efficiency in design and structural stability will go far toward reducing the disk. Of these two needs we will write at greater length after considering the lighter-than-air, or dirigible type of commercial aircraft.

2. *Mr. Cowles sees danger in the use of inflammable gas in dirigibles.* Lighter-than-air machines are not yet used to any extent commercially; that is, in America. Great Britain and Germany are leading the world at present in the use of dirigibles for the transportation of passengers and freight. Both countries favor the dirigible for long distance flying as well as for carrying great loads. They believe that lighter-than-air craft are more practical and economical where long hauls and heavy loads are to be considered. Yet, neither Great Britain nor Germany looks for a general employment of dirigibles within the next five years; this because of the prohibitive expense involved in construction of both machines, hangars and terminals. On the other hand, it is pointed out by airmen in all countries, that if the Government finds it profitable to employ dirigibles in time of war, their use commercially would create a valuable aerial reserve, one that could not be created over night.

The same care enters into the building of a dirigible that entails the production of a large steamship or man-of-war. It requires months to build dirigibles, even after the design has been proven feasible. It follows that much capital must be available to finance a dirigible transportation line.

Now, as to the fear of inflammable gases. The discovery of helium, a non-inflammable gas, in the latter months of the war, opened up a new field for the lighter-than-air machines. Helium, it is expected, can be manufactured in sufficient quantities and at a not unreasonable cost, providing factories are especially built. It is a by-product of natural gas. When we have facilities for the production of helium no longer will hydrogen be used either for reasons of economy or convenience. But on account of the great capital investment required no one looks for a general use of dirigibles within the next few years, though important plans are being completed for dirigible lines in this country to-day. It is therefore logical that we base our conclusions on the increasing popularity of the airplane, a motor-propelled body with wings.

3. *Mr. Cowles believes that the field for aircraft insurance is yet to be developed.* Mr. Cowles is of the opinion that the field for aircraft insurance is yet to be developed. We believe that he is correct in his opinion. But we would qualify his next statement to the effect that aircraft themselves "are yet to be manufactured and sold." This was doubtlessly true at the time of compiling the figures on which Mr. Cowles based his conclusions. But that was

ten months ago. The war machine was then being sold for sport and passenger-carrying purposes, invariably to ex-service men who saw in it a medium for continuing the fascinating sport which took thousands of our young men into the Air Service during the war.

Finding that they could realize quicker and larger profits from exhibition flights rather than from straight commercial flying, the pilots owning their own machines for several months continued to impress upon the public consciousness the erroneous belief that the flying machine was a dangerous toy and the person who travels through the air a daring adventurer. But the last few months have brought about a change, as romantic as it is radical.

The designers and manufacturers, realizing that the war machine, built with the element of speed and maneuverability as prime essentials, did not combine stability and airworthiness required in a safe commercial machine, set about to design machines for civil and commercial use. And they are succeeding. Here in America several companies have brought out, since January last, machines that are as safe, all things considered, as any other vehicle on earth. That is, the machine itself is as safe. Planes do not fall to earth any more, nor do they tear apart in mid-air. Expert engineering based on the laws of physics has provided against this. Before the war a machine when completed was taken out to the flying field and the entire staff gathered about to see if it would actually fly. To-day all the experimenting is done with models. When the finished machine goes out of the shop, the pilot knows it will fly. It may possibly require certain improvements, but these are confined to details.

There are at least ten American commercial machines of radically different types on the market to-day—all built for straight flying, all, by actual test, both on the field and in the air, as safe as a motor car or other conveyance that travels faster than fifteen miles an hour along congested highways. While there are fewer than two hundred of these machines in commercial use to-day, there are more than that number of persons endeavoring to sell commercial airplanes and flying boats.

These American machines carry from one to ten passengers. Some of them are designed to carry 1,500 pounds of cargo, mail or freight. They are in operation to-day, on the aerial mail routes between New York and Chicago, or on the aerial freight routes along the Atlantic or Pacific Coast or in the Middle West.

Airplanes are being used to-day for actual transportation purposes. The next generation will know as much about airplanes and flying as we to-day know about the automobile. The hardest task before us is to prove that the airplane is really as safe as we know it to be. Twenty years ago there was the same general apprehension regarding automobiles and fifty years earlier railroad travel was considered as a hazardous, uncertain adventure.

The field for aircraft insurance is yet to be developed to be sure, but if we do not attempt to do the thing over night and follow a plan dictated by ordinary business prudence the development should progress satisfactorily and certainly.

4. *Mr. Cowles says: "Will the history of the railroads, trolleys and automobiles be repeated in the future history of aircraft?"* The history of all transportation facilities is replete with scorn, doubt, wonder and then final acceptance on the part of the public. As Mr. Cowles states, we find something in the motor car more closely approximating aircraft than anything else. The motor car was at first used as a sport machine. The public at large feared and hated it, but years later came to accept it as a legitimate means of transportation. Present facts being more easily checked up than predictions of the future, what do we find to-day? Whenever a man considers flying, the first thought that strikes him is whether or not it is safe. He pauses to weigh his chances in the air, much as you and I in the early days of the motor car hesitated before climbing in alongside the driver of that old one-lunged red-devil.

To-day the deaths and injuries per mile via air are not only smaller but the percentage of contestants finishing is greater in the aerial races despite the unquestionable fact that conditions under which the two kinds of speed contests are conducted favor the motor car.

We venture to suggest, also, that the aerial races of to-day are real races, and not tours, and as such can no more be compared to commercial aviation than the speedway events of motordom can be used to prove the dangers of boulevard touring. Let us compare the conditions under which the air race and speedway competitions were held. Take the matter of distance. The New York-Toronto air race, held in the summer of 1919, covered 1,042 miles and the transcontinental air race later the same season 2,701 miles.

The longest motor contest of the year was the 500-mile race on the Indianapolis Speedway. The physical conditions of the course favored the motor car. The drivers on a bright clear day, after weeks of practice which had made them familiar with every well-paced inch, circled a banked course which had been prepared for them at a cost of hundreds of thousands of dollars. In their kits were everything which their skilled mechanics required to insure perfect performance of man and machine.

Compare these conditions with those under which the transcontinental race between New York and San Francisco was held. The army fliers were required to make twenty stops on as many fields, no two landing places alike, and on all twenty fields practically a negligible sum of money expended. Many of the pilots had never flown cross-country before. Individual resourcefulness was put to the hardest test.

The following figures have been set down, computing the number of miles per fatality, considering the full distance, of course, for the machines which finished and half the distance as a general average travelled by the machines which failed to complete the course.

Indianapolis Auto Race 1919:

Distance	500 miles
Highest speed	110 m.p.h.
Winning average	88 m.p.h.
Machines started	33
Machines finished	15 (45 per cent.)
Deaths	3 (4,000 machine miles per death)

New York-Toronto Aerial Race:

Distance	1,042 miles
Highest speed	135 m.p.h.
Winning average	128 m.p.h.
Machines started	52
Machines finished	30 (59 per cent.)
Deaths	0 (42,722 machine miles and no deaths)

New York-San Francisco Aerial Race:

Distance	2,710 miles
Highest speed	135 m.p.h.
Winning average	120 m.p.h.
Machines started	62
Machines finished	31 (50 per cent.)
Deaths	7 (17,940 machine miles per death)

In the transcontinental air race, all other conditions forgotten for the moment, the number of entries and distance travelled gave ten times the opportunity for accidents and failures that the Indianapolis race gave. The speed figures show that the strain on airplane and engine are as great as that on racing car and motor.

The transcontinental air race served two purposes. (1) Educational, that is, showing the public the possibilities in aerial travel, and (2) experimental. It served to lay the first transcontinental aerial route in any way resembling the aerial highway of the future. The experiment of laying out a course across the United States with twenty stops at intervals of not more than 180 miles and then having the pilots find these spots by means of maps, compasses and general sense of direction was successful. When these fields are properly leveled off, connected up by emergency fields, and directional wireless and then flown over by pilots familiar with their every feature, aerial traffic will have entered into its own.

Inasmuch as the belief has been expressed that great danger attends dirigible flights, the following is offered:

The history of real airship (lighter-than-air craft) flying in the United States dates from early in 1917. There have been constructed since then approximately 60 gas-filled airships for use in the United States. Nearly all of them, however, were for Naval use, and were flown under war or training conditions. Pilots have taken chances with these ships that one could hardly expect them to take in peace times. Yet it is interesting to note the kind of accidents occurring and their causes.

(1) *Fire.* Three cases:

(*a*) Car caught fire on the ground from careless handling of gasoline. Instead of extinguishing the fire, the crew liberated the airship; and the fire spread to the envelope. No fatalities.

(*b*) Ship caught fire on the ground from unknown causes. No fatalities.

(*c*) Ship caught fire in the air, presumably due to a chance electric spark at the gas valve at the moment when gas was escaping. Two passengers saved by parachute. Three others killed.

Static electricity around the value is practically the only danger in connection with using hydrogen gas. This subject is being intensively studied and promises to eliminate such accidents in the future. Airship builders assert that all available figures prove that in the whole history of airship flying there have been only four disasters due to the ship catching fire in the air. It is considered possible that only one of these was caused by use of hydrogen gas.

(2) *Damage in forced landings (not at regular flying fields).* Ten cases on record. No fatalities.

(3) *Damage due to wind.* Two cases:

(*a*) Ship was landed and held by ground crew without aid of hangar or mooring attachments. Later blown loose in wind storm and lost at sea.

(*b*) Ship was blown against hangar by strong gust of wind which ripped the envelope. No fatalities.

Note—Such accidents will be eliminated by efficient mooring and housing apparatus.

(4) *Collision.*

One case: A student pilot steered into a kite balloon which was anchored. No fatalities.

(5) *Loss of control at sea. Rescue and salvage by steamship, involving damages to airship.*

(*a*) Due to rudder failure. No fatalities.

(*b*) Due to engine failure. No fatalities.

Practically all airships now carry wireless, thereby making rescue as certain as that in the case of steamships, with the added advantage of more often being able to keep longer afloat and adrift in the air than in the case of vessels.

The following figures of dirigible balloon flights at the Goodyear

Balloon School and Naval Air Station at Akron, Ohio, may be of interest. They cover flying during the entire period of the war:

Number of dirigible flights	769
Number of passengers carried	2,711
Time in the air	632 hours
Approximate distance flown	25,280 miles

Those figures show the activities at only one air station.

As to the types of policies desirable for airships, aeronautical experts believe that fire, tornado or wind damage and travelers' accident tickets are most important.

It is considered most desirable to base a travelers' accident ticket on the length of time to be occupied in making the flight, such as making the premium so much per hour or fraction thereof.

So much for the safety of aerial travel. What are its needs at present?

5. *Mr. Cowles asks if aircraft will be as economical to operate as the auto truck and motor car.* The cost per mile of operating aircraft is somewhat more than that of any other vehicle. But when one considers that aircraft travel in a straight line this overhead is reduced considerably. In fact, there are persons who assert that given any known distance, the airplane can traverse it at less expense per pound load than any other vehicle, this due to the economy of the straight line. The question remains, will the airplane get its load there surely and safely?

Aviation makes such universal appeal that it is comparatively easy to raise capital by popular subscription for new aeronautical enterprises. One of the perils of the present situation lies in the fact that it is easy to capitalize this enthusiasm and public spirit, providing money for the development of aviation enterprises that are doomed to failure because a proper foundation has not been prepared.

It is hardly fair to assume that the airplane can get its load to a given destination swiftly and surely without outside aid. At the present time the right to fly over your property has not been definitely determined. No law has been passed establishing and limiting the liability of the operators of private aircraft. Suitable terminals, airports, landing fields or repair stations do not exist. Methods of signalling must be improved, and new methods adopted for flying in bad weather and navigating at night. There must be a vast amount of preparatory work before an aerial transport company can operate on a regular schedule.

Lives and money unnecessarily sacrificed in premature attempts to force the development of commercial aviation upon conditions wholly artificial, will tend to restrict a permanent and reasonable growth of aviation in America. One of the great handicaps of aeronautical development for many years has been a tendency to overstate and promise successes far beyond the actual power of possible achievement.

If we are going to successfully establish in the public mind through experience the truth that aviation is something more than the use of dangerous toys, we must recognize facts, face them and state them. Popular imagination starting with the true facts will provide all the stimulus required. Manufacturers and designers of aircraft are united in their opinion that something along these lines must occur if commercial aviation is to develop to the point of popularity and pecuniary success.

(*a*) The enactment of a federal code establishing control of the air and authority for the administration of such control; followed by concurrent legislation in all the several states providing "Rules of the Air Lanes," and inspection and licensing of machines and pilots.

(*b*) Establishment of aerial harbors for land and water craft, development of a system of meteorological and radio information for all aviators.

(*c*) Coördination in Washington of the various conflicting Government activities having to do with the purchase, operation or scientific development of aircraft.

(*d*) Adoption by the government of a constructive aeronautical policy which will make it possible for capital to remain in the business and thus insure maintenance of a basic aircraft industry for purposes of commercial development and national defense.

All of these present needs are imperative if aeronautics is to be worth the attention of insurance companies. Wise and just laws protecting aviators, property owners and passengers of aircraft will automatically form the basis for insurance rates. Until we have these laws, there is no telling just what irresponsible persons will do with flying craft. They may endanger themselves, their machines, their passengers or the property and property owners over which they attempt to fly. Aviation must be controlled by the government as far as inspection and licensing of aircraft and pilots are concerned. The government must see to it that aerial law is made uniform throughout the United States.

Then we will find the safety first program in operation, inventors encouraged to develop new and improved types of flying machines and the public less prone to doubt their practical value. Once remove this subconscious antagonism on the part of the general public and insurance companies will add aeronautical departments just as they were compelled to establish automotive branches a few years ago.

MR. EDW. R. HARDY:

The President asked me to discuss the fire phase of Mr. Cowles' paper on Aircraft Insurance.

You will recall that when the Century Dictionary was first published, in its preface or introduction the editor stated that in many cases the origin of words could not be discovered. He added that the fullest search in many cases only led to what they must call a negative result, that is, the origin of the word must be put down as unknown. I am very sorry, but that is the report I must bring to you on the fire phase of the aircraft insurance. I have communicated with all those both here and abroad who were engaged in this form of insurance only to ascertain a negative result. There is no experience, which of course is what we ought to have, to make my remarks of any value. Like the editor of the Century Dictionary I can only report a negative result. So far as any experience in this form of loss is concerned, it is still to be experienced.

ON THE GRADUATION OF FREQUENCY DISTRIBUTIONS—H. C. CARVER.

VOL. VI, PAGE 52.

WRITTEN DISCUSSION.

MR. A. H. MOWBRAY:

Mr. Carver's very interesting paper begins with the observation that the graduation of frequency distributions is that branch of actuarial and statistical theory which has been most neglected—in America, at least. It is true that the graduation of frequency curves has not been given much study in America; although American actuaries have given very considerable attention to the correlative but more specialized work of graduation of mortality tables. In our casualty work, so far, we have had little occasion for the use of such graduations, but I believe that in the course of time the occasion for such work is apt to develop not infrequently.

In dealing with the Pearson system of curves, Mr. Carver raises no question as to their practical sufficiency, but calls attention to the vast amount of work involved in making such a graduation. From having undertaken one or two such graduations, I can testify that he does not overestimate the amount of work so required.

In Section II of his paper Mr. Carver very frankly admits his indebtedness to Prof. Pearson's original suggestion in his approach to the problem, which is indeed evident from the similarity of his basic equation to Prof. Pearson's. It is rather surprising to find that the finite difference equation works out a simpler method of graduation than is developed by the method based upon the infinitesimal calculus. I think it is a usual experience that finite difference formulæ are more complicated and solutions by this method apt to be more difficult.

While the simplicity of Mr. Carver's formula is a great advantage over Prof. Pearson's, it seems to me there are certain apparent advantages about the Pearson system which may, however, in the end prove not to be real advantages after all. The several distinct forms of Pearson's equations indicate a law of formation determined by the constants and a different law of formation in each case. Mr. Carver's formula is universal and he does not distinguish, in setting forth its forms, the different types of distributions. It must be true, however, that varying relations between his constants, similar to the varying relations between the constants in Prof. Pearson's basic equation, define the different types of curves quite as sharply as they are differentiated in the Pearson system, and it is perhaps to be regretted he has not dealt with this phase of the matter and the determination of characteristic points in the distribution in terms of the constants or moments.

A mathematical law of formation with certain constants to be determined from the characteristics of the particular material to be used is sometimes of substantial advantage, even though the law itself is somewhat complex. As an illustration of this, we may recall the long search for a law of mortality, and while a definite law has not been established, the approximations thereto developed by Makeham have been of very great value to actuaries. In Prof. Whitney's reasoning underlying the present experience rating plan, frequency curves were also made use of, assuming a certain law of formation of a curve, although for the purpose he had in hand it was not necessary to know the precise constants. Mr. Carver's proposals do not appear to lead to a law which may be used in this way. On the other hand, some of Prof. Pearson's formulæ are so complex that it hardly seems likely they would be of great use in a similar investigation of basic principles.

The equations between the constants developed in Section III by Mr. Carver are developed in accordance with what is known as the "method of moments." It occurs to me there may be a sufficient number of our members not familiar with this method that a brief word of explanation may be valuable. The name of the method is derived from the mechanical principle of moments and the fact that in calculating physical moments the force applied at a given point is multiplied into its lever arm. The abscissa of a given function may be considered as equivalent to the lever arm and the ordinate to the measure of force applied at that point. In accordance with this analogy, the first moment of a given function about a given point is taken as the sum of the products of the successive values of the function (the ordinate) and the first power of the variable (the abscissa). The second moment is the similar sum of the products of the function and the second power of its variable, and so on. The method of determining the constants of the formula by this means is by calculating the successive moments from the original data and from the formula to which the data are to be fitted and equating the successive moments in the two distributions. The moments calculated from the formula must of necessity be expressed in terms of its constants, and in this way enough simultaneous equations are formed involving the constants of the formula to enable their determination.

There is an unfortunate printer's error in some of the group of equations (II). The exponent of the factor $\overline{r-1}$ in the second term on the right of each of these equations should be the same as the exponent of the factors $\overline{s-1}$ in the first term. The shortening of the work so indicated may perhaps be made more clear if the work which is actually to be undertaken were set forth in schematic form. The following column heads would indicate how the work might be done:

(1)	(2)	(3)	(4)	(5)	
x	f_x	xf_x	x^2	x^2f_x	etc.

The powers of the successive terms may be copied on the sheet from tables and then each term be multiplied by f_x, setting up the value in a machine and multiplying across the line before dropping to the next line. The totals then being summed, the formulæ here quoted may be applied to the total. It occurs to me that perhaps Mr. Carver may be able to suggest a summation method of determining the moments after the manner frequently used in fitting data to Pearson frequency curves.

There is also an unfortunate printer's error in equations on page 56, with the exception of Equation IV, which error likewise occurs in several places subsequently in the text, namely, the writing of v for ν. The error is perhaps excusable in view of the similarity of the characters, but it is regrettable that it was not caught when the proof was read.

The comparison between the constants in Mr. Carver's graduation formula and Pearson's basic differential equation is very interesting and suggests that in this way Mr. Carver might be able to set forth the relations between the constants in his formula which will distinguish the different types of curves and perhaps indicate in terms of the constants, as is possible, with the Pearson system, the location of the mean and the mode and the measure of skewness, the characteristics above referred to.

The illustrations used seem well adapted to show the use to which the system may be put, but Example II seems to me ill chosen from another point of view. On examination of Chart II, it will be noted that the greatest maximum is at \$24; the next greatest at \$21, with other pronounced maxima at \$14, \$12 and \$30. The graduated curve does not show any such distinctive points. When we note that \$24 per week is the weekly wage corresponding to \$4 per day for a six-day week, that \$21 per week is the weekly wage corresponding to \$3.50 per day for a six-day week or \$3 for a seven-day week, that \$14 is the weekly wage corresponding to \$2 per day and seven days per week, that \$12 is the weekly wage corresponding to \$2 per day for a six-day week, and \$30 is the weekly wage corresponding to \$5 per day and a six-day week, we will appreciate that these are characteristic variations in the curve, and the general rule as to graduation that characteristic features are not to be graduated out ought generally to apply.

If by graduation of a series of figures the correlation in the variation of two variable attributes may be made more clear, then we are justified in graduating and destroying to some extent certain characteristic features, and even then the value of the correlation study is to that extent diminished. But unless the graduation can serve some useful purpose along these lines, I think the general opinion is that it is better to attempt no graduation. I happen to know that about a year ago the Actuarial Committee of the New Jersey Rating Board was working on a correlation problem along these lines (upon which, however, we did not get satisfactory results) and Mr. Carver made this graduation for us to assist in that work. I think it is a bit unfortunate that he has chosen it as an illustration of his method and I have criticized the graduation of these figures mainly to bring sharply to the attention of students first taking up this type of work the impropriety of so graduating statistical data. As Mr. Carver puts the graduation forward as an illustration of method rather than a result for use, the criticism does not apply with the same force.

Mr. Carver's Example IV, where one basic series is split into two others, is interesting and illuminating, and particularly his expression of views, in comparison with those held by Mr. Fisher, are also interesting. My own personal view does not agree entirely with either that expressed by Mr. Carver or Mr. Fisher. I am willing to grant that a mortality table might be constructed from the records of death only if all the deaths were properly assigned to causes and if the varying causes were found to have clustering points about certain ages and the frequency distribution of deaths from those causes around the clustering point could be fully worked out. The difficulty is that it is clearly impossible to work out a frequency curve for each particular cause of death and that variation in the judgment exercised in selecting the causes to be grouped may give entirely different age distributions of death entering into the several frequency curves. It would also appear that the relative scales of the subsidiary curves would depend on the age composition of the population contributing the deaths, and this would not be known. My view, therefore, is that while it might be possible so to construct tables, it is exceedingly dangerous to do so without much greater knowledge than I believe we now possess, and the danger is enhanced by the absence of a criterion by which to judge results when the table is done.

The comparisons in Section V of the paper between Charlier's Type A curve and Prof. Pearson's system and Charlier's Type B curve and Mr. Carver's are most interesting.

It is to be hoped that Mr. Carver's paper may lead some of our other members with a natural inclination for higher mathematics to give us further papers along these lines and thus in time develop a facility for handling the mathematical side of our statistical work by the most highly developed modern methods.

MR. R. HENDERSON:

Mr. Carver has rendered a very practical service to all of us by calling attention to the possibility of using a law of frequency distribution based on finite differences in place of Pearson's system of curves based on a differential equation. While many of us will continue to feel that Pearson's system is from the theoretical side more satisfactory than that proposed, we will all recognize that in practical application the final results are likely to be for all practical purposes identical. While Mr. Carver's formula presents advantages arising from the uniformity of method applicable to all distributions, I cannot but think, however, that he feels somewhat too strongly the difficulty of applying the Pearson system.

In the monograph on Graduation published by the Actuarial Society as No. 4 of its series of actuarial studies the authors give a set of equations for determining the constants and an examination of these equations will show that the only point where difficulty arises in practical application is in connection with the constant k and here Mr. Carver's method of using an arbitrary constant and then applying whatever ratio is necessary to bring out the correct total is equally applicable.

It is unfortunate for the comfort of the student who wishes to read this paper that the printer has apparently substituted the italic v for the Greek letter ν over a considerable proportion of page 56 of the text. I also regret that the author found it convenient to designate by the letter δ a function analogous to the reciprocal of the function designated by that letter in the Actuarial Society study.

The analogy between the difference equation of Mr. Carver and the differential equation of Pearson is to my mind more clearly brought out if the former is put in central form, thus

$$\frac{\delta y_x}{\mu y_x} = \frac{y_{x+\frac{1}{2}} - y_{x-\frac{1}{2}}}{\frac{1}{2}(y_{x+\frac{1}{2}} + y_{x-\frac{1}{2}})} = \frac{(c_1 - c_3)(x - \frac{1}{2}) + (c_2 - c_4)}{(x - \frac{1}{2})^2 + \frac{c_1 + c_3}{2}(x - \frac{1}{2}) + \frac{c_2 + c_4}{2}}$$

$$= \frac{-\frac{\nu_3}{\nu_2} - 2(5 - 6y)x}{\frac{1}{4}(8 - 9y) + \nu_2(4 - 3y) + \frac{\nu_3}{\nu_2}x + (2 - 3y)x^2},$$

where

$$y = \frac{\beta_1 + 4 - \frac{1}{\nu_2}}{\beta_2 + 3 - \frac{1}{\nu_2}}.$$

Compare this with Pearson's equation in the form

$$\frac{1}{y_x}\frac{dy_x}{dx} = \frac{-\frac{\mu_3}{\mu_2} - 2(5 - 6y)x}{\mu_2(4 - 3y) + \frac{\mu_3}{\mu_2}x + (2 - 3y)x^2},$$

where

$$y = \frac{\beta_1 + 4}{\beta_2 + 3}.$$

The differences here reduce to

(*a*) the use of uncorrected moments in one case and corrected moments in the other,
(*b*) the subtraction in Carver's equation of $1/\nu_2$ from β_1 and β_2 and
(*c*) the extra term $\frac{1}{4}$ $(8 - 9\gamma)$ in the denominator.

In the case of a suggested general law of this kind a particular case already known always adds to the interest. Such a case is that where n individuals are taken at random from a large number N of whom pN possesses a certain attribute and y_x represents the relative probability that exactly x of the N will possess the attribute.

Then

$$\frac{y_{x+1}}{y_x} = \frac{(n-x)(pN-x)}{(1+x)\{(1-p)N-n+1+x\}},$$

$$\frac{\Delta y_x}{y_x} = \frac{(n+1)(pN+1)-(N+2)(1+x)}{(1+x)\{(1-p)N-n+1+x\}}.$$

The success achieved by Mr. Carver in the graduation of the United States Life Table for males is remarkable even without any allowance for the fact that only two frequency curves are used and much more so when that fact is taken into consideration. The faithfulness with which the graduated values of l_x follow the general trend of the ungraduated values is almost uncanny and the small absolute departures and frequent change of sign testify to the goodness of fit. Our only regret is that the author did not give us more details of the actual work so that the student might be better able to follow it.

MR. MERVYN DAVIS:

Two standard plans for applying the method of moments to the graduation of frequency distributions have hitherto been generally recognized, these plans being based on the assumption that the frequency function either satisfies a fundamental differential equation or that it may be expressed as a function or series of known form. Mr. Carver's paper presents a third plan in which the frequency function is defined by a difference equation of the first order analagous to the fundamental differential equation of the Pearson types of frequency curve, the graduation being based on the central ordinates of the areas representing the actual distribution. The Society is, therefore, indebted to Mr. Carver for an interesting and instructive study in the method of movements while actuaries and statisticians in general are still further indebted to him for a method of graduating frequency distributions which should produce results approximating those obtained by Pearson's method but at the cost of considerably less labor.

It cannot be claimed, however, that Mr. Carver's method rests on so complete and logical a basis as Pearson's. In the first place the graduation is based not on the areas which naturally represent the frequency distribution but on the midordinates of those areas. And in the second place the integration of the difference equation will not in general determine a frequency function which will represent the distribution throughout the range covered; all that the difference equation will furnish being the values of certain equidistant ordinates, the curve being completed by drawing a series of parabolic curves through the extremities of these ordinates.

To illustrate this latter point it may be noted that Mr. Carver's difference equation may be written in the form

$$\frac{y_{x+1}}{y_x}=\frac{x^2+c_1x+c_2}{x^2+c_3x+c_4}=\frac{(x+\alpha)(x+\beta)}{(x+\gamma))x+\delta)},$$

the integral of which is

$$y=K\frac{\Gamma(x+\alpha)\Gamma(x+\beta)}{\Gamma(x+\gamma)\Gamma(x+\delta)},$$

where

$$K=y_0\cdot\frac{\Gamma(y)\Gamma(\delta)}{\Gamma(\alpha)\cdot\Gamma(\beta)}.$$

Now if α, β, γ, δ are all imaginary, equations (2) will represent a true frequency curve extending to infinity in both directions and asymptotic to the axis provided $c_3 > c_1$: this will always be true in cases of high contact, for from equations IV, c_2 being positive, $1+2\delta$ must be and hence $4+2\delta$ or c_3-c_1. The values of y in this case are, therefore, completely defined by equation (2); the same remark holds in cases where such of these values as are real are also positive, the curve in such cases extending to infinity in the positive direction and terminating on the negative side where the abscissa taken negatively equals the numerically smallest of these values, y being infinite at that point if this value be found in the numerator and zero if taken from the denominator.

Where, however, any one of the quantities α, β, γ, δ is real and negative the curve given by equation (2) no longer resembles a frequency curve. As an example, the difference equation of Professor Carver's example III, may be written

$$\frac{y_{x+1}}{y_x}=\frac{(x-9.622)(x+4.441)}{(x=12.277)(x+4.114)}$$

and the integral in the form of equation (2) gives infinite values for y when $x = n + .622$, where n is any negative integer or positive integer not greater than 9, and zero values for y when $x = m + .277$, m being an integer not greater than 12.

It may be noted that no general expression can be found for the position of the mode which will have to be approximated either graphically or by interpolation.

In cases where there is not high contact at either end of the range the determination of the terminal areas will require special treatment. For not only are the quadrature formulæ quoted no longer available but allowance must be made for the fact that the area rests on a fractional, not a unit, length of the axis. The simplest method of estimating these areas would appear to consist in passing a parabolic curve through the points given by the succeeding or preceding ordinates and in determining the area required from the equation obtained.

It is somewhat remarkable that Mr. Carver implies, as he appears to have done in the examples presented, that the midordinates of ungraduated statistics may be taken as the average ordinates; the more so since he points out the necessity of applying a quadrature formulæ to the graduated ordinates. Such an assumption involves appreciable errors and it is therefore, desirable to devise adjustments to offset them.

The quadrature form quoted on page 61, is based on the assumption that y is a parabolic function of the fourth degree in n, i.e., that

$$y = a + bx + cx^2 + dx^3 + ex^4.$$

On the same assumption it may be readily shown that a formula giving midordinates in terms of areas is furnished by

$$y = Y - \tfrac{1}{24}(\Delta^2 - \Delta^3 + \tfrac{71}{80}\Delta^4)Y,$$

where

$$Y = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} y\,dx.$$

We thus have a formula for adjusting the ungraduated statistics in cases where there is high contact at each end of the range. At first sight it might seem that these adjustments would apply in all cases where there is high contact at the positive end. A little reflection, however, at once makes it evident that this is not so: for in such cases the initial area rests on only a fractional length of axis and until the graduation has been performed we have no way of determining at what point the curve starts.

Where, however, there is high contact at each end of the range, it will be simpler to apply the adjustments direct to the moments calculated from the adjusted figures. For in such cases from the equation:

$$\Sigma x^m \cdot \Delta^n f_x = \Sigma x^m (f_{x+n} - n \cdot f_{x+n-1} + nC_2 f_{x+n-2} \cdots)$$

by writing $\Sigma x^m \cdot f_{x+r}$ in the form $\Sigma (x + r - r)^m \cdot f_{x+r}$ and expanding, it may be shown that

$$\Sigma x^m \cdot \Delta^n f_x = 0, \quad \text{where} \quad m < n,$$
$$\Sigma x^2 \cdot \Delta^2 f_x = 2\Sigma f_x,$$
$$\Sigma x^3 \cdot \Delta^2 f_x = 6\Sigma x f_x - 6\Sigma f_x,$$
$$\Sigma x^4 \cdot \Delta^2 f_x = 12\Sigma x^2 \cdot f_x - 24\Sigma x \cdot f_x + 14\Sigma f_x,$$
$$\Sigma x^3 \cdot \Delta^3 f_x = -6\Sigma f_x,$$
$$\Sigma x^4 \cdot \Delta^3 f_x = -24\Sigma x \cdot f_x + 36\Sigma f_x,$$
$$\Sigma x^4 \cdot \Delta^4 f_x = 24\Sigma f_x,$$

the general equations being

$$\Sigma x^n \cdot \Delta^n f_x = (-1)^n \cdot \underline{|n}\, \Sigma f_x,$$
$$\Sigma x^{n+1} \cdot \Delta^n f_x = (-1)^n \underline{|n+1}\, \Sigma x \cdot f_x + (-1)^{n+1} \cdot \tfrac{1}{2} n \underline{|n+1}\, \Sigma f_x,$$
$$\Sigma x^{n+2} \Delta^n f_x = (-1)^n \tfrac{1}{2} \underline{|n+2}\, \Sigma x^2 \cdot f_x + (-1)^{n+1} \cdot \tfrac{1}{2} n \underline{|n+2}\, \Sigma x \cdot f_x$$
$$+ (-1)^{n+2} \cdot \frac{n}{24}(3n+1) \underline{|n+2}\, \Sigma f_x$$

If then μ's represent the moments of the adjusted ordinates we have

$$\mu_0' = \nu_0',$$
$$\mu_1' = \nu_1',$$
$$\mu_2' = \nu_2' - \tfrac{1}{12} \cdot \nu_0',$$
$$\mu_3' = \nu_3' - \tfrac{1}{24}[(6\nu_1' - 6\nu_0') + 6\nu_0']$$
$$= \nu_3' - \tfrac{1}{4}\nu_1',$$
$$\mu_4' = \nu_4' - \tfrac{1}{24}[12\nu_2' - 24\nu_1' + 14\nu_0' + 24\nu_1' - 36\nu_0' + \tfrac{71}{80} \cdot 24\nu_0']$$
$$= \nu_4' - \frac{\nu_2'}{2} + \frac{7}{240}\nu_0',$$

or, with unit total frequency, transferring the origin to the mean

$$\mu_2 = \nu_2 - \tfrac{1}{12},$$
$$\mu_3 = \nu_3,$$
$$\mu_4 = \nu_4 - \nu_2 + \tfrac{7}{240}.$$

These are Sheppard's adjustments so that it would appear that the errors introduced in the moments of the areas through considering these areas as concentrated along the central ordinates is equal to that produced in the areas of the central ordinates through taking these ordinates as equal to the average ordinates.

The presentation of the new method ends with an interesting example of its application to the graduation of a population table. The graduation of the deaths is given in full while it is stated that that of the number living may be made in the same way. A little further information on this latter point would be of interest, as also would be the completion of the illustration by the standard comparison of the actual and expected deaths. The weak point, if there be any, in the graduation of the death curve lies in the lack of any definite method, other than by inspection, of fixing the limiting point of the supplementary series β and the fact that series α is necessarily determined from a consideration of only the data at the older ages. To describe the entire population, a further supplementary series would evidently be required to cover the infantile ages, though, speaking offhand, I am inclined to the opinion that the inclusion of the supplementary series β in the example given is probably made necessary by unduly high exposures at the younger ages due to immigration.

I am glad to note that Professor Carver repudiates the idea that it is possible to correctly construct mortality tables from a consideration of deaths only.

The last section shows the connection between the Pearson and Charlier systems. I am unable, however, to agree with the author that we can, from this comparison, arrive at any reliable conclusion as to the comparative effectiveness of the two systems; for when considering the first few terms of a series as representing that series it makes a vast difference whether the series in question is the numerator or denominator of a fraction. In this connection it may be noted that the Charlier series fails completely in cases of appreciable skewness.

AUTHOR'S REVIEW OF DISCUSSIONS.

MR. H. C. CARVER:

I desire to take advantage of this opportunity to thank the gentlemen who have so kindly and ably led this discussion. Time will not permit me to answer in detail all the points which have been raised, but I shall venture a few remarks which may assist some of our members who are working along mathematical statistical lines.

There are two general classes of frequency distributions, distributions of "graduated variates" and distributions of "integral variates." The frequencies of the former should logically be represented by areas under a curve, those of the latter by ordinates of a curve. It follows, then, that from a theoretical standpoint Pearson's system is better adapted to deal with distributions of the first class than the method suggested in the paper under review, but the reverse is true for distributions of the second class, which are far less frequently met in the actuarial field of statistics.

The error that Mr. Davis refers to when he notes that I implied "that the midordinates of ungraduated statistics may be taken as average ordinates" is the same type of error that would be made if one attempted to graduate a distribution of integral variates by representing the frequencies by areas under a curve,—yet it may be noted that Pearson on page 401 of Vol. 186 of the *Philosophical Transactions of the Royal Society of London* has done this same thing, producing results thereby which seem to warrant the procedure. Furthermore, Charlier invariably treats the midordinates as average ordinates. A translation of a section of his "Uber Das Fehlergesetz" runs "we divide the errors into groups with constant dimension—α—and assume that all errors in such a group are of the same magnitude. The smaller we make the interval α, the smaller becomes the error which results from this assumption." I had this in mind when I made the statement that midordinates could be substituted for areas provided "the class interval is small as compared with the visible range." Thus it would be folly to use a quadrature formula for the distribution of Table II, though advisable for Table III of the paper. Even in the latter case it might be dispensed with, without entailing any appreciable error, as a comparison of columns 5 and 6 of Table IV will show. Again, it should be noted that in example III no quadrature formula was used, yet despite this we obtained a fit which, using χ^2 as a criterion, is somewhat superior to Elderton's graduation which made use of a quadrature formula. To be frank, however, I can tell of other cases where the reverse is true—possibly that influenced my choice of this particular illustration. There can be no doubt but that many problems of large class dimension will be met where it will be necessary to apply a quadrature formula, and Mr. Davis' careful and excellent treatment of this phase must be given very thoughtful consideration.

As regards the graduation of enumerated populations concerning which a question is raised; we may proceed along two distinct lines. First, we may choose as an α series all enumerations above age 70 and then graduate the residual series β, or secondly we may make use of the fact that the differences of the population frequencies may be graduated in the same manner as the recorded deaths and the results summed to produce the desired results. I do not want to go on record as believing that this is the proper method of constructing a mortality or life table. I do not believe it is—at least I know several better methods. The results were rather interesting, but from a theoretical standpoint it seems to me that such a method is but little more logical than an attempt to construct a mortality table from a record of deaths alone, or, going that one better, by using enumerated populations only. If we can construct a table from death alone as in *Proc.*, Vol. IV, and by dividing these deaths by q_x, determine the unenumerated populations—why not the converse?

I am afraid that I cannot accept Mr. Davis' suggestion that immigration may be responsible for series β. A graph of log q_x or log (colog p_x) for almost any life table will reveal the presence of a similarly situated supplementary series—a frequency shaped distribution—which when applied to the stationary population yields a strikingly similar series of expected deaths. As a result of certain experiments, I am led to believe that the equation

$$\mu_x = ab^x + \text{a frequency series } \beta$$

affords about the best representation of the law of mortality above age 10; in other words, instead of modifying Gompertz's law by the introduction of a supplementary series in the l_x column it may be advisable to introduce the series in either of the functions μ_x, log (colog p_x), or log q_x.

As regards the question concerning the position of the mode for a distribution defined by our difference equation it is quite evident that if we write the equation as

$$\frac{\Delta y_x}{y_x} = \frac{a - x}{b_0 + b_1 x + b_2 x^2}$$

the mode must lie approximately at $a + \frac{1}{2}$, that is at

$$x = \frac{-\frac{\nu_3}{\nu_2}\left(\beta_2 + 3 - \frac{1}{\nu_2}\right)}{2\left(5\beta_2 - 6\beta_1 - 9 + \frac{1}{\nu_2}\right)}.$$

The calculation of the mode by this result and by Pearson's formula

$$x = \frac{-\frac{\nu_3}{\nu_2}(\beta_2 + 3)}{2(5\beta_2 - 6\beta_1 - 9)}$$

will in general produce practically the same result: therefore Pearson's measure for skewness, etc., may be considered to hold for the system determined by the difference equation.

I regret that I cannot agree at all with the statements made by Mr. Davis in his last paragraph. For one thing I believe that Charlier's method is the only method which rests on a sound philosophic basis. On the other hand, I believe that Pearson's system and the proposed method can graduate any distribution that Charlier's can, and in the problem of smoothing the stump of a distribution are more powerful than Charlier's. We must remember, however, that

(a) The two methods referred to are empirical; we assume that the series in the denominator of the differential or difference equations are convergent.

(*b*) The coefficients of additional terms which may be required are readily computed in Charlier's series independently of the coefficients of the other terms. This is a great advantage. In the proposed method additional terms can be computed only at the expense of much additional labor, whereas in Pearson's differential equation the addition of an added term or so will render the resulting integral so complex that using it is almost out of the question. An objection to the use of additional terms is that by so doing higher moments with high probable errors are introduced. In this respect Pearson's method and the proposed are rather superior to Charlier's, since it is possible where additional terms are needed (that is where the number of classes is large, such as in graduating a mortality table) to break the distribution up into several series, from which the number of required equations may be obtained using a modification of the method of averages. In this way the number of required higher moments may be reduced.

The statement that Charlier's series fails completely to graduate in cases of appreciable skewness is an error. It frequently happens that the type *A series,*

$$A_0\varphi_{(x)} + A_3\varphi_{(x)}^{\text{III}} + A_4\varphi_{(x)}^{\text{IV}} + \cdots,$$

as generally understood, fails, but in those cases type *B* series can yield the desired results. The most important reason why series *A* fails frequently to be useful is that in calculating the graduated frequencies we are obliged to make use of an approximation. The true value for ϕ_x in the type *A* series is

$$\frac{1}{\pi}\int_0^\pi e^{-\frac{\sigma^2\omega^2}{2}}\cos x\omega d\omega$$

instead of the generally accepted value

$$\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma^2}},$$

which is the value of first named function when the upper limit is assumed to be infinity instead of π. Even assuming that the functions are for all practical purposes interchangeable it does not follow that their derivatives also are. Thus, if two functions intersect at right angles, they will have the same value at that point, but their derivatives may be zero and infinity respectively. It is not difficult however, to obtain an expression for the errors involved by making these assumptions in Charlier's series.

Too much significance ought not be placed upon the relations between the constants of the difference and differential equations. It is true, as Mr. Mowbray suggests, that the interrelated values define different types of curves. Thus we have the general solution

$$y = y_0 \frac{\Gamma(x-\alpha)\Gamma(x-\beta)}{\Gamma(x-\gamma)\Gamma(x-\delta)},$$

which Mr. Davis points out, but this is only useful for practical purposes when the roots α, β, γ and δ are real quantities. If the roots are integers or are replaced by the nearest integers, we have a hypergeometric series of the form

$$y_0 \frac{\underline{|x+a}\,\underline{|x+b}}{\underline{|x+c}\,\underline{|x+d}}$$

and it can easily be seen that a parabola also satisfies the difference equation. Although our idea of a frequency distribution excludes the idea of the parabola, nevertheless the logarithms of the frequencies may satisfy this function, since this merely implies that the distribution is Gaussian.

Again, when $b_2 = 0$, i.e., when

$$3\beta_1 - 2\beta_2 + 6 - \frac{1}{\nu_2} = 0,$$

the difference equation satisfies a binomial series or even a straight line. It is because of this wide degree of freedom that Pearson's differential equation and our difference equation are so widely applicable.

The summation method that Mr. Mowbray desires may readily be obtained by substituting the results which Elderton obtains on page 21 of his "Frequency Curves and Correlation" in either equations II or IV appearing on pages 55 and 56 of the paper under discussion.