DISCUSSION BY GEORGE M. LEVINE

1. INTRODUCTION

Messrs. Pinto and Gogol have written a paper rich with practical techniques for determining excess loss development by layer of loss for liability lines. I have used their novel approach for analyzing reporting patterns by liability layer, and had success in tailoring their patterns to determine expected development for various reinsurance programs. Before presenting my results, I will summarize their technique and present some goodness of fit tests comparing the actual data to their fitted curves. In addition, some limitations of the use of their method will be offered.

2. SUMMARY OF PINTO/GOGOL TECHNIQUE

The authors begin by describing the lack of available published information by layer for reported and paid excess loss development. Although the Reinsurance Association of America (RAA) publishes accident year reported loss development studies every two years, and the Insurance Services Office (ISO) annually distributes policy year reported loss development patterns, empirical loss detail by layer is generally not available. This lack of published data dictates the use of theoretical loss distributions (like the Pareto distribution). The Pinto/Gogol technique, although theoretically supported by the properties of the Pareto distribution, has the advantage of being applicable to empirical data.

From ISO excess loss development data by subline, Pinto/Gogol smooth the data two ways—by liability limit (retention), and by development interval. From the "Actual Factors" matrices that they present in Exhibits 1 through 3, they initially smooth the data horizontally by retention, for the monthly intervals 27–99, 39–99, 51–99, 63–99, 75–99, and 87–99. The curve selected to fit the data is $y = ax^b$, where x is the retention divided by \$10,000. Next, the authors convert the factors by retention back to the report-to-report intervals (27–39, 39–51, etc.), and smooth the data vertically (by retention) using a normal power approximation developed by Sherman [1].

By fitting the curve ax^b to the excess loss development within each development interval by layer, the authors have provided an easy method to determine development for layers of retention not published (for example, \$75,000 and \$150,000). In Section 4, the authors explain that the motivation of the selection of the curve ax^b to fit loss development factors was the single parameter Pareto distribution's good fit to the tail of casualty loss distributions. Another interesting parallel is the technique that Rosenberg and Halpert applied regarding their analysis of methods to adjust historical loss distributions for trend [2]. In their research, Rosenberg and Halpert chose the model $tr(x_t) = ax_t^b$ over two other models because that model provided the best fit to the trend data via a least squares regression test.

The parallel use of this same curve to fit actual liability data for trend and loss development is noteworthy. After rereading the sections of Rosenberg and Halpert regarding their fitting techniques of their model to trend, one can understand the relevancy of their comments regarding the function ax_t^b for trend to Pinto and Gogol's comments regarding ax^b for loss development. By setting $b > 0$, Rosenberg and Halpert have allowed trend to increase by claim size, and Pinto/Gogol have allowed excess loss development to increase by size of retention. Although much discussion has centered around the alleged "overlap" of trend and loss development, the use (and we will see later, the good fit) of the same theoretical function in both instances illustrates the similarity of the forces impacting trend and loss development.

Through the application of the Sherman normal power curve approximation, tail factors for development beyond 99 months have been determined. The authors offer several reasons why the fitted ISO tails are faster than the development based on RAA data. In addition, Pinto/Gogol offer a method to use the RAA development with the ISO fitted patterns, for development beyond 99 months, if the actuary believes that the RAA development is more appropriate.

3. GOODNESS OF FIT TESTS

The authors' intent is to find a loss distribution which will fit the three actual ISO loss development data matrices reasonably well. In their smoothing technique, cumulative intervals (27–99, 39–99, etc.) are used by the authors for smoothing development by retention. Additionally, Pinto/Gogol present the cumulative comparison of actual and fitted factors, for the 27–99 interval, showing the apparent similarity of those cumulative factors. For that reason, the goodness of fit tests are performed for the development of all six cumulative intervals.

The goodness of fit tests are applied to the actual and fitted cumulative data on Exhibit 1, Sheets 2–4, for OL&T BI, M&C BI, and Products BI, respectively. The formula for percentage error is as follows:

$$\left\{\left(\frac{\text{Fitted Cumulative Factor}}{\text{Actual Cumulative Factor}}\right) - 1\right\} \times 100 \qquad (3.1)$$

For each subline, three groups of mean percentage errors are calculated and shown on Exhibit 1, Sheet 1:

- by development interval, retention layers excess of \$10,000 to excess of \$250,000;
- by development interval, retention layers excess of \$10,000 to excess of \$1,000,000; and,
- by retention, all development intervals (27–99 to 87–99).

Restated, the goodness of fit tests are performed by row (development interval) twice, for the \$10,000 to \$250,000 retention columns and \$10,000 to \$1,000,000 retention columns; and by column (retention) once, for all development intervals. Also, mean percentage errors are calculated for the entire data matrices (all development intervals and retentions), to provide an indicator of the goodness of fit for the overall technique.

The conclusions from the goodness of fit tests are as follows:

- Excluding the excess development for retentions greater than \$250,000, the mean percentage errors for OL&T BI, M&C BI, and Products BI are −1.7%, −0.4%, and −0.6%, respectively. Therefore, the fitted cumulative development errors are at most 2% below the actual cumulative development errors, averaged over all retentions and development intervals.
- For every subline, the mean absolute percentage errors (MAPE) by retention (columns) exceed 1.5% for the following areas:
  OL&T BI: \$10,000 and \$250,000 retentions
  M&C BI: \$10,000 and \$250,000 retentions
  Products BI: \$100,000 retention
- For OL&T and M&C, the MAPEs for \$500,000 and \$1,000,000 retentions are between 8% and 27%. The Products MAPE for the \$1,000,000 retention is 8% in contrast to the other retentions' MAPEs of less than 2%. These observations show that the Sherman normal power approximation works reasonably well for the development through \$250,000 retentions, but is inconsistent for retentions in excess of \$250,000.
- The fit of the actual data by development interval (row) for retention intervals \$10,000 through \$250,000 is fine for M&C and Products, with all MAPEs below 1.5%.

EXHIBIT 1
SHEET 1

GOODNESS OF FIT TESTS
MEAN PERCENTAGE ERRORS

OL&T BI EXCESS LOSS & ALAE DEVELOPMENT FACTORS

By Development Interval			By Retention	
	Retention Layers			
Development Interval	10,000 to 250,000	10,000 to 1,000,000	Retention	27–99 to 87–99
27–99	−4.0%	−5.1%	10,000	−3.7%
39–99	−4.8%	−10.4%	25,000	−0.8%
51–99	−2.4%	−8.9%	50,000	0.2%
63–99	−1.3%	−5.5%	100,000	−0.3%
75–99	−0.8%	−3.6%	250,000	−3.8%
87–99	3.2%	2.7%	500,000	−10.0%
All	−1.7%	−5.1%	1,000,000	−17.4%

M&C BI EXCESS LOSS & ALAE DEVELOPMENT FACTORS

By Development Interval			By Retention	
	Retention Layers			
Development Interval	10,000 to 250,000	10,000 to 1,000,000	Retention	27–99 to 87–99
27-99	−1.1%	−3.4%	10,000	−1.9%
39–99	−1.1%	−7.7%	25,000	0.4%
51–99	−0.3%	−6.4%	50,000	1.2%
63–99	0.9%	−5.3%	100,000	0.7%
75–99	−0.8%	−6.4%	250,000	−2.5%
87–99	−0.2%	−2.3%	500,000	−7.9%
All	−0.4%	−4.5%	1,000,000	−26.8%

PRODUCTS BI EXCESS LOSS & ALAE DEVELOPMENT FACTORS

By Development Interval			By Retention	
	Retention Layers			
Development Interval	10,000 to 250,000	10,000 to 1,000,000	Retention	27–99 to 87–99
27–99	−1.3%	−2.6%	10,000	−0.3%
39–99	−1.3%	−4.3%	25,000	−0.9%
51–99	−0.1%	−3.3%	50,000	−1.0%
63–99	−0.8%	3.5%	100,000	−1.6%
75–99	−1.5%	3.8%	250,000	0.6%
87–99	1.0%	8.0%	500,000	1.0%
All	−0.6%	0.8%	1,000,000	7.9%

EXHIBIT 1
SHEET 2

OL&T BI EXCESS LOSS & ALAE DEVELOPMENT FACTORS

Fitted Cumulative Factors

Development Interval	Retention 10,000	25,000	50,000	100,000	250,000	500,000	1,000,000
27–99	1.89995	2.24203	2.54120	2.88023	3.39887	3.85234	4.36635
39–99	1.39133	1.62686	1.83122	2.06120	2.41015	2.71284	3.05358
51–99	1.20769	1.36148	1.49075	1.63223	1.84012	2.01478	2.20603
63–99	1.11799	1.20318	1.27195	1.34462	1.44711	1.52979	1.61720
75–99	1.06375	1.10490	1.13711	1.17024	1.21554	1.25095	1.28741
87–99	1.02691	1.04222	1.05396	1.06583	1.08173	1.09391	1.10623

Actual Cumulative Factors

Development Interval	Retention 10,000	25,000	50,000	100,000	250,000	500,000	1,000,000
27–99	2.01337	2.31925	2.61370	2.97050	3.57977	4.28524	4.62738
39–99	1.50747	1.67467	1.85962	2.11861	2.64932	3.36309	4.32951
51–99	1.27762	1.37268	1.48189	1.63700	1.95999	2.41254	3.31484
63–99	1.15559	1.20390	1.25968	1.34533	1.51199	1.71345	2.05444
75–99	1.08364	1.10683	1.13199	1.16925	1.24752	1.34000	1.50376
87–99	1.01180	1.01460	1.01670	1.02350	1.03830	1.06130	1.11110

Development Interval	Percentage Errors Retention 10,000	25,000	50,000	100,000	250,000	500,000	1,000,000	Mean Percentage Errors For Retentions: 10,000 to 250,000	10,000 to 1,000,000
27–99	−5.6%	−3.3%	−2.8%	−3.0%	−5.1%	−10.1%	−5.6%	−4.0%	−5.1%
39–99	−7.7%	−2.9%	−1.5%	−2.7%	−9.0%	−19.3%	−29.5%	−4.8%	−10.4%
51–99	−5.5%	−0.8%	0.6%	−0.3%	−6.1%	−16.5%	−33.4%	−2.4%	−8.9%
63–99	−3.3%	−0.1%	1.0%	−0.1%	−4.3%	−10.7%	−21.3%	−1.3%	−5.5%
75–99	−1.8%	−0.2%	0.5%	0.1%	−2.6%	−6.6%	−14.4%	−0.8%	−3.6%
87–99	1.5%	2.7%	3.7%	4.1%	4.2%	3.1%	−0.4%	3.2%	2.7%
All Intervals Mean Percentage Errors:	−3.7%	−0.8%	0.2%	−0.3%	−3.8%	−10.0%	−17.4%	−1.7%	−5.1%

EXHIBIT 1
SHEET 3

M&C BI EXCESS LOSS & ALAE DEVELOPMENT FACTORS

Fitted Cumulative Factors

Development Interval	Retention 10,000	25,000	50,000	100,000	250,000	500,000	1,000,000
27–99	2.46773	2.79319	3.06752	3.36890	3.81312	4.18770	4.59910
39–99	1.50464	1.66601	1.79943	1.94359	2.15200	2.32437	2.51057
51–99	1.19734	1.29235	1.36917	1.45059	1.56567	1.65876	1.75739
63–99	1.09365	1.15195	1.19808	1.24608	1.31249	1.36506	1.41976
75–99	1.04479	1.07786	1.10355	1.12987	1.16563	1.19343	1.22190
87–99	1.01728	1.03168	1.04270	1.05385	1.06876	1.08018	1.09173

Actual Cumulative Factors

Development Interval	Retention 10,000	25,000	50,000	100,000	250,000	500,000	1,000,000
27–99	2.51971	2.79852	3.06557	3.40139	3.90759	4.21684	5.59436
39–99	1.55097	1.66420	1.78220	1.94054	2.23533	2.61753	3.98005
51–99	1.22801	1.28272	1.34202	1.42866	1.62275	1.89060	2.54821
63–99	1.10631	1.13355	1.16606	1.21371	1.32860	1.51006	2.10075
75–99	1.06366	1.07823	1.09572	1.12505	1.20694	1.33162	1.75913
87–99	1.02670	1.03190	1.03820	1.04910	1.07820	1.11920	1.23830

Development Interval	Percentage Errors Retention 10,000	25,000	50,000	100,000	250,000	500,000	1,000,000	Mean Percentage Errors For Retentions: 10,000 to 250,000	10,000 to 1,000,000
27–99	−2.1%	−0.2%	0.1%	−1.0%	−2.4%	−0.7%	−17.8%	−1.1%	−3.4%
39–99	−3.0%	0.1%	1.0%	0.2%	−3.7%	−11.2%	−36.9%	−1.1%	−7.7%
51–99	−2.5%	0.8%	2.0%	1.5%	−3.5%	−12.3%	−31.0%	−0.3%	−6.4%
63–99	−1.1%	1.6%	2.7%	2.7%	−1.2%	−9.6%	−32.4%	0.9%	−5.3%
75–99	−1.8%	−0.0%	0.7%	0.4%	−3.4%	−10.4%	−30.5%	−0.8%	−6.4%
87–99	−0.9%	−0.0%	0.4%	0.5%	−0.9%	−3.5%	−11.8%	−0.2%	−2.3%
All Intervals Mean Percentage Errors:	−1.9%	0.4%	1.2%	0.7%	−2.5%	−7.9%	−26.8%	−0.4%	−4.5%

EXHIBIT 1
SHEET 4

PRODUCTS-BI EXCESS LOSS & ALAE DEVELOPMENT FACTORS

Fitted Cumulative Factors

Development Interval	Retention 10,000	25,000	50,000	100,000	250,000	500,000	1,000,000
27–99	3.07710	3.53866	3.93323	4.37181	5.02759	5.58824	6.21139
39–99	1.70416	1.87414	2.01387	2.16403	2.37988	2.55735	2.74804
51–99	1.33631	1.41189	1.47186	1.53438	1.62115	1.69003	1.76182
63–99	1.17969	1.21552	1.24332	1.27177	1.31039	1.34037	1.37105
75–99	1.09317	1.10981	1.12255	1.13544	1.15271	1.16595	1.17933
87–99	1.03817	1.04438	1.04909	1.05383	1.06013	1.06492	1.06973

Actual Cumulative Factors

Development Interval	Retention 10,000	25,000	50,000	100,000	250,000	500,000	1,000,000
27–99	3.07724	3.63668	3.99647	4.47700	5.01208	6.36789	6.20299
39–99	1.72000	1.90512	2.04287	2.21557	2.38070	2.66038	3.44113
51–99	1.33271	1.40485	1.47564	1.55795	1.60967	1.76208	2.17147
63–99	1.18190	1.22150	1.25736	1.29905	1.30857	1.25210	1.13446
75–99	1.11164	1.13354	1.15058	1.16350	1.14255	1.07384	0.93959
87–99	1.02930	1.03690	1.04050	1.04210	1.04400	0.96050	0.76570

Development Interval	Percentage Errors Retention 10,000	25,000	50,000	100,000	250,000	500,000	1,000,000	Mean Percentage Errors For Retentions: 10,000 to 250,000	10,000 to 1,000,000
27–99	−0.0%	−2.7%	−1.6%	−2.3%	0.3%	−12.2%	0.1%	−1.3%	−2.6%
39–99	−0.9%	−1.6%	−1.4%	−2.3%	−0.0%	−3.9%	−20.1%	−1.3%	−4.3%
51–99	0.3%	0.5%	−0.3%	−1.5%	0.7%	−4.1%	−18.9%	−0.1%	−3.3%
63–99	−0.2%	−0.5%	−1.1%	−2.1%	0.1%	7.1%	20.9%	−0.8%	3.5%
75–99	−1.7%	−2.1%	−2.4%	−2.4%	0.9%	8.6%	25.5%	−1.5%	3.8%
87–99	0.9%	0.7%	0.8%	1.1%	1.5%	10.9%	39.7%	1.0%	8.0%
All Intervals Mean Percentage Errors:	−0.3%	−0.9%	−1.0%	−1.6%	0.6%	1.0%	7.9%	−0.6%	0.8%

Based upon the above observations, several areas for understatement of actual development exist for these three sublines' data. For OL&T, fitted development at $10,000 and $250,000 retentions is at least 7% below actual development for the development interval 39–99 months. The M&C actual development is understated at least 3% for these same cells. Products fitted development data is about 2% understated for the $100,000 retention; this difference is not substantial. These differences might be adjusted for on an ad-hoc basis after application of the technique.

I initially performed these same tests on the data as of successive (e.g., non-cumulative) intervals, and discovered that the goodness of fit can reverse with the accumulation of development. For example, for OL&T BI, the percentage error for development interval 27–39, retention $250,000, is +4.4%; the corresponding factor for the interval 27–99 is −5.1%. This shows that random variations in reporting do not always get smoothed out when the development for successive intervals is accumulated.

In conversations with the authors, they indicated their goal was to produce an intuitively reasonable, natural, and smooth sequence of curves for development to provide knowledge where published information is not available. Based upon these goodness of fit tests, and ignoring pockets of discrepancies, the authors have met their goal.

4. RETENTIONS IN EXCESS OF $250,000

Based on the goodness of fit tests, it is obvious that the fit to the actual data for retentions in excess of $250,000 is poor. The authors mention that the tendency for development to increase as retention increases is *reversed* at $500,000 and $1,000,000 for the 27-39 month intervals.

The authors suggest this may be due to a credibility problem of the data for these large claims. However, there may be another reason. Some claims people feel that, for very large claims, an estimate of the loss put up in the first year often is not revised until several years later, closer to a jury trial. The following comparison for Products, based on goodness of fit tests for excess development factors at the $1,000,000 retention, is interesting:

DEVELOPMENT INTERVAL	PERCENTAGE ERROR
27–39 (SUCCESSIVE)	+25.4%
27–99 (CUMULATIVE)	+ 0.1%
63–75 (SUCCESSIVE)	− 3.7%
63–99 (CUMULATIVE)	+20.9%

The inclusion of the later 39–99 month development provides a better fit for the data than the 27–39 month development alone. At 63 months, however, the inclusion of the 75–99 month development provides a much poorer fit than the 63–75 month development alone.

In summary, this "catch-up" theory is supported by the goodness of fit tests. For less mature data, the inclusion of later development tends to smooth out the random variations; for more mature data, including the tail provides a poorer fit. For either reason, a lack of credibility or differing reserving practices, it seems wise to exclude the very high retentions when applying this technique.

5. THE METHOD APPLIED

In Section 5, the authors introduce the formula for the excess development factor as follows:

$$(f(c) - f(d)) \div (e_{c,n} - e_{d,n}) \tag{5.1}$$

with $f(c)$ being the ratio of the excess losses to the "ground-up" projected ultimate losses, and $e_{c,n}$ representing the excess loss ratio divided by the loss development factor to ultimate, for retention c and month n. The function $f(x)$ is a very familiar one to actuaries—it is merely an excess loss function. In standard actuarial terminology, these $f(x)$'s are also noted as $X3(x)$ [3]. For workers' compensation, the excess loss premium factors could be a readily available published source for these excess ratios.

At first, I found this formula to be somewhat problematic, but found the proof to be somewhat straightforward. (The proof is presented in Appendix A.) From that formula, other powerful formulas can be derived to estimate other development patterns.

For example, loss development data sometimes is available only for basic limits and total limits, but is not available for excess limits. Setting the basic limit equal to B (in 000's), the formula for development from 12 months to ultimate in the layer \$0 to \$$B$, the basic limits "layer," is the following:

$$LDF_{BASIC,\,12} = \frac{1 - f(B)}{\dfrac{1}{LDF_{0,12}} - \dfrac{f(B)}{LDF_{B,12}}} \tag{5.2}$$

with $LDF_{BASIC,12}$ representing the basic limits loss development factor, $LDF_{0,12}$ the total limits loss development factor, and $LDF_{B,12}$ being the excess loss development factor, all from 12 months to ultimate. Also, $f(0) = 1$ and $f(B) =$ the excess loss ratio for losses in excess of \$$B$, the basic limits. Here, basic limits development is treated as development for the layer of losses in excess of \$0 retention minus the losses in excess of the basic limit of \$$B$. This also leads to the formula:

$$LDF_{EXCESS} = LDF_{B,12} = \frac{f(B)}{\dfrac{1}{LDF_{0,12}} - \dfrac{(1 - f(B))}{LDF_{BASIC,12}}} \tag{5.3}$$

From this discussion, it can be inferred that primary loss development, from ground up, can be considered a special case of excess development. Therefore, primary development factors by layer can be produced using the Pinto/Gogol formula, as long as total limits "ground-up" loss development is available (that is, retention = \$0). Since the raw ISO development was presented on the exhibits, I extrapolated to 363 months the M&C BI "ground-up" development data using the Sherman method that Pinto and Gogol used for the excess development for other retentions. The application of formula (5.2) for primary development (\$15,000, \$25,000, up to \$250,000) is presented on Exhibit 2.

The results from this technique for M&C primary development are disappointing. The reason for these intuitively disturbing factors may be given by Rosenberg and Halpert. In their study for trend, they found that "a concern with the trend function ax_t^b is that it tends to underestimate the trend for small x_t, that is, small sizes of losses." This conclusion may extend to conclusions for excess loss development as well. Although the ground-up development is not part of the smoothing technique, its use with the excess development (that had been smoothed and tested for goodness of fit) to produce primary development factors does not provide sensible results.

EXHIBIT 2

DETERMINATION OF PRIMARY LOSS LAYER LOSS DEVELOPMENT FACTORS
M&C BI BASIC LIMITS LOSSES & ALAE

Loss Layer	Development Interval 27:ult	39:ult	51:ult	63:ult
\$0–\$15,000	1.201	0.962	0.955	0.936
\$0–\$25,000 (from ISO)	2.007	1.379	1.164	1.075
\$0–\$25,000 (computed)	1.123	0.862	0.830	0.818
\$0–\$35,000	1.137	0.853	0.806	0.792
\$0–\$50,000	1.189	0.874	0.811	0.793
\$0–\$75,000	1.288	0.927	0.844	0.819
\$0–\$100,000	1.385	0.984	0.884	0.888
\$0–\$250,000	1.727	1.187	1.032	0.975

Formula: $(f(low) - f(high))/((f(low)/low{:}ult) - (f(high)/high{:}ult))$

Development Interval	Cumulative Fitted Factors 0	15000	25000	35000	50000	75000	100000	250000
27:ult	2.2093	2.8639	3.2210	3.4802	3.7778	4.1472	4.4309	5.4702
39:ult	1.4769	1.7293	1.9212	2.0591	2.2161	2.4092	2.5563	3.0872
51:ult	1.2475	1.3607	1.4903	1.5824	1.6862	1.8126	1.9079	2.2461
63:ult	1.1521	1.2295	1.3284	1.3979	1.4755	1.5690	1.6389	1.8829
$f(x)$ ratios:	1.000	0.786	0.755	0.721	0.674	0.605	0.543	0.319

Development Interval	Report-to-Report Fitted Factors 0	15000	25000	35000	50000	75000	100000	250000
27– 39	1.49590	1.65613	1.67658	1.69018	1.70472	1.72141	1.73335	1.77192
39– 51	1.18395	1.27092	1.28912	1.30125	1.31424	1.32916	1.33984	1.37446
51– 63	1.08278	1.10671	1.12188	1.13199	1.14281	1.15523	1.16412	1.19291
63– 75	1.04456	1.05643	1.06873	1.07691	1.08565	1.09567	1.10284	1.12597
75– 87	1.02687	1.03484	1.04476	1.05134	1.05837	1.06641	1.07215	1.09065
87– 99	1.01720	1.02363	1.03168	1.03702	1.04271	1.04921	1.05385	1.06877
99–111	1.01180	1.01705	1.02367	1.02806	1.03272	1.03806	1.04186	1.05405
111–123	1.00850	1.01288	1.01839	1.02204	1.02592	1.03035	1.03351	1.04363
123–135	1.00630	1.01006	1.01470	1.01778	1.02105	1.02478	1.02743	1.03593
135–147	1.00490	1.00807	1.01204	1.01466	1.01744	1.02062	1.02288	1.03012
147–159	1.00380	1.00661	1.01004	1.01230	1.01470	1.01744	1.01939	1.02562
159–171	1.00314	1.00551	1.00849	1.01045	1.01254	1.01492	1.01661	1.02202
171–183	1.00256	1.00466	1.00728	1.00900	1.01084	1.01293	1.01441	1.01908
183–195	1.00211	1.00399	1.00631	1.00784	1.00946	1.01131	1.01262	1.01681
195–207	1.00177	1.00345	1.00552	1.00688	1.00832	1.00997	1.01114	1.01487
207–219	1.00149	1.00301	1.00486	1.00608	1.00737	1.00884	1.00989	1.01322
219–231	1.00127	1.00266	1.00433	1.00543	1.00659	1.00792	1.00886	1.01187
231–243	1.00110	1.00235	1.00386	1.00485	1.00591	1.00711	1.00796	1.01068
243–255	1.00095	1.00210	1.00347	1.00437	1.00533	1.00642	1.00719	1.00966
255–267	1.00083	1.00189	1.00314	1.00396	1.00484	1.00583	1.00654	1.00879
267–279	1.00073	1.00171	1.00286	1.00361	1.00441	1.00533	1.00598	1.00804
279–291	1.00064	1.00155	1.00260	1.00330	1.00403	1.00487	1.00547	1.00737
291–303	1.00057	1.00141	1.00238	1.00302	1.00370	1.00448	1.00503	1.00678
303–315	1.00051	1.00128	1.00218	1.00278	1.00341	1.00412	1.00463	1.00625
315–327	1.00046	1.00119	1.00202	1.00258	1.00316	1.00383	1.00431	1.00582
327–339	1.00041	1.00109	1.00187	1.00239	1.00294	1.00356	1.00400	1.00541
339–351	1.00037	1.00101	1.00173	1.00221	1.00272	1.00330	1.00371	1.00501
351–363	1.00034	1.00093	1.00161	1.00206	1.00253	1.00307	1.00346	1.00468

Rosenberg and Halpert may provide the solution, however, by suggesting that the understatement could "be corrected by changing the model to . . . $a(x_t + c)^b$, or by using the function ax_t^b only for claim sizes greater than a selected value and using empirical data to trend small losses." In this case, the fitted development for lower retentions (say below $35,000) may need to be adjusted to produce reasonable primary development. That study is beyond the scope of this discussion.

However, I also applied their data for some standard excess development layers, and obtained satisfyingly reasonable results for the excess layers. These results are presented in Exhibit 3.

6. SUMMARY

Messrs. Pinto and Gogol have written a fine paper with practical and useful applications. Although excess development at very low retentions (for use in primary development) or large retentions (in excess of $250,000) may be dubious, application for development at retentions between those extremes are easy to apply and tailor for today's "mix and match" reinsurance program environment.

EXHIBIT 3

DETERMINATION OF PRIMARY LOSS LAYER LOSS DEVELOPMENT FACTORS
M&C BI EXCESS LIMITS LOSSES & ALAE

Loss Layer	Development Interval 27:ult	39:ult	51:ult	63:ult
\$15,000–\$515,000	2.551	1.552	1.234	1.126
\$35,000–\$535,000	3.143	1.874	1.454	1.297
\$50,000–\$550,000	3.430	2.027	1.557	1.375
\$75,000–\$575,000	3.789	2.217	1.683	1.470
\$100,000–\$600,000	4.066	2.363	1.779	1.542
\$250,000–\$750,000	5.102	2.897	2.124	1.794
\$35,000–\$500,000	3.112	1.857	1.442	1.288
\$50,000–\$500,000	3.386	2.003	1.540	1.363
\$75,000–\$500,000	3.721	2.181	1.659	1.452
\$100,000–\$500,000	3.971	2.312	1.745	1.516
\$250,000–\$500,000	4.851	2.768	2.041	1.733

Formula: $(f(low) - f(high))/((f(low)/low{:}ult) - (f(high)/high{:}ult))$

Development Interval	Cumulative Fitted Factors 500000	515000	535000	550000	575000	600000	750000
27:ult	6.4157	6.4595	6.5164	6.5579	6.6253	6.6905	7.0429
39:ult	3.5610	3.5828	3.6110	3.6316	3.6650	3.6973	3.8712
51:ult	2.5413	2.5547	2.5721	2.5848	2.6054	2.6252	2.7317
63:ult	2.0913	2.1007	2.1129	2.1218	2.1361	2.1499	2.2238
$f(x)$ ratios:	0.148	0.142	0.135	0.130	0.122	0.114	0.078

Development Interval	Report-to-Report Fitted Factors 500000	515000	535000	550000	575000	600000	750000
27– 39	1.80167	1.80295	1.80460	1.85080	1.80772	1.80957	1.81930
39– 51	1.40124	1.40240	1.40389	1.40497	1.40671	1.40837	1.41715
51– 63	1.21516	1.21612	1.21735	1.21825	1.21970	1.22108	1.22837
63– 75	1.14380	1.14457	1.14555	1.14627	1.14743	1.14853	1.15436
75– 87	1.10486	1.10547	1.10626	1.10683	1.10775	1.10863	1.11326
87– 99	1.08020	1.08069	1.08132	1.08178	1.08251	1.08322	1.08694
99–111	1.06337	1.06377	1.06428	1.06466	1.06526	1.06584	1.06886
111–123	1.05134	1.05167	1.05210	1.05241	1.05291	1.05338	1.05588
123–135	1.04240	1.04268	1.04304	1.04330	1.04371	1.04411	1.04621
135–147	1.03562	1.03586	1.03616	1.03638	1.03673	1.03707	1.03886
147–159	1.03036	1.03056	1.03082	1.03101	1.03132	1.03161	1.03314
159–171	1.02613	1.02631	1.02653	1.02670	1.02696	1.02722	1.02854
171–183	1.02267	1.02282	1.02302	1.02316	1.02339	1.02362	1.02477
183–195	1.02000	1.02013	1.02031	1.02044	1.02064	1.02084	1.02187
195–207	1.01770	1.01782	1.01798	1.01809	1.01827	1.01845	1.01936
207–219	1.01575	1.01586	1.01600	1.01610	1.01627	1.01642	1.01724
219–231	1.01415	1.01425	1.01438	1.01447	1.01461	1.01475	1.01549
231–243	1.01274	1.01283	1.01294	1.01302	1.01316	1.01328	1.01395
243–255	1.01153	1.01161	1.01171	1.01179	1.01191	1.01202	1.01263
255–267	1.01050	1.01057	1.01067	1.01073	1.01084	1.01095	1.01150
267–279	1.00961	1.00968	1.00976	1.00982	1.00992	1.01002	1.01053
279–291	1.00881	1.00887	1.00895	1.00901	1.00910	1.00919	1.00965
291–303	1.00811	1.00816	1.00823	1.00829	1.00837	1.00845	1.00888
303–315	1.00748	1.00754	1.00760	1.00765	1.00773	1.00781	1.00820
315–327	1.00696	1.00701	1.00707	1.00712	1.00719	1.00726	1.00763
327–339	1.00648	1.00652	1.00658	1.00662	1.00669	1.00676	1.00710
339–351	1.00600	1.00605	1.00610	1.00614	1.00620	1.00626	1.00658
351–363	1.00561	1.00565	1.00570	1.00574	1.00580	1.00585	1.00615

REFERENCES

[1] Richard E. Sherman, "Extrapolating, Smoothing, and Interpolating Development Factors," *PCAS* LXXI, 1984, p. 122.

[2] Sheldon Rosenberg and Aaron Halpert, "Adjusting Size of Loss Distributions for Trend," *Inflation Implications for Property-Casualty Insurance,* Casualty Actuarial Society 1981 Discussion Paper Program, p. 458.

[3] J. Gary LaRose, "A Note on Loss Distributions," *PCAS* LXIX, 1982, p. 15.

APPENDIX A

PROOF

$f(c)$ = Excess loss ratio at retention c, the lower retention.

$f(d)$ = Excess loss ratio at retention d, the upper retention.

$$e_{c,n} = \frac{f(c)}{LDF_{c,n}}$$

with $LDF_{c,n}$ the excess loss development factor at retention c from n months to ultimate.

$$e_{d,n} = \frac{f(d)}{LDF_{d,n}}$$

with $LDF_{d,n}$ the excess loss development factor at retention d from n months to ultimate.

We know, however, that:

$$f(c) = \frac{\text{Ultimate Losses Excess of Retention } c}{\text{Ultimate Ground-Up Losses}} = \frac{ULT_c}{ULT_g}$$

$$f(d) = \frac{\text{Ultimate Losses Excess of Retention } d}{\text{Ultimate Ground-Up Losses}} = \frac{ULT_d}{ULT_g}$$

$$LDF_{c,n} = \frac{\text{Ultimate Losses Excess of Retention } c}{\text{Reported Losses Excess of Retention } c} = \frac{ULT_c}{REP_c}$$

$$LDF_{d,n} = \frac{\text{Ultimate Losses Excess of Retention } d}{\text{Reported Losses Excess of Retention } d} = \frac{ULT_d}{REP_d}$$

(I have dropped the subscript n for months of development.)

So:

$$\frac{f(c) - f(d)}{e_{c,n} - e_{d,n}}$$

$$= \frac{f(c) - f(d)}{\dfrac{f(c)}{LDF_{c,n}} - \dfrac{f(d)}{LDF_{d,n}}}$$

$$= \frac{\dfrac{ULT_c}{ULT_g} - \dfrac{ULT_d}{ULT_g}}{\left[\dfrac{ULT_c}{ULT_g} \times \dfrac{REP_c}{ULT_c}\right] - \left[\dfrac{ULT_d}{ULT_g} \times \dfrac{REP_d}{ULT_d}\right]}$$

$$= \frac{\dfrac{ULT_c - ULT_d}{ULT_g}}{\dfrac{REP_c}{ULT_g} - \dfrac{REP_d}{ULT_g}} = \frac{\dfrac{ULT_c - ULT_d}{ULT_g}}{\dfrac{REP_c - REP_d}{ULT_g}}$$

$$= \frac{ULT_c - ULT_d}{REP_c - REP_d}$$

It is obvious that the excess loss development factor for the layer c to d is the ultimate losses greater than c minus ultimate losses in excess of d, divided by the reported losses in excess of c minus the reported losses in excess of retention d.