FURTHER TABLES ADAPTED FOR MACHINE COMPUTATION

BY

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In my previous paper, Proceedings, Casualty Actuarial Society, XXV, entitled "Tables Adapted for Machine Computation"—which will be referred to hereafter as "the previous paper"—I gave ten place tables of logarithms adapted to be used on modern calculating (multiplying) machines. I also gave some auxiliary tables of interest functions. The logarithm tables, however, were a little trouble to use, although of course we must expect a little of that with such condensed tables, so I sought to get an arrangement easier to use. This I found and it is described in Part II of this paper. Again, in the previous paper, what I gave by way of interest tables were really only tables of the fundamental values and not tables of values for immediate use: the paper gave the procedure for obtaining the required values from the given fundamental functions. Here again, I was not satisfied and endeavored to get a set of tables giving the final values directly; I was principally interested in tables of weekly annuity values as these occur frequently in Casualty work. The set-up I finally devised is given in Part I of this paper.

Perhaps I should say a word here in anticipation of a type of criticism that may be levelled against the results given here, by some hasty critics, on the grounds that ten place logarithm tables and ready to use weekly annuity tables are unnecessary luxuries or give needless accuracy. I don't regard the matter thus. The tables I give are, I believe, useful additions to the tools of our profession: and it is a fitting example of the principle of division of labor for one person like myself, who is interested in these things and likes working them out, to undertake the work of preparing these tools and presenting them to the profession. If the few pleasant hours I spent in putting this paper together save members of the profession a few minutes work from time to time, then my labor was useful as well as pleasant.

Part I

As stated above, in the previous paper I gave some tables of interest functions which were merely tables of fundamental values that did not give directly the values actually required in practice, such as values of annuities certain.

One example in the previous paper—(8) on p. 142—did give, almost directly, with the aid of a calculating machine, weekly annuity values at

3½% and this example contains the germ of the idea for a table to give such values for any rate of interest. The first part of this paper will develop this idea and give the necessary tables.

I will give the full tables for finding weekly annuities and indicate how they can be used also for annual annuities payable either yearly or at more frequent intervals.

The present value of an annuity certain of 1 per annum payable r times a year for m years at rate of interest i is

$$\frac{1-e^{-\delta m}}{j_r} \qquad \text{(I)}$$

where, as usual, $\delta = \log_e (1+i)$ and $j_r = r\,[(1+i)^{\frac{1}{r}} - 1]$. The present value of an annuity certain of 1 per week, for m weeks at rate of interest i, is

$$\frac{r\,(1-e^{-\delta \frac{m}{r}})}{j_r} \qquad \text{(II)}$$

where δ and j_r have the same values as before and r is the number of weeks assumed to be in a year.

As in the previous paper, we will give tables for $r = 52$ and for $r = 52.1775$ together with the interpolation procedure to be used for other values of r, such as $52\frac{1}{7}$.

Both (I) and (II) are of the form $C\,(1-e^{-Bm})$ where B and C depend only on the rate of interest and the value of r. So we can get annuity values if we have tables of

(A) $\{n\} = 1 - e^{-kn}$ for all values of n, k being a fixed constant.

(B) $B = \dfrac{\delta}{rk}$ for weekly annuities or $B = \dfrac{\delta}{k}$ for annual annuities.

(C) $C = \dfrac{r}{j_r}$ for weekly annuities

or $C = \dfrac{1}{j_r}$ for annual annuities

for if we calculate $n = Bm$ then the annuity value is

$$C\,(1-e^{-kn}) \text{ or } C\,\{Bm\}.$$

The tables for B and C are easily constructed and can readily be given for all usual rates of interest and values of r required. On the other hand, the construction of $\{n\}$ requires some preliminary considerations. First of all, in theory, it should proceed from $n = 0$ to $n =$ infinity so as to give annuity values for extended terms, but in practice it is desirable to limit the table to a reasonable size, say to 200 entries. Fortunately, we can do this conveniently by choosing k, which can be any arbitrary number, so that $e^{-200k} = \frac{1}{2}$. If we do this and tabulate $\{n\}$ for values of n from 0 to 200, the table will

(i) directly give values for periods up to the number of years in which money doubles itself at compound interest at rate i; namely, about $70/i$ years (or about 70 years at 1%, about 35 years at 2%, etc.).

(ii) give an easy formula for n greater than 200: this is because

$$\{n + 200\} = 1 - e^{-(n+200)k} = 1 - \tfrac{1}{2}e^{-nk} = \tfrac{1}{2} + \tfrac{1}{2}\{n\}$$

and $\{200\} = \frac{1}{2}$

Similarly $\{n + 200\,s\} = 1 - \dfrac{e^{-nk}}{2^s} = \dfrac{2^s - 1}{2^s} + \dfrac{\{n\}}{2^s}$

and $\{200s\} = \dfrac{2^s - 1}{2^s}$

So if $n > 200$ $\quad \{n + 200\} = \frac{1}{2} + \frac{1}{2}\{n\}$

if $n > 400$ $\quad \{n + 400\} = \frac{3}{4} + \frac{1}{4}\{n\}$

if $n > 600$ $\quad \{n + 600\} = \frac{7}{8} + \frac{1}{8}\{n\}$, etc.

So we put $e^{-200k} = \frac{1}{2}$ which gives

$$k = \frac{\log_e 2}{200} = .003465735903.$$

then B, which is $\dfrac{\delta}{rk}$, will be approximately $\dfrac{i}{.18}$ and will vary from about .06 for $i = 1\%$ to about .4 for $i = 7\frac{1}{2}\%$.

Also for weekly annuities C which is $\dfrac{r}{j_r}$ will be approximately $\dfrac{52}{i}$ and vary from about 5200 for $i = 1\%$ to about 700 for $i = 7\frac{1}{2}\%$.

Now since $\{n\}$ will go from 0 at $n = 0$ to $\frac{1}{2}$ at $n = 200$, I find it more convenient to multiply $\{n\}$ by 1000 and divide C by 1000 and so tabulate

(A) $\quad \{n\} = 1000\,(1 - e^{-nk})$

(B) $\quad B = \dfrac{\delta}{rk}$

(C) $\quad C = \dfrac{r}{1000 j_r}$

Thus $\{n\}$ will go from 0 at $n = 0$ to 500 at $n = 200$, C will go from about 5.2 at 1% to about .70 at $7\frac{1}{2}\%$, while B, as before, will vary from about .06 at 1% to .4 at $7\frac{1}{2}\%$.

One more difficulty remains and that is that of interpolating in the table for $\{n\}$. Even though m is usually a whole number, the number n, with which we want to enter the table, will usually, except by sheer chance, consist of an integer plus a decimal portion for it is equal to Bm and B is not integral.

Now as $\{n\} = 1000\,(1 - e^{-kn})$

$$\Delta\{n\} = \{n+1\} - \{n\} = 1000\,e^{-kn}\,(1 - e^{-k})$$

$$\Delta^2\{n\} = \Delta\{n+1\} - \Delta\{n\} = -1000\,e^{-kn}\,(1 - e^{-k})^2$$

So $-\Delta^2\{n\}$ is about .012 at $n = 0$ and decreases to about .006 at $n = 200$. Thus if we use ordinary first difference interpolation in the table for $\{n\}$ the maximum error, which is equal to $\left|\frac{\Delta^2\{n\}}{8}\right|$ would be between .0015, the value for $n = 0$ and .00075, the value for $n = 200$.

This is not accurate enough; for example, if n is less than 100 the maximum error may exceed .001 and thus the result is not reliable to 5 significant figures.

We must thus use a more powerful interpolation procedure. We could use second difference interpolation but it is awkward to do this, even with a calculating machine, especially for the inverse interpolation which we have to use to obtain n from $\{n\}$. Fortunately, we can get over the difficulty by a simple procedure based on the fact that the differences of $\{n\}$ are in geometrical progression.

This procedure is arrived at as follows:

If we wish to obtain $\{n+t\}$ where n is integral and t is less than 1, then

$$\{n+t\} - \{n\} = 1000\,e^{-kn}\,(1 - e^{-kt})$$

$$\Delta\{n\} = 1000\,e^{-kn}\,(1 - e^{-k})$$

So $$\frac{\{n+t\} - \{n\}}{\Delta\{n\}} = \frac{1 - e^{-kt}}{1 - e^{-k}}$$

Put $\bar{t}$ for the expression on the right hand side of the last equation. This is independent of n and depends solely on t, as k is fixed. That this is so, arises from the fact that $\Delta\{n\}$ is a geometric series. Thus $\{n+t\} = \{n\} + \bar{t}\,\Delta\{n\}$. In other words, we use first difference interpolation, putting however $\bar{t}$ for t. Now $\bar{t}$ and t are equal if $t = 0$ or 1 and are nearly equal if t is between 0 and 1, as it is.

$$\bar{t} = \frac{kt - \frac{k^2t^2}{\lfloor 2} + \frac{k^3t^3}{\lfloor 3} - \cdots}{k - \frac{k^2}{\lfloor 2} + \frac{k^3}{\lfloor 3} - \cdot}$$

$$= t + \frac{t\,(1-t)\,k}{2} + \frac{t\,(1-t)\,(1-2t)\,k^2}{12} + \cdots$$

So if we put $u = \bar{t} - t$

$$u = \frac{t\,(1-t)k}{2} + \cdots$$

and the maximum value of u is at $t =$ approximately $\frac{1}{2}$

$$\left(\text{more exactly } \tfrac{1}{2} - \frac{k}{24} - \cdots \right)$$

when $u =$ approximately $\frac{k}{8}$ (more exactly $\frac{k}{8} - \frac{k^3}{526} + \cdots$) or $.00043322 \cdots$

(See the appendix to this paper for mathematical details.)

Thus if we wish to have t and $\bar{t}$ correct to five decimal places, when $\bar{t}\,\Delta\,\{n\}$ will be correct to five significant figures or at least four decimal places, we can get $\bar{t}$ by adding to t a correction u of less than .00044.

So we prepare an auxiliary table giving the ranges of t, to five decimal places, for which an addition of .00001, or .00002, etc. up to .00043 must be made to t to get $\bar{t}$. Naturally, the end points of these ranges are those for which u is equal to .000005, .000015, etc. For example, the range of t for which u equals .00012, that is to say, that in which .00012 must be added to t, runs from the value of t for which $u = .000115$ to that for which $u = .000125$ and the table, Table 2, is so calculated. There will be two ranges for a given value of u, one between 0 and the value of t for which u is a maximum and the other between this value and unity (see Table 2). Table 2 is an example of a so-called "critical table".

Our complete procedure and a simple one can now be set forth. First, we have the following tables given at the end of this Part I:—

Table 1 giving the values of $\{n\}$ and $\Delta\,\{n\}$ for $n = 0, 1, 2$, etc., up to 200.
Table 2 giving the value of u for all values of t and $\bar{t}$.
Table 3 giving the values of B and C for weekly annuities of 1 for $r = 52$ and $r = 52.1775$ for all rates of interest at intervals of ¼% from ¼% to 7½%.

Second, to find the present value of an annuity for 1 per week for m weeks at a given rate of interest, r being specified:

(i) Take out of Table 3, the value of B for the given i and r.
(ii) Multiply m by B and express the result in the form $n + t$ where n is a whole number and t a pure decimal to five places.
(iii) From Table 2, get the addition u to be added to t to get $\bar{t}$.
(iv) From Table 1, calculate $\{n + t\} = \{n\} + \bar{t}\,\Delta\,\{n\}$
(v) Multiply $\{n + t\}$ by C from Table 3 for the given i and r and the result is the required present value.

The answer will be correct to four figures more than the number of figures in the integral portion of $\{n + t\}$ but in the calculation find $\{n + t\}$ to five decimal places.

If n exceeds 200, use the formula given at the foot of Table 1. The process can obviously be reversed to find m, for a given i and r, from the present value by dividing the given present value by C, entering Table 1 inversely with the result to get $n + \overline{t}$, subtracting the proper value of u to get $n + t$, and dividing $n + t$ by B to get m.

For working purposes, the method can be expressed a little differently and perhaps a little more clearly, as follows:—

> To find the present value of P per week for M weeks (r and i being given) take B and C from Table 3 and get
>
> $BM =$ the adjusted number of weeks
>
> and $CP =$ the adjusted weekly payment
>
> Enter Table 1 with the adjusted number of weeks and multiply by the adjusted weekly payment. The result is the required present value, namely $CP\,\{BM\}$.
>
> To find, on the other hand, the number of weeks M, for which the present value has given value X, find the adjusted number of weeks BM from $\{BM\} = \dfrac{X}{CP}$ and then divide by B to get M.
>
> The whole procedure, whether direct or inverse, is very readily done with a calculating machine.

If we wish to use for r a value other than those given in Table 3, this value will be obviously somewhere near 52 and we proceed by interpolating in Table 3 for C. For example, for $r = 52\frac{1}{7}$, if the values of C for $r = 52$ and $r = 52.1775$ are C_1 and C_2 respectively, we use $C = C_1 + \dfrac{\frac{1}{7}}{.1775}(C_2 - C_1)$. As for B we use $\dfrac{52\,B_1}{52\frac{1}{7}}$ where B_1 is the value for $r = 52$.

A similar procedure can be used for annual annuities, i.e. 1 per annum payable annually, half-yearly, quarterly, etc. The only change required is in the B and C values. We must put $B = \dfrac{\delta}{k} = \delta \times 288.5390082$, which does not vary with the frequency of payment, and $C = \dfrac{1}{1000 j_r}$ which does change with the frequency of payment. It is usually, however, easier to use tables of $a_{\overline{n|}}^{m}$ and $\dfrac{i}{j_r}$, of which there are many available, but for completeness I give in Table 4 the necessary information for calculating B and C for annual annuities.

We can also use the tables for obtaining other interest functions. In fact, given the values of i and r, if for m weeks we calculate the value of $\{n+t\} = Bm$ we have

$$\{n+t\} = 1000\,(1 - v^{\frac{m}{r}})$$

so the present value of 1 due m weeks hence is $1 - \dfrac{\{Bm\}}{1000}$

and the amount of 1 accumulated for m weeks is $\dfrac{1000}{1000 - \{Bm\}}$

and the sum of an annuity of 1 per week accumulated for m weeks is $C\,\dfrac{1000\,\{Bm\}}{1000 - \{Bm\}}$

to which we add our first result, the present value of an annuity of 1 per week for m weeks is $C\,\{Bm\}$

We get similar expressions in connection with annual payments.

In Table 5, I have collected these formulas together for ready reference.

Some examples for the use of these tables follow. Most of these deal with the same data as in the interest examples in the previous paper.

Examples of the Use of the Tables

(1) Find the present value at 3¾% per annum compound interest of an annuity certain of 12.83 a week for 400 weeks. (52.1775 weeks to the year).

From Table 3 the "adjusted no. of weeks" is

$$400 \times .20357946 = 81.43178$$

and the "adjusted weekly payment" is

$$12.83 \times 1.4168287 = 18.177912$$

Now $\{81.43178\} = 244.76371 + 2.61292\,(.43178 + .00043)$

$$= 245.89304$$

so the required present value is $18.177912 \times 245.89304$

$= 4469.822$ to seven significant figures.

(2) Find the accumulated amount of the annuity in (1) at the end of the 400 weeks.

If V is the present value of 1 due 400 weeks hence, with the given data, then $V = 1 - \dfrac{\{81.43178\}}{1000} = .75410696$ and the required amount is the present value found in (1) divided by V or 5927.305.

(3) Find the present value of 1625.14 at 3¾% per annum compound interest due 400 weeks hence (52.1775 weeks to the year).

The value is $1625.14 \times V$ or 1225.529.

(4) Find the amount of 1625.14 accumulated for 400 weeks (52.1775 to the year) at $3\frac{3}{4}\%$ per annum compound interest.

The amount is $1625.14 \div V$ or 2155.052.

(5) Find the present value of the annuity in (1) if a year be assumed to have 52 weeks.

The "adjusted no. of weeks" is $400 \times .20427437$ or 81.70975 and the "adjusted weekly payment" is 12.83×1.4120072 or 18.116052 and the present value is

$$18.116052 \{81.70975\}$$
$$\text{or } 18.116052 [244.76371 + (.70975 + .00036)\, 2.61292]$$
$$\text{or } 18.116052 \times 246.61917$$

or 4467.766 to seven significant figures.

(6) For how many weeks will a payment of 1000 suspend an annuity certain for 12 per week, at 3% per annum compound interest, allowing 52.1775 weeks to the year?

If w is the number of weeks we have

"adjusted no. of weeks" is $.16345872 \times w = n$

"adjusted weekly payments" is 12×1.7647103 or 21.176524

and $21.176524 \{n\} = 1000.$

So $\{n\} = 47.22210$

$$n = 13 + (.95770 - .00007)$$
$$\text{for } \{n\} - \{13\} = 3.16742 = .95770 \times \Delta \{13\}$$
$$= 13.95763$$

so $w = 85.3893$

(7) By how many weeks will a payment of 1000 now shorten an annuity certain of 12 per week payable for 300 weeks, at 3% compound interest per annum, 52.1775 weeks to the year?

If x is the number of weeks in the shortened annuity we have

$$B = .16345872$$
$$C = 1.7647103$$

and $1000 = 12\, [C \{300\, B\} - C \{Bx\}]$

$$\{Bx\} = -47.22210 + \{300\, B\} = -47.22210 + \{49.03762\}$$
$$= 109.07210$$

So $Bx = 33.32388$

and $x = 203.8673$

So the annuity is shortened by 96.1327 weeks.

(8) Find the present value of an annuity of 1 per week payable for 467 weeks (52.1775 to the year) at $3\frac{1}{2}\%$ per annum compound interest.

Adjusted no. of weeks $= 88.84124$
Adjusted weekly payments $= 1.5162249$
Present value is $1.5162249 \{88.84124\}$ or 401.8169.

(9) Find the present value and amount of an annuity of 1, payable half-yearly, for 50 years at $2\frac{1}{2}\%$ per annum.

B ($= 52\,B$ for weekly annuities 52 weeks to year) $= 7.1247821$
Adjusted number of years $= 356.23911$

$$\{356.23910\} = 500 + \tfrac{1}{2}\{156.23910\} = 709.05781$$

Divide this by $1000\, j_{(2)}$ (from Table II of the previous paper) which is 24.845673 and we get the required result of 28.53848.

TABLE 1

n	$\{n\}$	$\Delta\{n\}$	n	$\{n\}$	$\Delta\{n\}$
0	0.00000	3.45974	33	108.07148	3.08584
1	3.45974	3.44776	34	111.15732	3.07516
2	6.90750	3.43584	35	114.23248	3.06452
3	10.34334	3.42396	36	117.29700	3.05392
4	13.76730	3.41210	37	120.35092	3.04336
5	17.17940	3.40030	38	123.39428	3.03282
6	20.57970	3.38854	39	126.42710	3.02234
7	23.96824	3.37681	40	129.44944	3.01187
8	27.34505	3.36513	41	132.46131	3.00146
9	30.71018	3.35349	42	135.46277	2.99107
10	34.06367	3.34189	43	138.45384	2.98072
11	37.40556	3.33032	44	141.43456	2.97041
12	40.73588	3.31880	45	144.40497	2.96014
13	44.05468	3.30732	46	147.36511	2.94989
14	47.36200	3.29588	47	150.31500	2.93969
15	50.65788	3.28447	48	153.25969	2.92951
16	53.94235	3.27311	49	156.18420	2.91939
17	57.21546	3.26179	50	159.10359	2.90928
18	60.47725	3.25050	51	162.01287	2.89921
19	63.72775	3.23926	52	164.91208	2.88919
20	66.96701	3.22805	53	167.80127	2.87918
21	70.19506	3.21688	54	170.68045	2.86923
22	73.41194	3.20575	55	173.54968	2.85930
23	76.61769	3.19466	56	176.40898	2.84941
24	79.81235	3.18361	57	179.25839	2.83955
25	82.99596	3.17259	58	182.09794	2.82973
26	86.16855	3.16162	59	184.92767	2.81993
27	89.33017	3.15067	60	187.74760	2.81018
28	92.48084	3.13978	61	190.55778	2.80046
29	95.62062	3.12892	62	193.35824	2.79077
30	98.74954	3.11809	63	196.14901	2.78111
31	101.86763	3.10730	64	198.93012	2.77149
32	104.97493	3.09655	65	201.70161	2.76191
33	108.07148	3.08584	66	204.46352	2.75234

$\{n + \bar{t}\} = \{n\} + \bar{t}\,\Delta\{n\}$

where $\bar{t} = t + u$ from table 2

$\{n\} = 1000\,(1 - e^{-kn})$ $\quad e^{200k} = 2 \quad k = .002465735903$

TABLE 1 (Continued)

n	$\{n\}$	$\Delta\{n\}$	n	$\{n\}$	$\Delta\{n\}$
66	204.46352	2.75234	100	292.89322	2.44640
67	207.21586	2.74283	101	295.33962	2.43794
68	209.95869	2.73333	102	297.77756	2.42951
69	212.69202	2.72388	103	300.20707	2.42110
70	215.41590	2.71446	104	302.02817	2.41272
71	218.13036	2.70506	105	305.04089	2.40438
72	220.83542	2.69570	106	307.44527	2.39605
73	223.53112	2.68638	107	309.84132	2.38777
74	226.21750	2.67709	108	312.22909	2.37951
75	228.89459	2.66782	109	314.60860	2.37127
76	231.56241	2.65859	110	316.97987	2.36307
77	234.22100	2.64940	111	319.34294	2.35490
78	236.87040	2.64022	112	321.69784	2.34674
79	239.51062	2.63110	113	324.04458	2.33863
80	242.14172	2.62199	114	326.38321	2.33054
81	244.76371	2.61292	115	328.71375	2.32247
82	247.37663	2.60388	116	331.03622	2.31444
83	249.98051	2.59487	117	333.35066	2.30693
84	252.57538	2.58589	118	335.65709	2.29845
85	255.16127	2.57694	119	337.95554	2.29050
86	257.73821	2.56804	120	340.24604	2.28258
87	260.30625	2.55914	121	342.52862	2.27468
88	262.86539	2.55029	122	344.80330	2.26681
89	265.41568	2.54147	123	347.07011	2.25896
90	267.95715	2.53268	124	349.32907	2.25115
91	270.48983	2.52391	125	351.58022	2.24336
92	273.01374	2.51518	126	353.82358	2.23561
93	275.52892	2.50648	127	356.05919	2.22786
94	278.03540	2.49781	128	358.28705	2.22016
95	280.53321	2.48917	129	360.50721	2.21248
96	283.02238	2.48055	130	362.71969	2.20482
97	285.50293	2.47197	131	364.92651	2.19719
98	287.97490	2.46342	132	367.12170	2.18960
99	290.43832	2.45490	133	369.31130	2.18201
100	292.89322	2.44640	134	371.49331	2.17447

$\{n + \bar{t}\} = \{n\} + \bar{t}\,\Delta\{n\}$
where $\bar{t} = t + u$ from table 2

TABLE 1 (Continued)

n	$\{n\}$	$\Delta\{n\}$	n	$\{n\}$	$\Delta\{n\}$
134	371.49331	2.17447	167	439.41696	1.93947
135	373.66778	2.16695	168	441.35643	1.93276
136	375.83473	2.15944	169	443.28919	1.92007
137	377.99417	2.15198	170	445.21526	1.91941
138	380.14615	2.14453	171	447.13467	1.91277
139	382.29068	2.13711	172	449.04744	1.90015
140	384.42779	2.12972	173	450.95359	1.89956
141	386.55751	2.12235	174	452.85315	1.89298
142	388.67986	2.11501	175	454.74613	1.88644
143	390.79487	2.10769	176	456.63257	1.87991
144	392.90256	2.10040	177	458.51248	1.87340
145	395.00296	2.09313	178	460.38588	1.86692
146	397.09609	2.08589	179	462.28280	1.86047
147	399.18198	2.07867	180	464.11327	1.85403
148	401.26068	2.07148	181	465.96730	1.84761
149	403.33213	2.06431	182	467.81491	1.84122
150	405.39644	2.05717	183	469.65613	1.83485
151	407.45361	2.05006	184	471.49098	1.82850
152	409.50367	2.04296	185	473.31948	1.82218
153	411.54663	2.03590	186	475.14166	1.81587
154	413.58253	2.02885	187	476.95753	1.80959
155	415.61138	2.02183	188	478.76712	1.80333
156	417.63321	2.01483	189	480.57045	1.79709
157	419.64804	2.00787	190	482.36754	1.79087
158	421.65591	2.00092	191	484.15841	1.78468
159	423.65683	1.99399	192	485.94309	1.77850
160	425.65082	1.98710	193	487.72159	1.77235
161	427.63792	1.98022	194	489.49394	1.76621
162	429.01814	1.97337	195	491.26015	1.76011
163	431.59151	1.96655	196	493.02026	1.75402
164	433.55806	1.95974	197	494.77428	1.74794
165	435.51780	1.95296	198	496.52222	1.74191
166	437.47076	1.94620	199	498.26413	1.73587
167	439.41696	1.93947	200	500.00000	

$\{n+t\} = \{n\} + \bar{t}\,\Delta\{n\}$

where $\bar{t} = t + u$ from table 2

$\{n+200\} = 500 + \tfrac{1}{2}\{n\}$

$\{n+400\} = 750 + \tfrac{1}{4}\{n\}$

$\{n+600\} = 875 + \tfrac{1}{8}\{n\}$

etc.

$$\{n + 200\,s\} = 1000 - \frac{1000}{2^s} + \frac{\{n\}}{2^s}$$

TABLE 2

t	u	$\overline{t}$	t	u	$\overline{t}$
.00000	0	.00000	.43100	43	.43143
.00290	1	.00291	.56872	42	.56914
.00873	2	.00875	.60240	41	.60281
.01464	3	.01467	.62748	40	.62788
.02061	4	.02065	.64838	39	.64877
.02667	5	.02672	.66668	38	.66706
.03280	6	.03286	.68317	37	.68354
.03902	7	.03909	.69829	36	.69865
.04532	8	.04540	.71234	35	.71269
.05170	9	.05179	.72552	34	.72586
.05818	10	.05828	.73797	33	.73830
.06476	11	.06487	.74980	32	.75012
.07144	12	.07156	.76109	31	.76140
.07822	13	.07835	.77192	30	.77222
.08512	14	.08526	.78232	29	.78261
.09213	15	.09228	.79237	28	.79265
.09926	16	.09942	.80208	27	.80235
.10526	17	.10543	.81148	26	.81174
.11393	18	.11411	.82061	25	.82086
.12147	19	.12166	.82949	24	.82073
.12917	20	.12937	.83814	23	.83837
.13703	21	.13724	.84657	22	.84679
.14507	22	.14529	.85480	21	.85501
.15329	23	.15352	.86284	20	.86304
.16172	24	.16196	.87071	19	.87090
.17035	25	.17060	.87841	18	.87859
.17923	26	.17949	.88597	17	.88614
.18835	27	.18862	.89464	16	.89480
.19775	28	.19803	.90064	15	.90079
.20746	29	.20775	.90778	14	.90792
.21749	30	.21779	.91480	13	.91493
.22789	31	.22820	.92170	12	.92182
.23871	32	.23903	.92849	11	.92860
.25000	33	.25033	.93518	10	.93528
.26182	34	.26216	.94176	9	.94185
.27426	35	.27461	.94825	8	.94833
.28742	36	.28778	.95464	7	.95471
.30147	37	.30184	.96095	6	.96101
.31659	38	.31697	.96717	5	.96722
.33307	39	.33346	.97331	4	.97335
.35137	40	.35177	.97937	3	.97940
.37226	41	.37267	.98536	2	.98538
.39734	42	.39776	.99127	1	.99128
.43100	43	.43143	.99711	0	.99711

$$\overline{t} = t + u$$

For values of t, or $\overline{t}$, between the values given use for u the value for the next smaller value of t, or $\overline{t}$.

TABLE 3

Values of B and C for Weekly Annuities of 1

i	$r = 52$		$r = 52.1775$	
	B	C	B	C
¼ %	.013854756	20.825489	.013807625	20.896578
½ %	.027675005	10.425478	.027580859	10.461067
¾ %	.041460918	6.9588010	.041319874	6.9825563
1 %	.055212665	5.2254569	.055024840	5.2432955
1¼ %	.068930415	4.1854462	.068695924	4.1997348
1½ %	.082614336	3.4921021	.082333295	3.5040223
1¾ %	.096264594	2.9968534	.095937117	3.0070848
2 %	.10988135	2.6254142	.10950755	2.6343760
2¼ %	.12346478	2.3365147	.12304477	2.3444920
2½ %	.13701504	2.1053930	.13654893	2.1125814
2¾ %	.15053228	1.9629916	.15002020	1.9228345
3 %	.16401668	1.7587053	.16345872	1.7647103
3¼ %	.17746839	1.6253615	.17686467	1.6309113
3½ %	.19088202	1.5110653	.19023820	1.5162249
3¾ %	.20427437	1.4120072	.20357946	1.4168287
4 %	.21762896	1.3253301	.21688861	1.3298558
4¼ %	.23095148	1.2488491	.23016581	1.2531137
4½ %	.24424209	1.1808649	.24341121	1.1848974
4¾ %	.25750094	1.1200358	.25662496	1.1238607
5 %	.27072818	1.0652887	.26980721	1.0689267
5¼ %	.28392397	1.0157546	.28295811	1.0192235
5½ %	.29708846	.97072263	.29607781	.97403786
5¾ %	.31022178	.92960566	.30916645	.93278054
6 %	.32332410	.89191428	.32222420	.89496050
6¼ %	.33639554	.85723741	.33525118	.86016526
6½ %	.34943627	.82522723	.34824754	.82804582
6¾ %	.36244642	.79555093	.36121343	.79830486
7 %	.37542614	.76806409	.37414900	.77068755
7¼ %	.38837557	.74243822	.38705437	.74497420
7½ %	.40129485	.71852009	.39992970	.72097443

Weekly Annuity of 1 for W weeks $= \{W \times B\} \times C$

Values of $\{W \times B\}$ from Tables 1 and 2.

Note: $B = 288.5390082 \times \dfrac{C}{r}$ $\qquad C = \dfrac{r}{1000\, j_r}$

TABLE 4

Annual Annuities

Value of annuity of 1 per annum, payable r times a year for Y years,

$$= \{Y \times B\} \times C$$

where $B = 288.5390082 \times \delta$, $C = \dfrac{1}{1000\, j_r}$ — Value of δ $(= j_\infty)$ and j_r from Table II P.C.A.S. XXV p. 130-1.

Value of $\{Y \times B\}$ from Tables 1 and 2; or, preferably, value $= \dfrac{\{Y \times B\}}{1000\, j_r}$

where $B = 52 \times$ value of B for $r = 52$ in Table 3 and j_r is taken from Table II P.C.A.S. XXV p. 130-1.

TABLE 5

If $a_{\overline{n}|}$, value of an annuity $= C\,\{N\}$

then $s_{\overline{n}|}$, amount of the annuity $= \dfrac{1000}{1000 - \{N\}} \times C\,\{N\}$

and v^n, present value of 1 for the term of the annuity, $= \dfrac{1000 - \{N\}}{1000}$

and $(1 + i)^n$, amount of 1 for the term $= \dfrac{1000}{1000 - \{N\}}$

Part II

In the previous paper I also gave tables for taking out rapidly, with the aid of a calculating machine, logarithms and antilogarithms to ten figures. These tables are based on the well-known factorial method and are reasonably accurate and rapid. Nevertheless, their use requires the splitting of a number into four factors which takes a little time and care. If a method could be devised to reduce the number of factors, to say two, the procedure would be much simplified and speeded up. This method I shall give in this Part II; it rests on the same method of interpolation used in Part I for $\{n\}$. We have seen that this stems from the circumstance that $\Delta\,\{n\}$ is a geometrical series; if the differences of log nk for successive values of n formed a geometrical series we could use the same method; however, this is not so but the differences of antilog nk are, since *$\log^{-1} nk = 10^{nk}$ which is itself a geometric series for successive values of n and therefore its differences are also. In fact $\Delta \log^{-1} nk = \log^{-1} (n+1)\, k - \log^{-1} nk = 10^{nk}\,(10^k - 1)$. As a matter of fact $\{n\}$ in Part I is a kind of table of antilogarithms since $\{n\}$ is a thousand times the complement of the antilogarithm of n to the rather unusual base of $e^{-k} = .9965402$.

The circumstances that we have to tabulate antilogarithms instead of logarithms to be able to apply the special method of interpolation to obtain, from a short table, results to a large number of figures, is no disadvantage; for in using tables of logarithms or antilogarithms, we almost always have to enter the tables first directly or inversely to get a logarithm and then later, enter the table the other way to get an antilogarithm. In fact, some modern calculators find advantages in tabulating antilogarithms instead of logarithms. For example, Frederic Deprez, in the introduction to his "Tables for Calculating, By Machine, Logarithms to Thirteen Places of Decimals" published in Switzerland in 1939, states "It is well known that a table of logarithms may be used directly for finding logarithms and inversely for finding antilogarithms. After extensive research, the author has found it better, when viewing the subject with the availability of calculating machines in mind, to produce a table of antilogarithms, which is used directly to find antilogarithms and inversely, to find logarithms. The advantages of this method will be enhanced when the log of the final number desired can be formed from tabulated logarithms, as is usually the case in interest calculations; this means that the tables are used more frequently directly than inversely". Incidentally, it is interesting to note that Mr. Deprez' tables are constructed on the factorial method.

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* Note: We shall use, as convenient, both of the notations antilog N and $\log^{-1} N$ to denote the antilogarithm of N, i.e. the number whose logarithm is N.

As will be seen, our two tables for the two factors to which we shall reduce the taking out of logarithms and antilogarithms will be, one, a log table and the other an antilog table.

Let us then analyze a table of antilogarithms to base 10 with an interval of 10^{-r}, that is to say, a table of $\log^{-1} 10^{-r} n$ as follows:

$$\log^{-1} .00\cdots(r-1 \text{ zeros})\cdots 01$$
$$\log^{-1} .00\cdots(r-1 \text{ zeros})\cdots 02$$
$$\log^{-1} .00\cdots(r-1 \text{ zeros})\cdots 03$$
etc.

(it is, of course, only necessary to tabulate antilogarithms between 0 and 1).

The differences are shown below where R is put for

$$10^{-r} = .00\cdots(r-1 \text{ zeros})\cdots .001$$

	$\log^{-1}$	Δ	Δ^2
Rn	10^{Rn}	$10^{Rn}\ (10^R-1)$	$10^{Rn}\ (10^R-1)^2$
$R\ (n+1)$	$10^{R\ (n+1)}$	$10^{R\ (n+1)}\ (10^R-1)$	$10^{R\ (n+1)}\ (10^R-1)^2$
$R\ (n+2)$	$10^{R\ (n+2)}$	$10^{R\ (n+2)}\ (10^R-1)$	$10^{R\ (n+2)}\ (10^R-1)^2$

etc.

A little study of this shows that ordinary first difference interpolation gives about $2r$ places correct. Let us see how to increase this accuracy by a method similar to that in Part I. Let us for shortness put temporarily $[n]$ for 10^{Rn}.

Then
$$[n+1] = 10^{R(n+1)}$$
$$\Delta\,[n] = 10^{Rn}\,(10^R-1)$$
$$\frac{[n+t]-[n]}{\Delta\,[n]} = \frac{10^{Rt}-1}{10^R-1} = \bar{t} \text{ say}$$
and
$$[n+t] = [n] + \bar{t}\,\Delta\,[n]$$

where as before $t>0$ and <1 and $\bar{t}$ is nearly equal to t, for putting $k = R\log_e 10 = \mathrm{R}\times 2.30259\cdots$ so that $10^R = e^k$

$$\bar{t} = \frac{e^{kt}-1}{e^k-1} = t - \frac{t\,(1-t)\,k}{2} + \text{etc.}\cdots$$

and
$$u = t - \bar{t} = \frac{t\,(1-t)}{2}\,k - \frac{t\,(1-t)\,(1-2t)}{12}\,k^2 + \cdots$$

The maximum value of u is when

$$t = \tfrac{1}{2} + \frac{k}{24} - \cdots$$

$$\text{and is } u = \frac{k}{8} - \frac{k^3}{576} + \cdots$$

Now $k = R \times 2.30259\cdots = .00\cdots(r-1 \text{ zeros})\cdots 0230259\cdots$ So the maximum value of u is approximately $.00\cdots(r \text{ zeros})\cdots 02878\cdots$ which is its value when $t = .500\cdots(r+1 \text{ zeros})\cdots 096\cdots$ approximately.

Thus if in the formula

$$[n+t] = [n] + (t-u)\,\Delta\,[n]$$

we have $[n]$ tabulated to $2r+s$ significant figures then $\Delta\,[n]$ will contain $r+s$ or $r+s+1$ significant figures and $t-u$ should consist of $r+s$ significant figures, i.e. u must be found to s figures.

For example, to take the case of $r=4$ or $R=.0001$, k will equal $.0002302585\cdots$ and the maximum value of u is $.00002878\cdots$, so if we want results to 10 places, we must tabulate log .0001, log .0002, etc., to 10 places which will give first differences with 6 or 7 figures. Then we must have $\bar{t} = t - u$ to 6 places at least, which means we must calculate u to two significant figures and use t with 6 figures. Thus u will not be greater than .000029 and we will need a table of the values of t, for which u is to be taken as .000000, .000001, .000002, etc., up to .000029. What we do is to calculate t to 6 significant figures for $u = .0000005$, etc., in a manner similar to that explained in Part I. These values are given in Table III. As in Part I, there will, of course, be two ranges for which u is equal to a given value, the first between 0 and the value of t for which u is greatest and the other between that value and unity.

We could, when and if we wanted to, proceed similarly for other values of r and s. For example, we could get 14 place antilogarithms by taking $r=5$ and $s=4$ when we would have to calculate the ranges of t for u from .000000001 up to .000002878.

To return to our 10 place tables, all we need now is a table of antilogarithms of .0000, .0001, .0002, etc., up to .9999, together with the first differences (for they are convenient in performing the first difference interpolations to which we have reduced the taking of values out of the table). This would mean a table of 10,000 entries, too many to give in this paper. We can, however, reduce this by using the same factorial method given in the previous paper. The Table III of that paper gave the logarithms to 10 places of 1.00, 1.02, etc., up to 10.00 at varying intervals such that the difference between any two successive logarithms did not exceed the log of $1.022\cdots$ or $.009545\cdots$ so if we use this table—and it is given in this paper as Table 1—and a table of antilogarithms of .0000, .0001, etc., up to say .0096 (about 100 entries)—see Table II—we can rapidly obtain 10 place logarithms and antilogarithms.

The complete procedure is:

First, the tables are:

Table I. Values of $\log N$ for values of N from 1.00 to 10.00

Table II. Values of $\log^{-1} M$ and $\Delta \log^{-1} M$ for values of M by intervals of .0001 from .0000 to .0100.

Table III. Values of t, u and $\bar{t}$.

Second, to find the antilog of a log supposed given to 10 decimal places. Subtract the largest log in Table I that is less than the given log. The remainder will be less than .0096. The first four decimals will be n and the next six t. From Table III take out u and *subtract* it from t to get $\bar{t}$. Then $\log^{-1}(n+t)$ will be $\log^{-1} n + \bar{t}\,\Delta \log^{-1} n$. Multiply this by the number from Table I, whose log was subtracted in the first step and the product to 10 places is the required antilog.

Third, to find the log of a number supposed given to 10 significant figures, we reverse the process. Divide the number by the largest number in Table I that is less than the given number. Take the result to 10 significant figures and enter inversely in Table II getting a result of 10 decimals. Adjust the last six from Table III by *adding* the proper value of u. Finally add the log from Table I of the number used as divisor in the first step. This gives the desired log.

Except for (i) the preparation of dividing by a number or subtracting by a log from Table I and (ii) the completion by adding the corresponding logarithm or multiplying by the corresponding number from Table I, the whole operation consists of a first difference interpolation, either direct or inverse, in Table II using an adjusted fractional part (t adjusted to $\bar{t}$) from Table III and it is decidedly easier and quicker than the fractorial method of the preceding paper.

It might have been theoretically more consistent if instead of the table of logs in Table I, I had given a table of antilogs of .01, .02, etc., up to 1.00, as then we would be using antilog tables throughout and not one log and one antilog table; but in practice this would not be so convenient for when carrying out the multiplication in the last step of taking out an antilog, or the division in the first step of taking out a log, we would be multiplying or dividing by a 10 figure number whereas in the procedure given above the multiplication or division is by a 3 figure number and this is a little easier. Furthermore, as it happened, Table I was already available.

The arrangement of logarithmic tables as given here for 10 places is obviously, and as indicated above, applicable to tables with more or less figures. I trust the 10 figure tables will prove useful. Tables of this power are not very accessible.

A word as to the accuracy of these tables and logarithm tables in particular. No ten figure table is absolutely accurate to the tenth figure; it can't be, from the nature of things. The same is true of five or seven or any other figure tables. I will illustrate with seven figure logarithms for we can check these with our ten figure table. A complete seven figure table gives, in effect, the logs to seven decimals of all numbers from 1,000,000 to 9,999,999 or 10,000,000 and the logarithms go from .0000000 to 1.0000000: there are 9,000,000 possible numbers and 10,000,000 possible logarithms so there cannot be one unique number for each logarithm. In actual fact the situation is even worse because at the beginning of the table, say just over 1,000,000 the logarithms increase faster than the numbers and at the end of the table, say near 9,999,999 the situation is reversed. Thus we get this sort of thing—

Near beginning of table:

log N	N (10 places)	N (7 places)
.0004004	1.000922380	1.000922
.0004005	1.000922610	
.0004006	1.000922841	
.0004007	1.000923071	1.000923
.0004008	1.000923302	
.0004009	1.000923532	
.0004010	1.000923763	
.0004011	1.000923993	1.000924
.0004012	1.000924224	
.0004013	1.000924454	
.0004014	1.000924685	1.000925

Near end of table:

N	log N (10 places)	log N (7 places)
9.600002	.9822713233	.9822713
9.600003	.9822713684	.9822714
9.600004	.9822714136	
9.600005	.9822714588	
9.600006	.9822715039	.9822715
9.600007	.9822715491	
9.600008	.9822715943	.9822716

At the beginning, to one seven-place number there belong 4 or even 5, seven-figure logarithms: at the end, to one seven-figure logarithm there belong 2, or even 3, seven-place numbers.

Absolute accuracy to the last place accordingly is not possible even apart from the errors introduced by rounding in the course of the work. It is remarkable, however, how little inaccurate seven-figure logarithm work is; except for long calculations or special circumstances it is usually reliable to six figures but if absolute seven-figure reliability is needed larger tables must be used. That is one reason why I have prepared the ten-place tables: however, it must be remembered that, similarly, these do not produce results absolutely accurate to the last, tenth, figure.

In the appendix, I give for the more mathematically minded, some details for the formulae for t and u.

There follow now some examples of the use of these tables: most of them are based on the same data as used in the examples of the previous paper.

Examples of the Use of the Tables

(1) Find log 1.05:

Divide 1.05 by 1.04 the next largest number in Table I: the result is 1.009615385, to ten figures.
Antilog 1.009615385 comes between .0041 and .0042 in Table II. Subtract antilog .0041 from 1.009615385; we get 0.000130083 which we divide by Δ antilog .0041 or 0.000232469 getting .559571. This is $\bar{t}$ which we adjust to t from Table III by *adding* 28 units: $t = .559599$.
So log 1.009615385 = .0041559599 and to get log 1.05 we add log 1.04 from Table I.
So log 1.05 = .0211892992.

(2) Find $\log^{-1} .6$:

Subtracting from .6 the largest possible log from Table I, namely, log 3.92, we get .0067139330.
Then
$\log^{-1} .0067139330 = \log^{-1} .0067 + (.139330 - .000014)\, \Delta \log^{-1} .0067$
where the adjustment (*subtraction*) of .000014 is taken from Table III.
So from Table II $\quad \log^{-1} .0067139330 = 1.015546936 + .000032581$
$= 1.015579517$

and $\log^{-1} .6 = 1.015579517 \times 3.92$
$= 3.981071707$

(3) Evaluate $1.23456789^{9.87654321}$:

$$\begin{aligned} 1.23456789 \div 1.22 &= 1.011940893 \\ \log^{-1} .0051 &= \underline{1.011812406} \\ & .000128487 \end{aligned}$$

Divide by $\Delta \log^{-1} .0051 = .000233005$ and we get .551435 which we adjust by adding .000028.

$$\begin{aligned} \text{So log } 1.011940893 &= .0051551463 \\ \text{Add} \quad \log 1.22 &= \underline{.0863598307} \\ \log 1.23456789 &= .0915149770. \end{aligned}$$

So $\log 1.23456789^{9.87654321} = 1.0915149770 \times 9.87654321$
$= 10.7803948347.$

To find $\log^{-1} .7803948347$ we subtract $\log 6$ and get $.0022435843$ and from Table III adjust the last six figures downward to 435815.

So $$\log^{-1} .002235843 = \log^{-1} .0022 + .435815 \,\Delta \log^{-1} .0022$$
$$= 1.005179411$$

and $$\log .7803948347 = 6 \times 1.005179411 = 6.031076466$$

Thus $1.23456789^{9.87654321} =$ 60,310,764,660 nearly.

(The correct figure is 60,310,764,802.44$\cdots$)

(4) Find π^{19}: (a) assuming $\pi = \dfrac{355}{113}$ (b) using the true value of 3.141592654$\cdots$

We find $\log \dfrac{355}{113} = .4971499094$, $\log \pi = .4971498726$.

Taking the antilogarithms of 19 times these we get

$$\left(\frac{355}{113}\right)^{19} = 2{,}791{,}568{,}434$$

$$\pi^{19} = 2{,}791{,}563{,}937$$

(5) Find a number such that its common logarithm (i.e. to base 10) is one-tenth of the number:

Let N be the required number. Then we have $\dfrac{N}{10} = \log N$.

If we make N go from 0 to plus infinity we find that $\dfrac{N}{10} - \log N$ is plus infinity for $N = 0$, and plus infinity for $N =$ plus infinity.

Its differential coefficient with respect to N is $\dfrac{1}{10} - \dfrac{\log e}{N}$ which is negative from $N = 0$ to $N = 10 \log e = 4.343\cdots$ and positive from there to $N =$ plus infinity. At $N = 10 \log e$, $\dfrac{N}{10} - \log N = -.204$ so therefore $\dfrac{N}{10} - \log N$ decreases continually from a positive value for $N = 0$ to a negative value for $N = 4.343$ and then increases continually to a positive value for $N =$ infinity. So there are two values of N for which $\dfrac{N}{10}$ equals $\log N$, namely one value below $4.34\cdots$ and one above $4.34\cdots$.

As to the second value we recognize at once it is $N = 10$ since $\log 10 = 1$.

As to the first we find it is greater than 1 since $\dfrac{1}{10} - \log 1$ is positive.

From Table I we find N is between 1.36 and 1.38

$\frac{N}{10}$	$\log N$	$\frac{N}{10} - \log N$
.136	.1335389··	+ .0024611··
.138	.1398791··	− .0018791··
Difference	.0063402··	.0043402··

Since an increase in $\log N$ in the second column by .0063042 brings a change (decrease) of .0043402 in $\frac{N}{10} - \log N$ in the third column, a change of .0024611 (which will make $\frac{N}{10} - \log N = 0$) will be brought about by an increase in $\log N$ of approximately $.0063402 \times \frac{24611}{43402}$ or .0035952. So $\log N$ must be increased by an amount between .0035 and .0036.

Making use of Table II we get the following

$$\log 1.36 + .0035 = .1370389084 = \log (10 \times .1371004588)$$
$$\log 1.36 + .0036 = .1371389084 = \log (10 \times .1371320311)$$

Now what we want is to find t such that in

$$\log 1.36 + .0001\,(35 + t) = .137089084 + .0001\,t$$
$$= \log 10\,(.1371004588 + \bar{t}.0000315723)$$

we have

$$.1370389084 + .0001\,t = .1371004588 + \bar{t}\,.0000315723$$

or $t - .315723\,\bar{t} = .615504$.

As a first approximation we put $\bar{t} = t$; then we get $t = .899495$ and for this value of t the value of u is .000010 and $\bar{t} = t - .000010$. So putting this value of $\bar{t}$ in the equation we get $t = .899492$.

Thus $\log 1.371288576 = \log 1.36 + .0035899492 = .1371288576$.

So $1.371288576\cdots$ is the second value for which $\frac{N}{10} = \log N$.

These two values, 10 and $1.371288575\cdots$, are examples of numbers whose logs to base 10 have the same significant figures as the numbers.

TABLE I

Logarithms of Numbers from 1.00 to 10.00

N	$\log N$	N	$\log N$
1.00	.00000 00000	1.80	.25527 25051
1.02	.00860 01718	1.83	.26245 10897
1.04	.01703 33393	1.86	.26951 29442
1.06	.02530 58653	1.89	.27646 18042
1.08	.03342 37555	1.92	.28330 12287
1.10	.04139 26852	1.95	.29003 46114
1.12	.04921 80227	1.98	.29666 51903
1.14	.05690 48513	2.01	.30319 60574
1.16	.06445 79892	2.04	.30963 01674
1.18	.07188 20073	2.07	.31597 03455
1.20	.07918 12460	2.10	.32221 92947
1.22	.08635 98307	2.13	.32837 96034
1.24	.09342 16852	2.16	.33445 37512
1.26	.10037 05451	2.19	.34044 41148
1.28	.10720 99696	2.22	.34635 29745
1.30	.11394 33523	2.25	.35218 25181
1.32	.12057 39312	2.30	.36172 78360
1.34	.12710 47984	2.35	.37106 78623
1.36	.13353 89084	2.40	.38021 12417
1.38	.13987 90864	2.45	.38916 60844
1.40	.14612 80357	2.50	.39794 00087
1.42	.15228 83444	2.55	.40654 01804
1.44	.15836 24921	2.60	.41497 33480
1.47	.16731 73347	2.65	.42324 58739
1.50	.17609 12591	2.70	.43136 37642
1.53	.18469 14308	2.75	.43933 26938
1.56	.19312 45984	2.80	.44715 80313
1.59	.20139 71243	2.85	.45484 48600
1.62	.20951 50145	2.90	.46239 79979
1.65	.21748 39442	2.95	.46982 20160
1.68	.22530 92817	3.00	.47712 12547
1.71	.23299 61103	3.05	.48429 98393
1.74	.24054 92483	3.10	.49136 16938
1.77	.24797 32664	3.15	.49831 05538

TABLE 1 (Continued)

Logarithms of Numbers from 1.00 to 10.00

N	$\log N$	N	$\log N$
3.20	.50514 99783	5.68	.75434 83357
3.25	.51188 33610	5.76	.76042 24834
3.30	.51851 39399	5.88	.76937 73261
3.35	.52504 48070	6.00	.77815 12504
3.40	.53147 89170	6.12	.78675 14221
3.45	.53781 90951	6.24	.79518 45897
3.50	.54406 80444	6.36	.80345 71156
3.55	.55022 83531	6.48	.81157 50059
3.60	.55630 25008	6.60	.81954 39355
3.68	.56584 78187	6.72	.82736 92731
3.76	.57518 78449	6.84	.83505 61017
3.84	.58433 12244	6.96	.84260 92396
3.92	.59328 60670	7.08	.85003 32577
4.00	.60205 99913	7.20	.85733 24964
4.08	.61066 01631	7.32	.86451 10811
4.16	.61909 33306	7.44	.87157 29355
4.24	.62736 58566	7.56	.87852 17955
4.32	.63548 37468	7.68	.88536 12200
4.40	.64345 26765	7.80	.89209 46027
4.48	.65127 80140	7.92	.89872 51816
4.56	.65896 48427	8.04	.90525 60487
4.64	.66651 79806	8.16	.91169 01588
4.72	.67394 19986	8.28	.91803 03368
4.80	.68124 12374	8.40	.92427 92861
4.88	.68841 98220	8.52	.93043 95948
4.96	.69548 16765	8.64	.93651 37425
5.04	.70243 05364	8.76	.94250 41062
5.12	.70926 99610	8.88	.94841 29658
5.20	.71600 33436	9.00	.95424 25094
5.28	.72263 39225	9.20	.96378 78273
5.36	.72916 47897	9.40	.97312 78536
5.44	.73559 88997	9.60	.98227 12330
5.52	.74193 90777	9.80	.99122 60757
5.60	.74818 80270	10.00	1.00000 00000

TABLE II

M	Antilog M	Δ	M	Antilog M	Δ
.0000	1.000000000	230285	.0025	1.005773063	231614
.0001	1.000230285	230338	.0026	1.006004677	231668
.0002	1.000460623	230391	.0027	1.006236345	231721
.0003	1.000691014	230444	.0028	1.006468066	231775
.0004	1.000921458	230498	.0029	1.006699841	231828
.0005	1.001151956	230550	.0030	1.006931669	231881
.0006	1.001382506	230603	.0031	1.007163550	231935
.0007	1.001613109	230657	.0032	1.007395485	231988
.0008	1.001843766	230709	.0033	1.007627473	232041
.0009	1.002074475	230763	.0034	1.007859514	232095
.0010	1.002305238	230816	.0035	1.008091609	232149
.0011	1.002536054	230869	.0036	1.008323758	232202
.0012	1.002766923	230922	.0037	1.008555960	232255
.0013	1.002997845	230976	.0038	1.008788215	232309
.0014	1.003228821	231028	.0039	1.009020524	232362
.0015	1.003459849	231082	.0040	1.009252886	232416
.0016	1.003690931	231135	.0041	1.009485302	232469
.0017	1.003922066	231188	.0042	1.009717771	232523
.0018	1.004153254	231242	.0043	1.009950294	232577
.0019	1.004384496	231294	.0044	1.010182871	232630
.0020	1.004615790	231348	.0045	1.010415501	232683
.0021	1.004847138	231401	.0046	1.010648184	232737
.0022	1.005078539	231455	.0047	1.010880921	232791
.0023	1.005309994	231508	.0048	1.011113712	232844
.0024	1.005541502	231561	.0049	1.011346556	232898
.0025	1.005773063	231615	.0050	1.011579454	232952

$$\text{Antilog}\,(M + .0001\,t) = \text{Antilog}\,M + \bar{t}\,\Delta$$
where $\bar{t} = t - u$ from Table III

TABLE II (Continued)

M	Antilog M	Δ	M	Antilog M	Δ
.0050	1.011579454	232952	.0075	1.017419366	234297
.0051	1.011812406	233005	.0076	1.017653663	234350
.0052	1.012045411	233059	.0077	1.017888013	234404
.0053	1.012278470	233113	.0078	1.018122417	234459
.0054	1.012511583	233166	.0079	1.018356876	234512
.0055	1.012744749	233220	.0080	1.018591388	234566
.0056	1.012977969	233273	.0081	1.018825954	234621
.0057	1.013211242	233328	.0082	1.019060575	234674
.0058	1.013444570	233381	.0083	1.019295249	234729
.0059	1.013677951	233435	.0084	1.019529978	234782
.0060	1.013911386	233488	.0085	1.019764760	234837
.0061	1.014144874	233543	.0086	1.019999597	234890
.0062	1.014378417	233596	.0087	1.020234487	234945
.0063	1.014612013	233650	.0088	1.020469432	234999
.0064	1.014845663	233704	.0089	1.020704431	235053
.0065	1.015079367	233757	.0090	1.020939484	235107
.0066	1.015313124	233812	.0091	1.021174591	235161
.0067	1.015546936	233865	.0092	1.021409752	235215
.0068	1.015780801	233919	.0093	1.021644967	235270
.0069	1.016014720	233973	.0094	1.021880237	235324
.0070	1.016248693	234027	.0095	1.022115561	235378
.0071	1.016482720	234080	.0096	1.022350939	235432
.0072	1.016716800	234135	.0097	1.022586371	235486
.0073	1.016950935	234189	.0098	1.022821857	235540
.0074	1.017185124	234242	.0099	1.023057397	235595
.0075	1.017419366	234297	.0100	1.023292992	

$$\text{Antilog}\,(M + .0001\,t) = \text{Antilog}\,M + \bar{t}\,\Delta$$
$$\text{where } \bar{t} = t - u \text{ from Table III}$$

TABLE III

t	u	$\bar{t}$	t	u	$\bar{t}$
.000000	0	.000000	.450491	29	.450462
.004363	1	.004362	.549529	28	.549501
.013204	2	.013202	.605547	27	.605520
.022209	3	.022206	.640807	26	.640781
.031387	4	.031383	.668857	25	.668832
.040749	5	.040744	.692871	24	.692847
.050305	6	.050299	.714208	23	.714185
.060069	7	.060062	.733605	22	.733584
.070054	8	.070046	.751510	21	.751489
.080277	9	.080268	.768222	20	.768202
.090756	10	.090746	.783953	19	.783934
.101509	11	.101498	.798856	18	.798838
.112561	12	.112549	.813051	17	.813034
.123938	13	.123925	.826629	16	.826613
.135669	14	.135655	.839665	15	.839650
.147791	15	.147776	.852219	14	.852205
.160346	16	.160330	.864341	13	.864328
.173383	17	.173366	.876072	12	.876060
.186962	18	.186944	.887448	11	.887437
.201157	19	.201138	.898499	10	.898489
.216061	20	.216041	.909252	9	.909243
.231792	21	.231771	.919729	8	.919721
.248505	22	.248483	.929952	7	.929945
.266411	23	.266388	.939936	6	.939930
.285808	24	.285784	.949700	5	.949695
.307147	25	.307122	.959255	4	.959251
.331161	26	.331135	.968616	3	.968613
.359212	27	.359185	.977793	2	.977791
.394473	28	.394445	.986798	1	.986797
.450491	29	.450462	.995639	0	.995639

$$\bar{t} = t - u$$

For values of t, or $\bar{t}$, between the values given use for u the value for the next smaller values of t, or $\bar{t}$.

APPENDIX

In Part II we have

$$\left.\begin{aligned} u &= t - \bar{t} = t - \frac{e^{kt}-1}{e^k-1} \\ &= \frac{t\,(1-t)}{2}k - \frac{t\,(1-t)(1-2\,t)}{12}k^2 - \frac{t^2\,(1-t)^2}{24}k^3 \\ &\qquad + \frac{t\,(1-t)(1-2\,t)(1+3\,t-3\,t^2)}{720}k^4 - \cdots \end{aligned}\right\} \quad \text{(i)}$$

which is the fundamental equation.

In our case k is $.0002302585\cdots$ and comes from $e^k = 10^{.0001}$ so that $k = .0001 \times \log_e 10$.

Let the maximum value of u, for $t > 0$ and < 1, be denoted by U, and let the value of t that produces U be denoted by T.

Differentiating the second equation of (i) by t and setting du/dt equal to zero to get the maximum value we have

$$0 = 1 - \frac{ke^{kT}}{e^k - 1}$$

whence $e^{kT} = \dfrac{e^k - 1}{k}$

and $T = \dfrac{1}{k}\log_e \dfrac{e^k - 1}{k}$

$$= \tfrac{1}{2} + \frac{k}{24} - \frac{k^3}{2880} + \cdots + (-1)^{s-1}\,\frac{B_{2s}\,k^{2s-1}}{2\,s\,\underline{|2\,s}} + \cdots \qquad \text{(ii)}$$

where the numbers B_{2s} are Bernouilli's numbers.

Putting these values of T and e^{kT} in (i) we get

$$U = T - \frac{1}{k} + \frac{1}{e^k - 1}$$

$$= \frac{k}{8} - \frac{k^3}{576} + \cdots + (-1)^{s-1}\,\frac{(2\,s+1)\,B_{2s}\,k^{2s-1}}{2\,s\,\underline{|2\,s}} + \cdots \qquad \text{(iii)}$$

The main problem, however, that we have with equation (i) is to determine the values of t for which u has a given value. We could do this by solving directly the third equation of (i), by putting $\frac{8u}{k} = 1 - z^2 = w$ and we get

$$t = \tfrac{1}{2} \pm \frac{z}{2} + \frac{wk}{24} \pm \frac{w(2-w)}{288z} k^2 + \frac{w(2-w)}{4320} k^3 \cdots \tag{iv}$$

and for our purposes the first three terms of this are accurate enough and give

$$t = \tfrac{1}{2} + \frac{u}{3} \pm \sqrt{\tfrac{1}{4} - \frac{2u}{k}} \tag{v}$$

However if z is small, i.e. if u is near to U, the series in (iv) is divergent (this is because $U < \frac{k}{8}$ and so $z = 0$ gives imaginary roots in the third equation of (i)).

The correct method of finding t accordingly is: from the second equation of (i) and the first of (iii) we get

$$k(U-u) = e^{k(t-T)} - k(t-T) - 1 \tag{vi}$$

and we solve this for $t - T$: note it is satisfied by $u = U$, $t = T$.

If $e^x - x - 1 = \frac{y^2}{2}$

then $x = y - \frac{y^2}{6} + \frac{y^3}{36} - \frac{y^4}{270} + \frac{y^5}{4320} - \frac{y^6}{17010} + \cdots$

so putting $U - u = \frac{k\phi^2}{2}$ and $k(U-u) = \frac{k^2\phi^2}{2} = \frac{y^2}{2}$

we get $k(t-T) = y - \frac{y^2}{6} + \cdots$

and

$$t = T \pm \phi - \frac{U-u}{3} \pm \frac{\phi(U-u)}{18} k - \frac{2(U-u)^2}{135} k^2 \pm \cdots$$

$$= \tfrac{1}{2} + \frac{u}{3} - \frac{2(U-u)^2}{135} k^2 - \cdots \pm \phi\left[1 + \frac{U-u}{18} k + \cdots\right] \tag{vii}$$

The first three terms of this are accurate for the calculations of this paper and give

$$t = \tfrac{1}{2} + \frac{u}{3} \pm \sqrt{\frac{2(U-u)}{k}} \tag{viii}$$

which of course is practically the same as (v).

For the Part I calculations we have to change the sign of k, and of u, and we get similarly to the above

$$u = \frac{1 - e^{-kt}}{1 - e^{-k}} - t$$

$$T = \tfrac{1}{2} - \frac{k}{24} + \frac{k^3}{2880} - \cdot\cdot$$

$$U = \frac{k}{8} - \frac{k^3}{576} + \cdot\cdot$$

$$t = T \pm \phi + \frac{U - u}{3} \pm \cdots$$

$$\left.\begin{aligned} &= \tfrac{1}{2} - \frac{u}{3} \pm \sqrt{\frac{2\,(U-u)}{k}} \\ \text{or } &= \tfrac{1}{2} - \frac{u}{3} \pm \sqrt{\tfrac{1}{4} - \frac{2\,u}{k}} \end{aligned}\right\} \begin{aligned} &\text{near} \\ &\text{enough.} \end{aligned}$$