Thus the credibility premium may be written as the product:

$$\{(1 - z_1)k + z_1\bar{k}\} \ \{(1 - z_2)m + z_2\bar{m}\}$$

with:

$$z_1(t) = \frac{t}{t + \dfrac{k}{Var[\lambda]}}$$

$$z_2(n) = \frac{n}{n + \dfrac{E[\sigma^2(\theta)]}{Var[\mu(\theta)]}}.$$

We notice that the two credibilities are properly distinguished now.

For Hewitt's numerical example we find:

$$z_1(t) = \frac{t}{t + 30.1}$$

$$z_2(n) = \frac{n}{n + \dfrac{e^{\sigma^2} - 1}{e^{S^2} - 1} e^{S^2}} = \frac{n}{n + 54.9}$$

Finally, we remark that the assumption of independence between λ and θ is not necessary for the construction of the above described credibility premium. However, in the general case, it will not be possible to write the credibility premium:

$$akm + b\bar{k}m + ck\bar{m} + d\bar{k}\bar{m}$$

in product form (as it was possible in the case of independence).

DISCUSSION BY HANS BÜHLMANN*

This is an inspiring paper very clearly written and well presented. I hope that the point made by Mr. Hewitt comes home, namely that credibility is theoretically justifiable and eminently practical. The main contribution of this paper is the explicit application of general credibility techniques to the

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factorized pure premium, the factors being expected frequency and expected severity. So far, most other applications have been made by either applying the same techniques to the expected frequency alone or directly to the pure premium, without considering its product form.

Mr. Hewitt leads us through two examples in order to illustrate his approach, the discrete die-spinner model and the continuous frequency-severity model for automobile insurance. The first is ingenious from an educational angle, the second leads us right to the center of interest cherished by the casualty actuary. My discussion, therefore, concentrates on this second example.

Mr. Hewitt assumes the frequency of the individual exposure unit to be Poisson distributed with unknown parameter m, the severity of each claim to follow a log normal distribution with unknown "mean of the log" μ but known "variance of the log" σ^2. Assuming the parameters m and μ to be independent and to follow specific distribution functions as well as the usual independence of frequency and severity given these parameters, he arrives at an explicit formula for the constant K in $Z = \frac{n}{n + K}$. Most interesting are his applications of the formula thus obtained to auto merit rating and to a single-split experience rating plan. As shown in the case of Canadian private passenger data, credibility is considerably reduced by taking severity into account. Obviously Mr. Hewitt's approach can be carried through for any *discrete* model with an arbitrary structure (a priori distribution) on the parameter space; and, as he himself points out, the assumption of independence of frequency and severity parameters is not vital then. However, in the continuous case the calculations involved would become very cumbersome if a) the parametric assumptions would differ from the normal (for log of claims) — normal (for mean of log of claims) case and b) independence of frequency and severity parameters is not postulated.

May I say that the Hewitt approach is actually geared to the use of Bayes estimates rather than the use of credibility estimates because he builds up all the machinery needed to compute Bayes estimates. As we know that Bayes estimates are optimal for quadratic loss and that they have many other attractive properties, why not use them? Incidentally, the comparison of Bayes and credibility estimates for the die-spinner-model is very illuminating — and very encouraging for all credibility fans! Credibility estimates can be characterized as closest (in the sense of mean square deviation) approximation to the Bayes estimates provided one knows only *mean* and *variance* of all distributions (including the distributions of the parameters). Let me show how this works by reproducing the Hewitt approach without parametric assumptions (point "a" above) but still assuming independence of frequency and severity parameters (point "b" above),

1. The model (with a slight change of notation)

	random variable	mean	variance	parameter characterizing the distribution
frequency	k	E_k	σ_k^2	$\eta \in H$
severity (average claim)	$\bar{y}$	$E(\bar{y}\mid k) = E_y$	$Var(\bar{y}\mid k) = \frac{\sigma_y^2}{k}$	$\theta \in \Theta$
loss per unit	$x = k\bar{y}$	$E(x) = E_k \,.\, E_y$	$Var(x) = E_k\,\sigma_y^2 + (E_y)^2 \cdot \sigma_k^2$	$(\eta, \theta) \in H \times \Theta$

2. The parameters and their distribution

The formulae given under 1) are all to be understood for a fixed distribution of frequency and severity (fixed parameter values). We emphasize this by writing:

(1) $E(x \mid \eta,\theta) = E_k(\eta) \cdot E_y(\theta)$

(2) $Var(x \mid \eta, \theta) = E_k(\eta) \cdot \sigma_y^2(\theta) + (E_y(\theta))^2 \cdot \sigma_k^2(\eta)$

In the following the operations E[ ] and Var[ ] (square brackets as opposed to parentheses before !) mean expected value and variance with respect to the structure function (prior distribution) $U(\eta, \theta)$ over $H \times \Theta$. Let us assume independence of η and θ, an assumption which is equivalent with postulating the "product form" for U, i.e. $U(\eta, \theta) = U_1(\eta) \cdot U_2(\theta)$.

Then we obtain:

(3) $E[E(x \mid \eta, \theta)] = m_k \cdot m_y$

(4) $Var[E(x \mid \eta, \theta)] = w_k w_y + m_k^2 w_y + m_y^2 w_k$

(5) $E[Var(x \mid \eta, \theta)] = m_k v_y + (m_y^2 + w_y) v_k$

with the abbreviations:

$$m_k = E[E_k(\eta)] \qquad m_y = E[E_y(\theta)]$$

$$w_k = Var[E_k(\eta)] \qquad w_y = Var[E_y(\theta)]$$

$$v_k = E[\sigma_k^2(\eta)] \qquad v_y = E[\sigma_y^2(\theta)]$$

3. *Credibility*

We determine the constant K in the credibility formula $Z = \frac{n}{n+K}$

$$(6) \quad K = \frac{E[Var(x \mid \eta, \theta)]}{Var[E(x \mid \eta, \theta)]} = \frac{m_k v_y + (m_y^2 + w_y)v_k}{w_k w_y + m_k^2 w_y + m_y^2 w_k}$$

and if the w's are strictly positive:

$$(7) \quad K = \frac{\frac{K_y}{m_k} C_k + K_k(C_y + 1)}{C_k + C_y + 1}$$

where:

$$K_k = \frac{v_k}{w_k} \qquad K_y = \frac{v_y}{w_y}$$

$$C_k = \frac{m_k^2}{w_k} \qquad C_y = \frac{m_y^2}{w_y}$$

observe that K_k and K_y are the K-constants in the credibility formulae for the factors alone. Hence the K-constant in the credibility formula of the product appears as:

weighted average of $\frac{K_y}{m_k}$ and K_k

(Note the division by m_k which suggests the original assumption

$$Var(y \mid k) = \frac{\sigma_y^2}{k} \;!)$$

We may have assumed from Mr. Hewitt's example that credibility always *decreases* if severity is taken into account. However, this belief now turns out to be incorrect. The correct statement is that credibility *decreases* by taking severity into account exactly if:

$$(8) \qquad K_y > m_k K_k$$

Let me finish my discussion by thanking Mr. Hewitt for his stimulating paper. He has opened a new road of research into *full* credibility (credibility based on severity as well as frequency). For the pure Bayesian the road has already lead to its destination because to him the quantities m, v, w are to be assessed by underwriting judgement. The empirical Bayesian, however, still has a task ahead of him. He must find the appropriate estimates of m, v, w from observations. This can be done also in the non parametric version of the Hewitt approach by applying the method of estimation described in the paper by Mr. Straub and this reviewer.*

* *Editors Note:* This paper appeared in the second edition of ARCH 1972. Copies may be obtained from the Editor, David G. Halmstad, Aviation Reinsurance Unit, Metropolitan Life Insurance Company, 1 Madison Avenue, New York, New York 10010.