SOME NOTES ON CREDIBILITY

BY

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The following notes arose out of current problems in rate making, principally automobile, and while arranged in some sort of logical order, are more or less separate. They cover a wide range of aspects of credibility and do not pretend to be either a complete study of the subject or even a complete survey of the specific points discussed. Indeed, as regards a good many principles of credibility (including some of the fundamental) actuarial opinion does not seem to be formed and one of the reasons for these notes is to provoke discussion of the whole subject, which is of increasing importance today on account of the great attention being paid by the carriers, the supervising authorities and the public to rates, especially automobile liability rates. Increasing insistence is being placed on the desirability of basing, as far as possible, rates for the various territories on the territories' own experience and accordingly it has become much more important to have reliable guides as to how far local experience can be relied on and to what extent it must be supplemented by experience in other districts. Thus credibility in the sense it is used in casualty rate making is of much greater importance nowadays than it was some years ago.

There is an excellent summary of automobile rate making credibility in the late Mr. R. A. Wheeler's comparatively recent paper entitled "Credibility and Automobile Rate Making" (*Proceedings*, Vol. XVI, page 268). In it he discusses the genesis of the usual criterion for credibility in automobile rate making and this criterion is the subject of my first note.

Note 1

Usual criterion for credibility of accident frequency.

The usual formula for ascertaining the amount of exposure necessary in order that a certain experience may be relied on is arrived at by determining an exposure large enough so that there is a very great probability that the number of accidents actually observed is within a small percentage of the expected or most probable number. This is derived as follows:—

The probability P that the number of accidents is within $k(100k\%)$ of the most probable number nq, where n is the exposure and q is the accident frequency is, if $p+q=1$,

$$P = \frac{1}{\sqrt{2\pi\, npq}} \int_{-knq}^{+knq} e^{-\frac{x^2}{2npq}}\, dx \tag{a}$$

or changing the variable x to $t = \dfrac{x}{\sqrt{2\,npq}}$

$$P = \frac{2}{\sqrt{\pi}} \int_0^{\frac{knq}{\sqrt{2npq}}} e^{-t^2}\, dt$$

If the criterion is to be (say) that for full credibility $P = .99$ and $k = .05$, then from a table of $\dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^z e^{-t^2}\, dt$ we find that

$$.99 = \frac{2}{\sqrt{\pi}} \int_0^{1.82} e^{-t^2}\, dt \text{ and } \frac{knq}{\sqrt{2npq}} = 1.82 \text{ or } nq = 2650(1-q). \tag{b}$$

This is the usual credibility formula but some little time ago a question arose in connection with Owners', Landlords' and Tenants' area experience, as to why the unit of exposure should affect the result brought out by the formula since it is obvious from first principles that such a change should not affect the requirements for full credibility.

(For example, in the case of an accident frequency of one accident per annum for each 1,000 sq. ft., if the unit of exposure be a square foot, $q = .001$ and $n = \dfrac{2650 \times .999}{.001} = 2{,}647{,}350$ sq. ft., but if the unit be 100 sq. ft., $q = .1$ and $n = \dfrac{2650 \times .9}{.1} \times 1000 =$ 2,385,000 sq. ft.)

Now formula (a) is derived as an approximation for

$$P = \sum_{r=(1-k)nq}^{r=(1+k)nq} {}_nC_r\, (1-q)^{n-r}\, q^r \tag{c}$$

and is based on the assumption of n trials with a probability of happening in each trial of q, each trial being subject to the (implied) condition that in it the accident either happens (probability q) or does not happen (probability $p = 1 - q$). Now this is not the case for liability assurance where q is the yearly accident frequency. There are not, in the case of an exposure of n with yearly accident frequency q, n trials with probability q in the above sense, for each unit of exposure may have none, one, two, three, etc., accidents in a year. To make the necessary change in the formula we proceed as follows: The unit of exposure is one unit of coverage exposed for a year: for clarity we will take the case of one car exposed for a year—a "car year": alter the car year exposure to an exposure of a car for an s^{th} of a year where s is ultimately to be made so large that we can regard a car exposed for $\frac{1}{s}$ years as being a single trial with only the two possible results, namely, no accident or one accident. Then in (b) we write ns for n, $\frac{q}{s}$ for q, and get $ns \times \frac{q}{s} = 2650\left(1 - \frac{q}{s}\right)$ or $nq = 2650\left(1 - \frac{q}{s}\right)$

Now making s very large we get $nq = 2650$, the proper credibility requirement.

Similarly if the criterion is to be that n is large enough so that the probability is P that the number of accidents are within $100k\%$ of the expected value nq, then the value of n is determined as follows:—

From tables find z so that $P = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt$

Then $nq = 2\left(\frac{z}{k}\right)^2$

It will be seen that the criterion becomes simply that the number of expected accidents shall be a certain fixed amount regardless of the accident frequency. This is an eminently reasonable result for it says in effect that 10,000 cars for one year are as credible as 20,000 cars for six months, or five thousand cars for two years, if they are all exposed to the same (annual) hazard of accident, or in the case of different accident frequencies, that 10,000 car years at one accident rate are as credible as 20,000 car years at half the accident rate, or 5,000 car years at double the accident rate.

The effect of this adjustment to the usual credibility formula is to increase slightly the requirements for full credibility: for example, for a yearly accident frequency of 10% the requirement is increased 11% and for a yearly accident frequency of q the increase is $\frac{100\%}{1-q}$.

One further point ought to receive consideration:—

(a) is the usual approximation to the binomial (c) if n is large, if k is not too large (and in practice both these conditions are fulfilled) and *if q is not too small.* But using the above modification of the usual credibility theory we are using in effect (a) with $\frac{q}{s}$ in place of q and ns in place of n, or

$$\frac{2}{\sqrt{2\pi\, nq}}\int_0^{knq} e^{-\frac{x^2}{2nq}}\,dx$$

as an approximation to

$$\sum_{r=(1-k)ns\frac{q}{s}}^{r=(1+k)ns\frac{q}{s}} {}_{ns}C_r\left(1-\frac{q}{s}\right)^{ns-r}\left(\frac{q}{s}\right)^r \quad \text{with } s\to\infty$$

and the question is whether this is a proper approximation since $\frac{q}{s}$ will be very small. Appendix A shows that under these conditions the approximation is satisfactory.

The conclusion of this note is that the criterion for accident frequency credibility usually used in casualty rate making for such lines as automobile, and which criterion is based on the probability integral (a) should be modified by the use of nq in place of npq. This reduces the criterion to a certain number of accidents, which number is independent of the accident frequency.

Note 2

Expression of criterion in terms of standard deviation.

For some purposes it is most convenient to express the credibility criterion in terms of the more usual statistical constants, the mean and the standard deviation.

In a normal frequency distribution of a variable with a mean μ

and a standard deviation σ the probability that an observation differs from the mean by less than $\delta = k\mu$ is

$$P = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\delta}^{+\delta} e^{-\frac{x^2}{2\sigma^2}}\,dx = \frac{2}{\sigma\sqrt{2\pi}}\int_0^{\delta} e^{-\frac{x^2}{2\sigma^2}}\,dx$$

or putting $x = \sigma\, t\sqrt{2}$

$$P = \frac{2}{\sqrt{\pi}}\int_0^{\frac{k\mu}{\sigma\sqrt{2}}} e^{-t^2}\,dt$$

From a table of $\dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^z e^{-t^2}\,dt$ we can determine for a given value of P the value of z as a function of P, say $f(P)$, i.e.,

$$P = \frac{2}{\sqrt{\pi}}\int_0^{f(P)} e^{-t^2}\,dt$$

Thus if $\dfrac{k\mu}{\sigma\sqrt{2}} \geq f(P)$ or $\dfrac{\sigma}{\mu} \leq \dfrac{k}{\sqrt{2}\,f(P)}$ (a)

then the probability is greater than P that an observed value will differ from the mean μ by less than $k\mu$.

For example, if $P = .99$ $\quad k = .05 \quad f(P) = 1.8214$

$$\frac{k}{\sqrt{2}f(P)} = .01941$$

so that if $\dfrac{\sigma}{\mu} = .01941$ then the probability is 99 to 1 that an observation will be within 5% of the mean, and if $\dfrac{\sigma}{\mu}$ is less than .01941 the probability is greater than 99 to 1 that the deviation will be less than 5%.

Usually in rate making we are dealing with the mean of a large number, say n, of individual observations or experiences, and even if the frequency distribution of the individual observations is not normal the frequency distribution of the means will usually tend to be normal and the larger n is the more normal the frequency distribution (of the means) will be and the smaller the standard deviation will be. If the mean of one observation of the variable observed is μ and the standard deviation is σ then the mean of the mean of n observations will be μ and the standard deviation σ_n will equal $\frac{\sigma}{\sqrt{n}}$. The problem that we have to solve is how large must n be so that there is a probability P that the variation of the mean of the n observations from the true value is less than $100k\%$. In other words we must determine from (a) the value of n so that

$\frac{\sigma_n}{\mu} \leq \frac{k}{\sqrt{2} f(P)}$. Now if $\frac{k}{\sqrt{2} f(P)} \geq \frac{\sigma_n}{\mu} = \frac{\sigma}{\mu\sqrt{n}}$, then

$$n \geq \frac{2\sigma^2 \{f(P)\}^2}{\mu^2 k^2} \tag{b}$$

and this determines the minimum value for n.

It will be seen that if P and k are fixed, n depends on the ratio for the variable observed of the standard deviation to the mean.

A few values of $\{f(P)\}^2$ are given together with some values of $2\{f(P)\}^2 \div k^2$.

P	$\{f(P)\}^2$	$2\{f(P)\}^2 \div k^2$				
		$k = .025$	$k = .05$	$k = .075$	$k = .1$	$k = .125$
.99	3.316	10,611	2,653	1,179	663	424
.98	2.706	8,659	2,165	962	541	346
.97	2.356	7,539	1,885	838	470	302
.96	2.108	6,746	1,686	750	421	270
.95	1.921	6,147	1,537	683	384	246
.90	1.353	4,330	1,082	481	271	173
.85	1.036	3,315	829	368	207	133
.80	0.823	2,634	658	293	165	105
.75	0.661	2,115	529	235	132	085
.70	0.537	1,718	430	191	107	069

From this we can easily calculate n as soon as $\frac{\sigma}{\mu}$ is known.

We can also now readily see the effect of changes in the credibility requirements. If k is changed, P remaining constant, n varies inversely as the square of k, e.g., if k is doubled, n is decreased to one-fourth. If k remains constant and P is changed, n varies as $\{f(P)\}^2$, e.g., if P is decreased from .99 to .80 n is decreased in the ratio of 3.316 to .823 or is approximately decreased to one-fourth. If both k and P are changed the net effect depends on the change in $\frac{\{f(P)\}^2}{k^2}$: e.g., if k is doubled and P is increased from .99 to .80, n is approximately unchanged.

To apply the above to the accident frequency criterion discussed in Note 1 we can proceed

either (i) from Note 1 we see that for an exposure n and accident frequency q, the mean expected value is nq and the standard deviation is $\sqrt{nq}$. Thus for the n observations

$$\frac{\sigma}{\mu} = \frac{1}{\sqrt{nq}} \text{ and from (a) } \frac{1}{\sqrt{nq}} \leq \frac{k}{\sqrt{2}\,f(P)} \text{ or } nq \geq \frac{2\{f(P)\}^2}{k^2} \quad \text{(c)}$$

or (ii) from Note 1, for a single observation ($n = 1$) the mean expected value is q and the standard deviation is $\sqrt{q}$.

Thus for one observation $\frac{\sigma}{\mu} = \frac{1}{\sqrt{q}}$

and from (b) $n \geq \frac{2\{f(P)\}^2}{qk^2}$

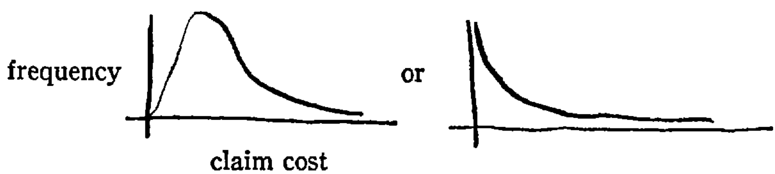
$$\text{or } nq \geq \frac{2\{f(P)\}^2}{k^2} \text{ as before}$$

We can apply the above table directly to this.
For example

if $P = .99$	$k = .05$	$nq \geq 2{,}653.$
$P = .80$	$k = .025$	$nq \geq 2{,}634.$

NOTE 3

In our formulae for full credibility we have considered so far only the variation in accident frequency and have derived criteria for determining whether the accident frequency shown by the experience can be relied on. This has been the usual procedure to date: with a credible accident frequency it has been assumed the pure premium is credible. But as the pure premium is the accident frequency multiplied by the average claim cost we must see how possible variations in the average claim cost affect the pure premium and how we must modify our credibility requirements accordingly. In a given territory for a given period the cost of a claim will form a frequency distribution which will usually when plotted graphically be something like



The frequency distribution will not be "normal" (i.e., not follow the "normal" curve of error), but will be either of a very skew cocked-hat type or else of a J or L type. Nevertheless the average of a large number of observations will tend to be normal (the larger the number the more normal) and if M, S are the mean and the standard deviation respectively of the claim cost distribution the mean and the standard deviation of the average of nq observations will be M and $\frac{S}{\sqrt{nq}}$ respectively.

Further, if nq is large so that the frequency distribution of the average claim cost is fairly normal the pure premium (arrived at by multiplying the accident frequency q which has a standard deviation of $\sqrt{\frac{q}{n}}$ and the average claim cost M which has a standard deviation of $\frac{S}{\sqrt{nq}}$) will have a mean value of Mq and a standard deviation of $\sqrt{\left\{M^2\frac{q}{n}+q^2\frac{S^2}{nq}\right\}}$ for the standard deviation of M_1M_2 where M_1, M_2 have standard deviations of σ_1, σ_2, respectively, is $\sqrt{\{M_1^2\,\sigma_2^2+M_2^2\,\sigma_1^2\}}$ if both M_1 and M_2 are normal. (This is a well known theorem in mathematical statistics). Thus for the pure premium

$$\frac{\text{Standard Deviation}}{\text{mean}}=\sqrt{\left\{\frac{1}{nq}+\frac{S^2}{M^2nq}\right\}}=\frac{1}{\sqrt{nq}}\sqrt{\left\{1+\frac{S^2}{M^2}\right\}}$$

Putting this value in formula (a) of Note (2) we must have for full credibility of the pure premium

$$\frac{1}{\sqrt{nq}}\sqrt{\left\{1+\frac{S^2}{M^2}\right\}}\leq\frac{k}{\sqrt{2}\,f(P)}$$

$$\text{or } nq\geq\frac{2\{f(P)\}^2\left(1+\frac{S^2}{M^2}\right)}{k^2} \qquad \text{(d)}$$

Comparing this with formula (c) of Note 2 we see that the volume of exposure required for full credibility of the pure premium requires the multiplication by the factor $1+\frac{S^2}{M^2}$ of the number of claims required for credibility of the accident frequency (notice the requirement is still expressed in number of claims).

This factor seriously increases the credibility requirements. For automobile public liability private passenger personal injury standard limits claims some countrywide figures show an average claim of about \$325 with a standard deviation of about \$715 so that $1+\frac{S^2}{M^2}=5.84$. Thus retaining the same requirements for full credibility, namely, .99 probability of a variation of less than 5%, the number of claims required would be increased from 2650 to 15500: or for an accident frequency of 5% the number of car years required is increased from 53,000 to 310,000! Corresponding automobile private passenger property damage figures show a \$40 average claim with a standard deviation of \$71 so that $1+\frac{S^2}{M^2}=4.15$ and the number of claims required would be increased from 2650 to 11,000 and the number of cars with a 12% accident frequency from 22,000 to 91,000!

Such great increases in credibility requirements could not very well be made in practice under present day conditions for they would greatly limit the employment of local data. It seems likely, however, that, as Mr. Wheeler remarks in his paper mentioned above, the present credibility requirements, which are based on accident frequency only, have been made unduly stringent to take care in a rough manner of the increased requirements for pure premium credibility. Thus a reference to Note 2 will show that, approximately, in the case of automobile property damage a requirement of .99 probability of a variation of less than 10% will quarter the number of claims required for 100% credibility and when applied to pure premiums as in this note will produce somewhat the same needed exposure as the requirement of .99 probability of a variation less than 5% applied to accident frequency. In the case of automobile public liability the requirement would have to be reduced to .99 probability of a variation of less than 12% to give the same number of claims.

The question of standard deviation of claim costs requires considerable further study. It would seem that claim conditions would vary considerably from state to state and accordingly a distribution of country-wide claims by size would form not a Bernoullian distribution but a Lexian distribution, and the standard deviation of a state's distribution of claims by size might very well be less than the standard deviation of the country-wide distribution. Theoretically when making rates for a given territory the frequency distribution of the claims for that territory shall be used. However, in most instances there would not be enough claims in the territory to give a reliable estimate of the average value and standard distribution.

The standard deviation of the average claim cost is greatly affected by the comparatively few large claims, particularly in the case of automobile personal injury. It has been suggested that probably better (that is, more stable) results could be secured by considering separately (i) the first $1,000 of each claim together with the complete cost of all claims of less than $1,000 and (ii) the excess of cost over $1,000 per claim for those exceeding that amount. By this method the pure premium would be split into two parts, namely, "normal" and "excess". (The terms "normal" and "excess" are here used in the experience rating sense.) This procedure would not entail much change in the calculation of the "normal" pure premium which would be calculated just as the full pure premium is at present. However, in the calculation of the "excess" pure premium it would be necessary (even if a lower credibility were used) either (I) to combine the experience of similar territories to get a sufficiently credible exposure (which procedure while easily defended is contrary to the present tendency to use local experience as much as possible) or else (II) use more years experience than for the "normal" pure premium (which procedure is open to many objections, chief of which is disinclination to go back any further than necessary into the past on account of rapidly changing conditions). The method mentioned in this paragraph has received some attention and warrants further study. However, it must be realized that credibility requirements would call for a considerable excess exposure as shown by the following example (which is based on actual experience).

$$P = .99 \qquad\qquad k = .05$$

(A) Full pure premium

$$q = .092 \quad M = 328 \quad 1 + \frac{S^2}{M^2} = 5.32 \quad \text{pure premium} = 30.2$$

Exposure required for 100% credibility = 153,000 car years.

(B) "Normal" pure premium

$q = .092 \quad M = 229 \quad 1 + \frac{S^2}{M^2} = 2.61 \quad \text{pure premium} = 21.1$

Exposure required for 100% credibility = 69,000 car years.

(C) "Excess" pure premium

$q = .0066 \quad M = 1379 \quad 1 + \frac{S^2}{M^2} = 1.75 \quad \text{pure premium} = 9.1$

Exposure required for 100% credibility = 702,000 car years.

The exposure required for full credibility for the "normal" pure premium is about 45% of that for the full pure premium, while that for the "excess" pure premium is over 450%.

Note 4

The next aspect of the subject of credibility which comes up for consideration is the treatment of experience which is not fully credible, according to the principles of the preceding notes or according to any other standards which may be used. The usual and most satisfactory procedure is to assign a credibility of less than 100% to the experience and then combine with some broader experience giving this latter the complement of the weight assigned. To illustrate, if we had pure premiums by class for a given state, and had also national pure premiums by class, then for classes for which the experience in the state had 100% credibility we would use the state experience and for a class for which the credibility of the state experience was less than 100% (say 60%) we would use 60% of the state pure premium plus 40% of the national pure premium. In practice we would allow for differences in conditions in the state and the rest of the country by a procedure which would put the national pure premiums on the state level, but I do not intend to go into many details here. The question of what credibility to assign to an exposure less than one warranting 100% credibility is usually decided by assigning a credibility of $\frac{1}{\sqrt{r}}$ to an exposure of $\frac{1}{r}$ of that required for 100% credibility. The reasons prompting the use of this do not appear very explicitly in casualty actuarial literature, but it seems to be based on the rule used in "combination of observations" (in such sciences as astronomy, engineering) that the best weight to be assigned an observation is the reciprocal of its standard deviation: according to this the relative weights of two experiences, one (exposure n) entitled to 100% credibility and other $\left(\text{exposure} \frac{n}{r}\right)$ would be in the ratios of the reciprocals of their standard deviations or as $\frac{\sqrt{n}}{\sigma}$ to $\frac{\sqrt{\frac{n}{r}}}{\sigma}$ that is 1 to $\frac{1}{\sqrt{r}}$.

The rule seems plausible and practical. It is to be noted, however, that the principles upon which it was derived for use in other branches of science are not especially applicable to casualty rate making. It is further to be remarked that in an analogous problem in casualty actuarial practice, namely, the determination of the weight to be assigned to a risk in experience rating, an entirely different rule was used. The experience rating method, however, would give roughly the same results as the use of the method mentioned above. It will be noticed from Note 2 that the above mentioned rule amounts to this: if the criterion for 100% credibility is a probability P of a variation of less than $k\%$ the criterion for $\frac{100\%}{t}$ credibility is a probability P of a variation of less than $tk\%$ or alternately the criterion can be expressed (less simply) as a probability of P^1 of a variation of less than $k\%$ where $t\,f(P^1) = f(P)$. For example, if $P = .99$, $k\% = 5\%$, then for 50% credibility $t = 2$ and the criterion for 50% credibility is .99 probability of 10% variation or .80 probability of 5% variation since $2f\,(.80) = f(.99)$.

Nevertheless let us examine the procedure a little more closely using the principles of the preceding notes.

Let us assume an indication on local experience of p with a credibility arrived at as above of $\frac{1}{t}$ or, what amounts to the same thing, with a standard deviation of σt where σ is the minimum standard deviation required for full credibility. We are to combine this indication with an indication based on wide experience of P, the standard deviation of which we will suppose is x. The final pure premium will thus be

$$\frac{1}{t}p + \left(1 - \frac{1}{t}\right)P$$

and the square of the standard deviation of this will be

$$\frac{1}{t^2}\,\sigma^2 t^2 + \left(1-\frac{1}{t^2}\right)^2 x^2 \text{ or } \sigma^2 + \left(1-\frac{1}{t}\right)^2 x^2$$

Now assuming (which will usually be the case) that p and P are nearly equal, the final pure premium will be entitled to full credibility (on the above principles) only if the square of the standard deviation is not greater than σ^2 which will only be the case if x is zero. This will not ever be quite true although we may *assume* x is very small if the wider experience is very extensive. It would appear then that we should give the local experience somewhat less weight than $\frac{1}{t}$, although it is not possible to give a simple rule.

The present rule is probably satisfactory enough in practice particularly as it is customary to limit the use of experience with very low credibility. However, this discussion shows that any changes in partial credibility formulas should not be in the direction of giving greater credibility.

It must be borne in mind that the above remarks are all based on the assumption of the existence of an available and applicable wider experience with which to make the combination of a partially credible experience and there is usually in practice some question as to the availability and applicability of wider experience.

It would take me too far from the objects of these notes to go into the difficulties—for they are many—of the treatment of experience with less than 100% credibility, and of the proper selection of wider experience with which to combine. Nevertheless the following will indicate the theoretical procedure for a simple ideal case. Suppose in one state, for a group of r territories all reasonably similar we have the following exposures, experience pure premiums, and credibilities,

Territory	1	2	3	..... r
Exposure	e_1	e_2	e_3	..... e_r
Experience pure premiums	p_1	p_2	p_3	..... p_r
Credibility	c_1	c_2	c_3	..... c_r

Then we wish to determine pure premiums solely from the group, we can proceed:—take for each territory pure premiums of

$$p_1 c_1 + (1-c_1)\,\pi,\ p_2 c_2 + (1-c_2)\,\pi, \text{ etc.}$$

where π is based on experience of the group so as to reproduce the experience as a whole.

Thus we must have $\sum_{s=1}^{r} e_s \{ p_s c_s + (1-c_s)\pi \} = \sum_{s=1}^{r} e_s p_s$

$$\text{or } \pi = \frac{\sum_{s=1}^{r} (1-c_s) p_s e_s}{\sum_{s=1}^{r} (1-c_s) e_s}$$

This procedure amounts to dividing the experience of any territory s into two parts c_s and $1-c_s$, and using the latter part to determine a broad pure premium.

In order that the final indications may be credible, it is necessary as shown earlier that $\Sigma (1-c_s) e_s \pi$ which equals $\Sigma (1-c_r) p_s e_s$ should be considerable larger than the exposure required for 100% credibility.

It will often be found in practice that if the r territories are, as assumed, reasonably similar, then practically the same results will be obtained by taking for π the average pure premiums of all the territories or $\frac{\Sigma p_s e_s}{\Sigma e_s}$

Note 5

Projection by Method of Least Squares.

The progressive increase in pure premiums for many coverages has given rise of late years to the use of "projection" methods aimed at using the trend of past experience to "project" the observed pure premium for past years so as to forecast the future pure premiums on the assumption that the observed trend will be continued. With the propriety of such attempts and the methods used, I am not concerned in these notes, my sole purpose here being to consider the effect on credibility of the method most widely used, namely, straight line projection by the method of least squares.

This method consists of setting down the pure premiums observed by years (usually policy years) and fitting by mathematical methods the straight line of best fit to these observations and reading off the projected value of the pure premiums for the required future year. Thus if we have for r years of observation the following pure premiums

Year	1	2	3 $\cdots$	r
Pure Premium	p_1	p_2	p_3 $\cdots$	p_r

we fit a "straight line" so that the adjusted pure premium for the year x is $a + xb$ where a and b are determined from the p's. The name of "least squares" comes from the method used to determine a and b, which consists of choosing these as to reduce to a minimum the sum of the squares of the deviations, i.e., the sum of $(a + xb - px)^2$ for $x = 1, 2, 3, \cdots r$ is made a minimum: according to the mathematical theory of statistics this gives the best fit for the "straight line" $a + xb$.

Details of the calculation and the resulting formulas are given in Appendix B. A refinement can (in theory) be made by giving the different years varying "weights" in accordance with the varying credibilities of the various years' experiences. Usually, however, in practice varying weights are not used and we will deal with the simple case of equal weights which is the one set out above. The number of years used (the value of r above) is usually 4 or 5. I may say in passing that projection is usually applied direct to pure premiums but in theory there is a case for applying projection separately, as respects a class of business such as automobile liability, to the observed accident frequencies and to the average cost of an accident. Each of these two factors of the pure premium usually have their own trends and the result of combining the factors projected separately will be rather different from that obtained by projecting the total pure premium. The difference, however, would not usually be large and to make the projections separately would greatly complicate the work.

Taking now the case of equal weights, the least squares methods give a value for year $\frac{r+1}{2}$ (the mean year) of

$$\frac{p_1 + p_2 + \cdots + p_r}{r}$$

with an increase for each year beyond that of

$$\frac{6}{r^3 - r}\{-(r-1)\,p_1 - (r-3)\,p_3 - \cdots\cdots + (r-3)\,p_{r-1} + (r-1)\,p_r\}$$

so that the adjusted value for year x is

$$\frac{6}{r^3-r}\left[\left\{\frac{r^2-1}{6}-\left(x-\frac{r+1}{2}\right)(r-1)\right\}p_1+\cdots+\right.$$
$$\left.+\left\{\frac{r^2-1}{6}+\left(x-\frac{r+1}{2}\right)(r-1)\right\}pr\right]$$

Notice that the sum of the coefficients of the P's is 1 and that some of their values can be and often will be negative. Thus for $r = 5$, if $p^1{}_x$ is the adjusted value.

$$\begin{aligned}
10\,p^1{}_1 &= 6p_1 + 4p_2 + 2p_3 + 0p_4 - 2p_5\\
10\,p^1{}_2 &= 4p_1 + 3p_2 + 2p_3 + 1p_4 + 0p_5\\
10\,p^1{}_3 &= 2p_1 + 2p_2 + 2p_3 + 2p_4 + 2p_5\\
10\,p^1{}_4 &= 0p_1 + 1p_2 + 2p_3 + 3p_4 + 4p_5\\
10\,p^1{}_5 &= -2p_1 + 0p_2 + 2p_3 + 4p_4 + 6p_5\\
10\,p^1{}_6 &= -4p_1 - 1p_2 + 2p_3 + 5p_4 + 8p_5\\
10\,p^1{}_7 &= -6p_1 - 2p_2 + 2p_3 + 6p_4 + 10p_5
\end{aligned}$$

Writing now $p^1{}_x = \Sigma\, k_s\, p_s \qquad (s = 1, 2, \cdots r)$
if the standard deviation of each p_s is σ_s, the square of the standard deviation of $p^1{}_x$ is $\Sigma\, k^2{}_s\, \sigma^2{}_s$ and if the standard deviations σ_s are all equal (or nearly so) to a common value σ (which we can assume if the volume of exposure each year is fairly constant), then the square of the standard deviation of $p^1{}_x$ is equal (or nearly so) to $\sigma^2\, \Sigma\, k^2{}_s$.

Now $\Sigma\, k^2{}_s$ is smallest when $x = \frac{r+1}{2}$ and the further away that x is from this value the larger is $\Sigma\, k^2{}_s$. In fact for $x = \frac{r+1}{2}$, $\Sigma\, k^2{}_s = \frac{1}{r}$ and for other values x

$$= \frac{1}{r}\left\{1 + \frac{12\left(x-\frac{r+1}{2}\right)^2}{r^2-1}\right\}$$

Thus for the middle of the range 1 to r, the adjusted value is the mean of the r observations, its standard deviation is then $\frac{\sigma}{\sqrt{r}}$ and is at its minimum. For other years the standard deviation increases as we get away from the middle of the range and when we project into the future the standard deviation of the projected value increases rapidly. The following are the figures for $r = 4$ and $r = 5$.

Year	$r = 4$		$r = 5$	
	Square of Stand. Dev. for Adjusted Value	Ratio to Minimum Value	Square of Stand. Dev. for Adjusted Value	Ratio to Minimum Value
1	$0.7\sigma^2$	2.8	$0.6\sigma^2$	3.0
2	$0.3\sigma^2$	1.2	$0.3\sigma^2$	1.5
3	$0.3\sigma^2$	1.2	$0.2\sigma^2$	1.0
4	$0.7\sigma^2$	2.8	$0.3\sigma^2$	1.5
5	$1.5\sigma^2$	6.0	$0.6\sigma^2$	3.0
6	$2.7\sigma^2$	10.8	$1.1\sigma^2$	5.5
7	$4.3\sigma^2$	17.2	$1.8\sigma^2$	9.0

Now, since the amount of exposure required for 100% credibility increases as the square of the standard deviation (see Note 2), the exposure required for projection into the future is considerably above that required if, say the mean of the r years observations is taken. For example, if $r = 4$, then for projection 2 years beyond the end of the range, that is, to year 6, as compared with the amount of exposure *each year*, that would be required if we did not project but took the mean of the observations, we would require 10.8 times in order to secure equal credibility. Or to put it another way:

If we have four years' experience of about equal amounts of exposure and we are using a criterion (see previous notes) of 2650 accidents required for 100% credibility, then if we take the mean of the four years' experience for the indicated pure premium, then we require about 660 accidents per year for complete credibility, but if we decide to recognize trend and project to two years beyond the last year's experience we increase the requirement to about 6890 accidents per year: and that amount of exposure is credible only to the extent that the assumption of a straight line trend is justifiable.

This and similar results for other values of r and n indicate the need for adequate volume of experience when applying trend formulae.

SUMMARY

A summary of the conclusions reached is as follows:

In the case of liability insurance the usual credibility formula should be modified, giving slightly higher requirements for full credibility. The introduction into the credibility requirements of the variations in claim costs in addition to consideration of the variation in the number of accidents, seriously increases the credibility requirements; to maintain anything like present criteria necessitates quite a sharp reduction in the stringency of the requirements for full credibility. The credibilities of experiences of less than full credibility need further study; they should be reduced, if anything. The methods of completing experiences of less than full credibility do not seem altogether satisfactory. Further consideration could well be given to this subject. The use of projection methods, such as the method of least squares, increases very greatly the volume of experience required for 100% credibility.

Appendix A

Required the approximate value of

$$P = \sum_{r=(1-k)nq}^{r=(1+k)nq} {}_{ns}C_r \left(1-\frac{q}{s}\right)^{ns-r} \left(\frac{q}{s}\right)^r$$

where nq is finite and $s \rightarrow \infty$.

Putting $ns = m \qquad \frac{q}{s} = u \qquad 1 - u = v.$

Let ${}_my_x = {}_mC_{um+x}\, v^{vm-x}\, u^{um+x}$.

Then by Stirling's formula, since m, $um + x$, $vm - x$ are not small

$${}_my_x = \frac{\left(1+\frac{x}{um}\right)^{-um-x-\frac{1}{2}} \left(1-\frac{x}{vm}\right)^{-vm-x-\frac{1}{2}}}{(2\pi m v u)^{\frac{1}{2}}} \text{ very nearly}$$

Now if $s \rightarrow \infty$ and $u \rightarrow 0$ $v \rightarrow 1$ and $nq \rightarrow$ finite

$$um = nq \qquad vm \rightarrow \infty$$

$$\text{and } \therefore \ {}_my_x = \frac{1}{\sqrt{2\pi nq}}\left(1+\frac{x}{nq}\right)^{-nq-x-\frac{1}{2}} e^x \text{ very nearly}$$

$$= \frac{1}{\sqrt{2\pi nq}} \; e^{x-(nq+x+\frac{1}{2})\log\left(1+\frac{x}{nq}\right)}$$

$$= \frac{1}{\sqrt{2\pi nq}} \; e^{x-(nq+x+\frac{1}{2})\left(\frac{x}{nq}-\frac{x^2}{2(nq)^2}+\frac{x^3}{3(nq)^3}-\cdots\right)}$$

$$= \frac{1}{\sqrt{2\pi nq}} \; e^{-\frac{x^2+x}{2nq}} \qquad \text{if } \left(\frac{x}{nq}\right)^2, \left(\frac{x}{nq}\right)^3, \text{ etc.}$$

and $\frac{x^3}{(nq)^2}$, $\frac{x^4}{(nq)^3}$, etc., are small and can be neglected from the exponent of e.

It is to be noted that we cannot neglect $\frac{x}{nq}$ for if x is small $\frac{x}{nq}$ will not be small in comparison with $\frac{x^2}{nq}$.

Thus $P = \sum_{x=-knq}^{x=+knq} {}_m y_x = \frac{1}{\sqrt{2\pi nq}} \int_{-knq}^{+knq} e^{-\frac{x^2+x}{2nq}} dx$ approximately.

But $\int_{-knq}^{+knq} e^{-\frac{x^2+x}{2nq}} dx = \int_0^{knq} e^{-\frac{x^2}{2nq}} \left(e^{-\frac{x}{2nq}} + e^{\frac{x}{2nq}}\right) dx$

$= 2\int_0^{knq} e^{-\frac{x^2}{2nq}} dx$ since $e^{-\frac{x}{2nq}} + e^{\frac{x}{2nq}} = 2\left\{1+\frac{1}{2}\left(\frac{x}{2nq}\right)^2 + \ldots\right\}$

$= 2$ approximately.

So $P =$ very nearly $\frac{2}{\sqrt{\pi nq}} \int_0^{knq} e^{-\frac{x^2}{2nq}} dx$

Appendix B

Method of Least Squares.

If $L = \sum_{x=1}^{x=r} (a + bx - p_x)^2$ is to be a minimum.

Differentiate L first with respect to a and then with respect to b and put the two results equal to 0.

Then $(a+b-p_1) + (a+2b-p_2) + \cdots + (a+rb-p_r) = 0$

$(a+b-p_1) + 2(a+2b-p_2) + \cdots + r(a+rb-p_r) = 0$

or $ra + \frac{2}{r(r+1)} b = \Sigma p_r$

$\frac{r(r+1)}{2} a + \frac{r(r+1)(2r+1)}{6} b = \Sigma r p_r$

whence $a = \frac{2(2r+1)\,\Sigma p_r - 6\,\Sigma r p_r}{r^2 - r}$ $b = \frac{12\,\Sigma r p_r - 6(r+1)\,\Sigma p_r}{r^3 - r}$

This obviously gives a minimum; and in any case $\frac{\delta^2 L}{\delta a^2}, \frac{\delta^2 L}{\delta a\, \delta b}, \frac{\delta^2 L}{\delta b^2}$ are all essentially positive and therefore L has no maximum for finite values of a, b. It has one absolute minimum, which is given by the above values of a, b.

From the above

$$a+\frac{r+1}{2}b=\frac{\{2(2+1)-3(r+1)\}\Sigma p_r-\{6-6\}\Sigma rp_r}{r^2-r}$$

$$=\frac{1}{r}\Sigma p_r=\frac{p_1+p_2+\ldots+p_r}{r} \qquad \text{(i)}$$

Also $b=\sum_{s=1}^{s=r}\frac{\{12_s-6(r+1)\}p_s}{r^3-r}$

$$=\frac{6}{r^3-r}\{-(r-1)p_1-(r-3)p_3-\ldots+(r-3)p_{r-1}+(r-1)p_r\} \qquad \text{(ii)}$$

From (i) and (ii) $a+xb$, for the value of x which is t over or under $\frac{r+1}{2}$ is equal to

$$\frac{p_1+\ldots+p_r}{r}\pm\frac{6t}{r^3-r}\{-(r-1)p_1-\ldots+(r-1)p_r\}$$

$$=\frac{1}{r^3-r}\left[\{(r^2-1)\mp 6t(r-1)\}p_1+\{(r^2-1)\mp 6t(r-3)\}p_3+\ldots\right.$$
$$\left.+\{(r^2-1)\pm 6t(r-1)\}p_r\right]$$

and the sum of the squares of the coefficients of the p's is

$$\frac{r(r^2-1)^2+36t^2\{(-r+1)^2+(-r+3)^2+\ldots+(r-3)^2+(r-1)^2\}}{r^2(r^2-1)^2}$$

$$=\frac{1}{r}+\frac{12t^2}{r^2-1} \qquad \text{(iii)}$$