QUANTIFYING THE UNCERTAINTY IN CLAIM SEVERITY ESTIMATES FOR AN EXCESS LAYER WHEN USING THE SINGLE PARAMETER PARETO

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Abstract

This paper addresses the question: How valuable is a sample of excess claims in determining the expected claim severity in an excess layer of insurance?

An established procedure to estimate this expected claim severity is to first fit a model distribution to claim size data and then, using the fitted distribution, estimate the expected claim severity in the given excess layer. One of the more popular models used is the single parameter Pareto. This paper provides a means of quantifying the uncertainty in these excess claim severity estimates when using the single parameter Pareto. This approach requires one to incorporate prior opinions about the distribution of the Pareto parameter using Bayes' Theorem.

1. INTRODUCTION

Ever since Robert Miccolis's [2] classic paper on increased limits ratemaking was published, it has been an established procedure among members of the Casualty Actuarial Society to estimate the expected claim severity in an excess layer of insurance by first fitting a model distribution function to claim severity data and, using the fitted distribution, to estimate the expected claim severity in the given excess layer. One of the more popular models used is the single parameter Pareto. Its properties have been discussed on many occasions and the reader can consult the *Proceedings* for a very readable account by Stephen Philbrick [3].

An often stated concern in excess limits pricing is the uncertainty of the estimates of the excess claim severity. The purpose of this paper is to describe a Bayesian method of quantifying the uncertainty in excess claim severity estimates. This method is very easy to apply in the case of the single parameter Pareto.

2. A REVIEW OF THE SINGLE PARAMETER PARETO

The single parameter Pareto describes claims that are above a given truncation point, k. The cumulative distribution function is given by:

$$F(x) = 1 - \left(\frac{k}{x}\right)^q \quad \text{for } x \geq k\,. \tag{2.1}$$

The probability density function is given by:

$$f(x) = \frac{qk^q}{x^{q+1}} \quad \text{for } x \geq k\,. \tag{2.2}$$

Let $x_1, x_2, \ldots, x_n$ be a sample of n claims that are larger than k. The likelihood function, $L(q)$, is given by:

$$L(q) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \frac{qk^q}{x_i^{q+1}}\,.$$

Solving for the $\hat{q}$ that maximizes the likelihood function yields:

$$\hat{q} = \frac{n}{\sum_{i=1}^{n} \ln(x_i) - n\ln(k)}\,. \tag{2.3}$$

3. THE CONDITIONAL DISTRIBUTION OF $\hat{q}$

Let us temporarily assume that q is known. The purpose of this section is to describe the distribution of $\hat{q}$ in terms of q, with the final result being given in Equation 3.7 below.

We first note that:

$$\mathrm{E}[\ln(x)] = qk^q \int_k^\infty \frac{\ln(x)}{x^{q+1}}\, dx = \ln(k) + \frac{1}{q}\,. \tag{3.1}$$

From Equations 2.3 and 3.1, we get:

$$\mathrm{E}\left[\frac{1}{\hat{q}}\right] = \frac{1}{q}\,. \tag{3.2}$$

We next note that:

$$\mathrm{E}[\ln(x)^2] = qk^q \int_k^\infty \frac{\ln(x)^2}{x^{q+1}}\, dx = \ln(k)^2 + \frac{2\ln(k)}{q} + \frac{2}{q^2}\,. \tag{3.3}$$

From Equations 3.1 and 3.3, we see that:

$$\mathrm{Var}\,[\ln(x)] = \mathrm{E}[\ln(x)^2] - \mathrm{E}[\ln(x)]^2 = \frac{1}{q^2}\,. \tag{3.4}$$

Thus from Equations 2.3 and 3.4, we get:

$$\mathrm{Var}\left[\frac{1}{\hat{q}}\right] = \mathrm{Var}\left[\frac{\sum_{i=1}^{n} \ln(x_i) - n\ln(k)}{n}\right] = \frac{1}{nq^2}\,. \tag{3.5}$$

Now the central limit theorem states that the distribution of

$$\sum_{i=1}^{n} \ln(x_i)$$

will have an approximately normal distribution for sufficiently large n.

Thus, for known q, $1/\hat{q}$ has an approximately normal distribution with mean $1/q$ and variance $1/nq^2$. The conditional distribution of $1/\hat{q}$ given q is:

$$c(1/\hat{q}|q) = \sqrt{\frac{n}{2\pi}}\, qe^{-\left(\frac{1}{\hat{q}} - \frac{1}{q}\right)^2 nq^2/2}. \tag{3.6}$$

Now the distribution of $\hat{q}$ given q is:

$$c(\hat{q}|q) = c\,(1/\hat{q}|q)\left|\frac{d}{d\hat{q}}\left(\frac{1}{\hat{q}}\right)\right| = \sqrt{\frac{n}{2\pi}}\,\frac{q}{\hat{q}^2}\,e^{-\left(\frac{1}{\hat{q}} - \frac{1}{q}\right)^2 nq^2/2}$$

$$= \sqrt{\frac{n}{2\pi}}\,\frac{q}{\hat{q}^2}\,e^{-\frac{n(q-\hat{q})^2}{2\hat{q}^2}}. \tag{3.7}$$

4. BAYESIAN ESTIMATION

The treatment above considers q as the known quantity and $\hat{q}$ as the random variable. In practice, $\hat{q}$ is known and q is unknown. However, in many instances, we will have some prior knowledge or beliefs about the distribution of q. In other instances, we may have very little prior knowledge of the distribution of q. Our task is to use our knowledge of $\hat{q}$ to refine our knowledge about the distribution of q. To accomplish this, we use Bayes' Theorem.

We first consider the discrete case where q can take on values q_0, $q_1, q_2, \ldots, q_m$. Let the prior probabilities be given by $\Pr(q = q_i) = p_i$. By Bayes' Theorem, the posterior probability function of q_i is given by:

$$b(q_i|\hat{q}) = \frac{c(\hat{q}|q_i)p_i}{\sum_{j=1}^{m} c(\hat{q}|q_j)p_j}. \tag{4.1}$$

For the continuous case, the posterior probability density for q is given by:

$$b(q|\hat{q}) = \frac{c(\hat{q}|q)p(q)}{\int_0^\infty c(\hat{q}|q)p(q)dq}, \tag{4.2}$$

where $p(q)$ is the prior probability density of q.

Now it is a common practice for Bayesian statisticians to express the conditional, the prior, and the posterior distributions in simplest terms by ignoring all coefficients that do not depend upon q in the probability or density functions. The distributions, with the coefficients removed, are referred to as weight functions. In keeping with this practice, we replace c with v, b with w, and p with r and write:

$$c(\hat{q}|q) \propto v(\hat{q}|q) \equiv qe^{-\frac{n}{2}\frac{(q-\hat{q})^2}{\hat{q}^2}} \tag{4.3}$$

in place of Equation 3.7,

$$b(q_i|\hat{q}) \propto w(q_i|\hat{q}) \equiv v(\hat{q}|q_i)r_i \tag{4.4}$$

in place of Equation 4.1, and

$$b(q|\hat{q}) \propto w(q|\hat{q}) \equiv v(\hat{q}|q)r(q) \tag{4.5}$$

in place of Equation 4.2.

It is often necessary to determine the constant of proportionality for Equations 4.4 and 4.5. This is usually done after the fact by finding the constant, T, which forces the total probability to be equal to 1. That is:

$$T = \frac{1}{\sum_{j=1}^{m} v(\hat{q}|q_j)r_j} \quad \text{or} \quad T = \frac{1}{\int_0^{\infty} v(\hat{q}|q)r(q)dq}. \tag{4.6}$$

An advantage of this practice is that it is no longer necessary to require that $r(q)$ be a proper distribution. The function $r(q)$ becomes a weighting function that can sum to anything, including infinity. The only requirement is that the sums in Equation 4.6 be finite. Prior distributions that sum to infinity are called improper, or diffuse, priors. They are useful when it is felt that there is little or no prior knowledge.

A rather interesting example can be constructed for the single parameter Pareto with the diffuse prior $p(q) \propto r(q) = 1/q$. We have

$$\begin{aligned} w(q|\hat{q}) &= v(\hat{q}|q)r(q) \\ &= qe^{\frac{-n(q-\hat{q})^2}{2\hat{q}^2}} \cdot \frac{1}{q} \\ &= e^{-\frac{n(q-\hat{q})^2}{2\hat{q}^2}}. \end{aligned} \tag{4.7}$$

By comparing Equation 4.7 with the standard normal distribution

$$\varphi(x) \propto e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

we see that the posterior distribution $b(q|\hat{q})$ is normal with mean $\hat{q}$ and variance $\hat{q}^2/n$.

This result should be compared to a standard "non-Bayesian" treatment. The distribution of $\hat{q}$, for a given q, is asymptotically normal with mean q and variance q^2/n. It has become common practice, using the methods demonstrated by Hogg and Klugman,[1] to express a confidence interval for q in terms of an approximately normal distribution with mean $\hat{q}$ and variance $\hat{q}^2/n$. This is admittedly an approximation. For the single parameter Pareto, the above Bayesian result provides a set of assumptions that make this approximation more meaningful.

5. THE DISTRIBUTION OF EXCESS CLAIM SEVERITY ESTIMATES

Obtaining the posterior distribution of q is only an intermediate step toward obtaining the posterior distribution of excess claim severity estimates. We now turn to the completion of this task.

In what follows, we will use a discrete prior distribution. This makes the procedure for getting the posterior distribution easy to set up on a spreadsheet program. The steps for constructing such a spreadsheet program are shown in Table 1 with the results given in various exhibits.

All examples in this paper assume that a maximum likelihood estimate of 1.75 has been obtained using data with a value of k equal to \$100,000. The task is to estimate the expected severity for a layer between \$1,000,000 and \$5,000,000.

It is important to note that the expected severity estimates in this paper will be conditional on the claim being greater than \$100,000. To use these results in practice, one must also consider the number of claims above \$100,000.

The prior means have little meaning if the prior distribution is improper and the means do not exist. The posterior mean and standard deviation of the q_is are given by:

[1] See, for example, Section 4 of Chapter 3 in *Loss Distributions*, by Hogg and Klugman [1]. Examples 1 and 4 in this section are very pertinent to this discussion.

TABLE 1

SPREADSHEET DEFINITIONS

Column	Description
q_i	Values of q_i that have a specified start and end. These are divided into m equally spaced intervals. (We use $m = 30$ for the examples in this paper.)
r_i	Prior weights for q_i .
$v(\hat{q}\|q_i)$	Conditional weights for $\hat{q}$ given q_i, as given by Equation 4.3.
$w(q_i\|\hat{q})$	Posterior weights for q_i, which equal the product of the prior two columns as given by Equation 4.4.
$b(q_i\|\hat{q})$	Posterior probabilities for q_i, which equal: $$w(q_i\|\hat{q})/\sum_{j=1}^{m} w\ (q_j\|\hat{q}).$$
$B(q_i\|\hat{q})$	Cumulative posterior probabilities for q_i, which equal the sum of the posterior probabilities of q_j for $j \leq i$.
$E[X\|q_i]$	Layer average severities given q_i, i.e., the expected severities for a given layer. For the single parameter Pareto, the layer average severities between retention R and limit L are given by the formula:

$$\int_R^L (1-F(x))dx = \begin{cases} \dfrac{k^{q_i}}{q-1}\left(\dfrac{1}{R^{q_i-1}} - \dfrac{1}{L^{q_i-1}}\right) & \text{for } q_i \neq 1 \\ k(\ln(L) - \ln(R)) & \text{for } q_i = 1 \end{cases} \tag{5.1}$$

$$\mathrm{E}[q_i|\hat{q}] = \sum_{i=1}^{m} q_i b(q_i|\hat{q}) \text{ and } \mathrm{Std}[q_i|\hat{q}] = \sqrt{\sum_{i=q}^{m} q_i^2\, b(q_i|\hat{q}) - \mathrm{E}[q_i|\hat{q}]^2}\ .$$

The posterior mean and standard deviation of the $\mathrm{E}[X|q_i]$s are given by:

$$\mathrm{E}[\mathrm{E}[X|\hat{q}]] = \sum_{i=1}^{m} \mathrm{E}[X|q_i] b(q_i|\hat{q}) \text{ and}$$

$$\mathrm{Std}\ [\mathrm{E}[X|\hat{q}]] = \sqrt{\sum_{i=1}^{m} \mathrm{E}[X|q_i]^2 b(q_i|\hat{q}) - \mathrm{E}[\mathrm{E}[X|\hat{q}]]^2}\ .$$

It is a rare event for a q_i or an $\mathrm{E}[X|q_i]$ to hit exactly on a t^{th} or a $(1-t)^{th}$ percentile, so we adopt the following convention for this paper.[2] The t^{th} percentile of q_i is the last q_i before the cumulative probability exceeds t. Similarly, the $(1-t)^{th}$ percentile of q_i is the last q_i before the cumulative probability exceeds $(1-t)$. We would proceed similarly for $\mathrm{E}[X|q_i]$ except that $\mathrm{E}[X|q_i]$ is a decreasing function of q_i. So in this case we replace t with $1-t$ for the t^{th} percentile and $(1-t)$ with t for the $(1-t)^{th}$ percentile. In the examples, we use $t = 2.5$ percent to calculate a 95 percent confidence interval.

For the sake of comparison, we also provide the "classical" estimates based on the estimate, $\hat{q}$, and the estimate of $\hat{q}$ put into Equation 5.1.

[2] It is likely that various textbooks will define percentiles differently than is done here. A possible alternative would be to interpolate between the q_is. The motivation here is that this definition is easy to implement with the typical spreadsheet program and the final decision made as the result of the confidence interval is unlikely to be affected by the choice of confidence interval definition.

It is often helpful to describe the posterior distributions of q_i and $E[X|q_i]$ graphically. This is straightforward for the posterior distribution of q_i. One simply plots q_i on the horizontal axis and $b(q_i|\hat{q})$ on the vertical axis. An additional consideration for plotting $E[X|q_i]$ is that the values of $E[X|q_i]$ are not evenly spaced. If we want the graph to have approximately the same shape as the corresponding continuous posterior distribution, we must plot $E[X|q_i]$ on the horizontal axis and

$$\frac{B(q_{i+1}|\hat{q}) - B(q_{i-1}|\hat{q})}{E[X|q_{i-1}] - E[X|q_{i+1}]} \approx B'(E[X|q_i])$$

on the vertical axis. The plots corresponding to Exhibit 1 are on Figures 1 and 2.

FIGURE 1

POSTERIOR DISTRIBUTION OF q

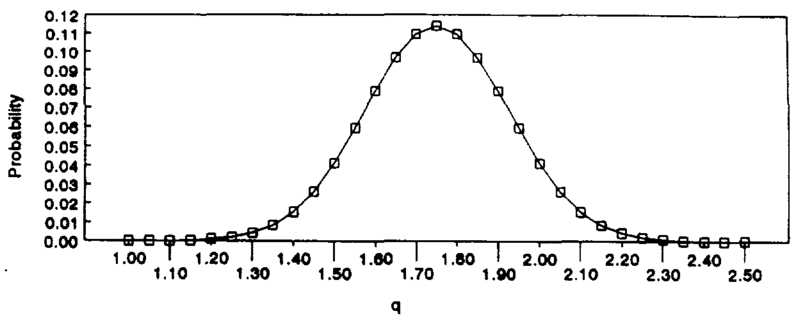
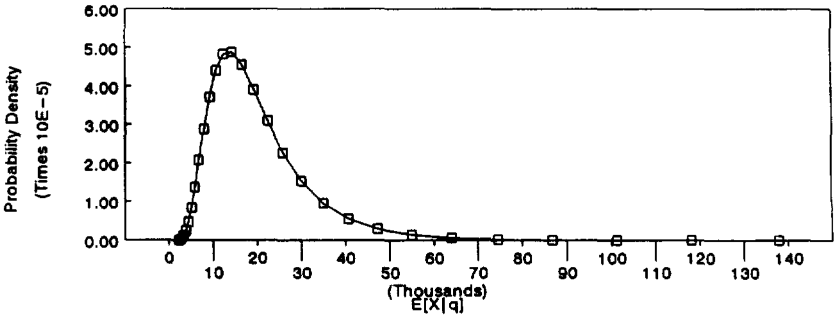


FIGURE 2

POSTERIOR DISTRIBUTION OF E[x|q]



6. THE EFFECT OF SAMPLE SIZE

What is noticeable about the example shown in Exhibit 1 is the large width of the confidence interval and the difference between the posterior mean and the classical estimate of the expected severity. Exhibits 2 and 3 show what happens when the sample size is increased to 1,000, and then 10,000 claims. Table 2 takes the key numbers from these exhibits.

TABLE 2

Source Exhibit	Sample Size	Classical Estimate	Posterior Mean	Posterior Std Dev	Approximate 95% Confidence Interval Low	High
1	100	16,619	19,062	10,640	6,922	54,964
2	1,000	16,619	16,847	2,769	12,755	23,711
3	10,000	16,619	16,642	859	15,213	18,703

Here we see that one must have a sample size of 10,000 to get the length of the confidence band down to 21 percent of the posterior mean claim severity.

Table 2 does point out a possible danger inherent in using the maximum likelihood estimate, $\hat{q}$, directly in Equation 5.1. If one truly believes that the prior distribution of q is proportional to $1/q$, then the classical estimate can produce a significant understatement of the posterior mean. This is especially true for small sample sizes.

7. ON THE CHOICE OF A PRIOR DISTRIBUTION

The bias for small sample sizes noted above may be a function of the prior distribution. In this section, we explore the implications of using different prior distributions.

In our first example, shown on Exhibit 1, we chose a prior distribution for q that was proportional to $1/q$. This had the effect of giving more weight to the smaller qs. Exhibit 4 shows the effect of choosing a prior distribution for q that is proportional to q. This has the effect of giving more weight to the larger qs. The summary of results for each exhibit is provided in Table 3.

TABLE 3

SUMMARY OF RESULTS

	Source Exhibit	Classical Estimate	Posterior Mean	Posterior Std Dev	Approximate 95% Confidence Interval Low	High
q_i	1	1.750	1.750	0.175	1.350	2.050
$E[X\|q_i]$	1	16,619	19,062	10,640	6,922	54,964
q_i	4	1.750	1.785	0.173	1.400	2.050
$E[X\|q_i]$	4	16,619	17,154	9,432	6,922	47,245

Perhaps the most notable observation is that the understatement of the posterior mean by the classical estimate of $E[X|q_i]$ is reduced with the second prior distribution. But we would caution against choosing a prior *for this reason.* The reason for choosing a prior distribution should be based on one's beliefs about the distribution of q.

Figures 3 and 4 provide graphical comparisons of the results of Exhibits 1 and 4.

A common sentiment of practitioners is: "I am extremely lucky to get 100 claims to analyze. Yet I can't go to my company and say: 'On the basis of (for example) Exhibit 1, my recommended value for the expected severity is \$19,000, but it could reasonably be as low as \$7,000 or as high as \$55,000.' "

Many practitioners are, at least intuitively, aware of the large potential variability of the results and frequently override any outlying estimates citing "judgment." As the following example shows, it is possible to blend the maximum likelihood estimate and one's prior judgment by choosing an appropriate prior distribution.

FIGURE 3

POSTERIOR DISTRIBUTION OF q

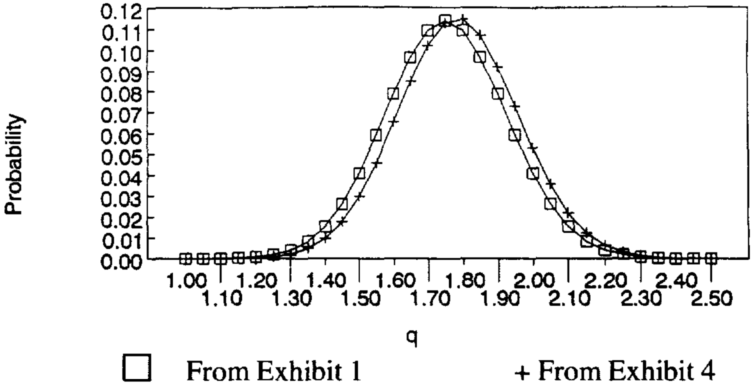
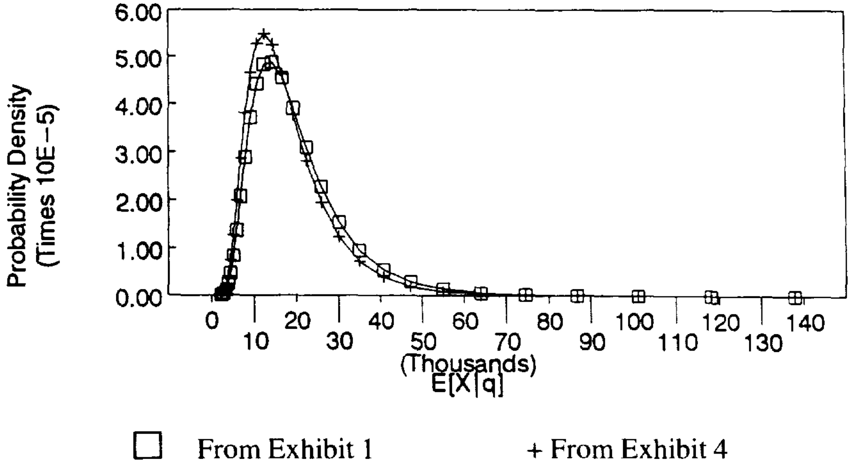


FIGURE 4

POSTERIOR DISTRIBUTION OF E[X | q]



Let us assume that one is willing to accept that the expected severity could be as low as \$17,500, or as high as \$25,000. Thus we allow q to be no lower than 1.612 or no higher than 1.732 (give or take some rounding error in q). Let us further assume that one feels that lower qs should be given more weight and selects a prior proportional to $1/q$. The resulting posterior distributions are in Exhibit 5.

Now, since we have a priori bounds on the range of the q_is, it makes sense to talk about prior means. The prior means of the q_is and the E[X|q_i]s are 1.671 and \$21,099, respectively. This should be compared with the posterior means of 1.675 and \$20,851, respectively. It should also be noted that the approximate 95 percent confidence intervals are pretty much the same as the a priori ranges of the q_is and the E[X|q_i]s. It would appear that, at least in this example, the information added by the 100 claims has a relatively minor impact on our estimated claim severity for the \$1,000,000 to \$5,000,000 layer.

This example also makes the point that it is possible to observe a $\hat{q}$ that is outside the prior range of q. However, the posterior mean of q, given $\hat{q}$, is within the prior range of q.

8. OTHER DISTRIBUTIONAL MODELS

In this section we indicate how one can proceed if a distributional model other than the single parameter Pareto is to be used. Let $\boldsymbol{\theta} = \{\theta_j\}$ be the parameter vector for the chosen model, $f(x|\boldsymbol{\theta})$, and let $\hat{\boldsymbol{\theta}} = \{\hat{\theta}_j\}$ be the maximum likelihood estimate of $\boldsymbol{\theta}$.

The procedure described in Section 5 will work with the following modifications.

1. The parameter q_i must be replaced with the vector $\boldsymbol{\theta}_i$.
2. Prior weights have to be assigned to each $\boldsymbol{\theta}_i$.
3. The conditional weights $v(\hat{\boldsymbol{\theta}}_i|\boldsymbol{\theta}_i)$ must be derived. Depending upon the distributional form selected, it may be possible to derive the weights directly as was done in Section 3. If this fails, there is an alternative approximation. As described by Hogg and Klugman,[3] the conditional distribution of $\hat{\boldsymbol{\theta}}$, given $\boldsymbol{\theta}$, is asymptotically a multivariate normal distribution with mean $\boldsymbol{\theta}$ and covariance matrix $\boldsymbol{\Sigma}^{-1}$, where $\boldsymbol{\Sigma} = \{a_{jk}\}$, and:

$$a_{jk} = -n\mathrm{E}\left[\frac{\partial^2 \ln(f(x|\boldsymbol{\theta}))}{\partial\theta_j\partial\theta_k}\right].$$

Then $$v(\hat{\boldsymbol{\theta}}|\boldsymbol{\theta}) = |\Sigma| e^{-(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta})^T\Sigma(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta})/2}.$$

[3]Section 4 of Chapter 3 of *Loss Distributions* [1].

4. The formula for $E[X|\theta_i]$ will depend upon the distributional form of $f(x|\theta_i)$.

A bit of soul searching may be necessary to come up with a prior distribution for the parameter vector θ. One suggestion would be to place a prior distribution on $E[X|\theta]$ and translate the results into a prior distribution for θ. We came close to doing this in Exhibit 5 by choosing qs that restrict the expected severity between \$17,500 and \$25,000.

A complaint often heard is that one should be just as concerned about the model uncertainty as with parameter uncertainty. To address this complaint, one can put any number of models into this procedure, as long as prior probabilities for each model are assigned.

The problems associated with other severity models may indeed be formidable. We are fortunate to have a simple and realistic model like the single parameter Pareto to provide us with a blueprint.

9. A CONCLUDING REMARK

As noted in the Introduction, it is currently a common practice to use a fitted claim severity distribution to estimate the expected claim severity for an excess layer of insurance. These fits are often obtained with sample sizes containing fewer than 100 claims.

These estimates take a prominent role in insurance (and reinsurance) price negotiations. Insurance buyers will often readily accept estimates based on "their own data." One expects a buyer with a relatively low estimate to cite this as evidence that they deserve a break in their rates, while those buyers with relatively high estimates are in much weaker negotiating positions. While there may be significant differences between insurance buyers, the examples given above illustrate the dangers of drawing such a conclusion based solely on a fitted distribution. Good prior information should also play an important role in these negotiations.

While many practitioners recognize this, they are often under pressure to recognize “real data” supplied by the (re)insured. This paper provides a way to recognize the data and integrate it with prior information.

REFERENCES

[1] Hogg, Robert V., and Stuart A. Klugman, *Loss Distributions*, John Wiley & Sons, 1984.

[2] Miccolis, Robert S., "On the Theory of Increased Limits and Excess of Loss Pricing," *PCAS* LXIV, 1977, pp. 27-59.

[3] Philbrick, Stephen W., "A Practical Guide to the Single Parameter Pareto Distribution," *PCAS* LXXII, 1985, pp. 44-123.

EXHIBIT 1

Prior Distribution			Given			Observed	
q_0	q_{30}	r_i	k	Retention	Limit	$\hat{q}$	n
1.000	2.500	$1/q_i$	100,000	1,000,000	5,000,000	1.75	100

q_i	r_i	$v(\hat{q} \mid q_i)$	$w(q_i \mid \hat{q})$	$b(q_i \mid \hat{q})$	$B(q_i \mid \hat{q})$	$E[X \mid q_i]$
1.000	1.000	0.0001	0.0001	0.0000	0.0000	160,944
1.050	0.952	0.0004	0.0003	0.0000	0.0000	137,822
1.100	0.909	0.0011	0.0010	0.0001	0.0002	118,085
1.150	0.870	0.0032	0.0028	0.0003	0.0005	101,229
1.200	0.833	0.0086	0.0072	0.0008	0.0013	86,826
1.250	0.800	0.0211	0.0169	0.0019	0.0032	74,512
1.300	0.769	0.0477	0.0367	0.0042	0.0074	63,979
1.350	0.741	0.0990	0.0734	0.0084	0.0158	54,964
1.400	0.714	0.1895	0.1353	0.0154	0.0312	47,245
1.450	0.690	0.3336	0.2301	0.0262	0.0574	40,631
1.500	0.667	0.5407	0.3604	0.0411	0.0985	34,961
1.550	0.645	0.8067	0.5205	0.0593	0.1578	30,099
1.600	0.625	1.1081	0.6926	0.0789	0.2368	25,926
1.650	0.606	1.4015	0.8494	0.0968	0.3336	22,343
1.700	0.588	1.6320	0.9600	0.1094	0.4430	19,265
1.750	0.571	1.7500	1.0000	0.1140	0.5570	16,619
1.800	0.556	1.7280	0.9600	0.1094	0.6664	14,344
1.850	0.541	1.5713	0.8494	0.0968	0.7632	12,387
1.900	0.526	1.3159	0.6926	0.0789	0.8422	10,702
1.950	0.513	1.0149	0.5205	0.0593	0.9015	9,251
2.000	0.500	0.7209	0.3604	0.0411	0.9426	8,000
2.050	0.488	0.4716	0.2301	0.0262	0.9688	6,922
2.100	0.476	0.2842	0.1353	0.0154	0.9842	5,992
2.150	0.465	0.1577	0.0734	0.0084	0.9926	5,189
2.200	0.455	0.0806	0.0367	0.0042	0.9968	4,496
2.250	0.444	0.0380	0.0169	0.0019	0.9987	3,897
2.300	0.435	0.0165	0.0072	0.0008	0.9995	3,380
2.350	0.426	0.0066	0.0028	0.0003	0.9998	2,932
2.400	0.417	0.0024	0.0010	0.0001	1.0000	2,545
2.450	0.408	0.0008	0.0003	0.0000	1.0000	2,210
2.500	0.400	0.0003	0.0001	0.0000	1.0000	1,920

SUMMARY OF RESULTS

	Classical Estimate	Posterior Mean	Posterior Std Dev	Approximate 95% Confidence Interval Low	High
For q_i	1.750	1.750	0.175	1.350	2.050
For $E[X \mid q_i]$	16,619	19,062	10,640	6,922	54,964

EXHIBIT 2

Prior Distribution				Given		Observed	
q_0	q_{30}	r_i	k	Retention	Limit	$\hat{q}$	n
1.540	1.990	$1/q_i$	100,000	1,000,000	5,000,000	1.75	1,000

q_i	r_i	$v(\hat{q}\|q_i)$	$w(q_i\|\hat{q})$	$b(q_i\|\hat{q})$	$B(q_i\|\hat{q})$	$E[X\|q_i]$
1.540	0.649	0.0011	0.0007	0.0001	0.0001	31,012
1.555	0.643	0.0031	0.0020	0.0002	0.0003	29,652
1.570	0.637	0.0079	0.0050	0.0005	0.0008	28,353
1.585	0.631	0.0186	0.0117	0.0013	0.0021	27,111
1.600	0.625	0.0406	0.0254	0.0027	0.0049	25,926
1.615	0.619	0.0824	0.0510	0.0055	0.0104	24,793
1.630	0.613	0.1553	0.0953	0.0103	0.0207	23,711
1.645	0.608	0.2719	0.1653	0.0179	0.0386	22,677
1.660	0.602	0.4424	0.2665	0.0288	0.0674	21,689
1.675	0.597	0.6686	0.3992	0.0432	0.1105	20,745
1.690	0.592	0.9389	0.5556	0.0601	0.1706	19,844
1.705	0.587	1.2250	0.7185	0.0777	0.2483	18,982
1.720	0.581	1.4850	0.8633	0.0934	0.3417	18,158
1.735	0.576	1.6724	0.9639	0.1042	0.4459	17,371
1.750	0.571	1.7500	1.0000	0.1081	0.5541	16,619
1.765	0.567	1.7013	0.9639	0.1042	0.6583	15,901
1.780	0.562	1.5368	0.8633	0.0934	0.7517	15,213
1.795	0.557	1.2897	0.7185	0.0777	0.8293	14,557
1.810	0.552	1.0056	0.5556	0.0601	0.8894	13,929
1.825	0.548	0.7285	0.3992	0.0432	0.9326	13,329
1.840	0.543	0.4903	0.2665	0.0288	0.9614	12,755
1.855	0.539	0.3066	0.1653	0.0179	0.9793	12,207
1.870	0.535	0.1782	0.0953	0.0103	0.9896	11,683
1.885	0.531	0.0962	0.0510	0.0055	0.9951	11,181
1.900	0.526	0.0482	0.0254	0.0027	0.9979	10,702
1.915	0.522	0.0225	0.0117	0.0013	0.9991	10,244
1.930	0.518	0.0097	0.0050	0.0005	0.9997	9,805
1.945	0.514	0.0039	0.0020	0.0002	0.9999	9,386
1.960	0.510	0.0015	0.0007	0.0001	1.0000	8,985
1.975	0.506	0.0005	0.0003	0.0000	1.0000	8,602
1.990	0.503	0.0002	0.0001	0.0000	1.0000	8,235

SUMMARY OF RESULTS

	Classical Estimate	Posterior Mean	Posterior Std Dev	Approximate 95% Confidence Interval Low	High
For q_i	1.750	1.750	0.055	1.630	1.840
For $E[X\|q_i]$	16,619	16,847	2,769	12,755	23,711

EXHIBIT 3

Prior Distribution			Given			Observed	
q_0	q_{30}	r_i	k	Retention	Limit	$\hat{q}$	n
1.670	1.820	$1/q_i$	100,000	1,000,000	5,000,000	1.75	10,000

q_i	r_i	$v(\hat{q}\mid q_i)$	$w(q_i\mid\hat{q})$	$b(q_i\mid\hat{q})$	$B(q_i\mid\hat{q})$	$E[X\mid q_i]$
1.670	0.599	0.0000	0.0000	0.0000	0.0000	21,055
1.675	0.597	0.0002	0.0001	0.0000	0.0000	20,745
1.680	0.595	0.0006	0.0003	0.0000	0.0001	20,440
1.685	0.593	0.0017	0.0010	0.0001	0.0002	20,140
1.690	0.592	0.0047	0.0028	0.0003	0.0005	19,844
1.695	0.590	0.0121	0.0072	0.0008	0.0013	19,552
1.700	0.588	0.0287	0.0169	0.0019	0.0032	19,265
1.705	0.587	0.0625	0.0367	0.0042	0.0074	18,982
1.710	0.585	0.1255	0.0734	0.0084	0.0158	18,703
1.715	0.583	0.2321	0.1353	0.0154	0.0312	18,429
1.720	0.581	0.3957	0.2301	0.0262	0.0574	18,158
1.725	0.580	0.6218	0.3604	0.0411	0.0985	17,892
1.730	0.578	0.9004	0.5205	0.0593	0.1578	17,630
1.735	0.576	1.2016	0.6926	0.0789	0.2368	17,371
1.740	0.575	1.4779	0.8494	0.0968	0.3336	17,117
1.745	0.573	1.6752	0.9600	0.1094	0.4430	16,866
1.750	0.571	1.7500	1.0000	0.1140	0.5570	16,619
1.755	0.570	1.6848	0.9600	0.1094	0.6664	16,376
1.760	0.568	1.4949	0.8494	0.0968	0.7632	16,137
1.765	0.567	1.2224	0.6926	0.0789	0.8422	15,901
1.770	0.565	0.9212	0.5205	0.0593	0.9015	15,668
1.775	0.563	0.6398	0.3604	0.0411	0.9426	15,439
1.780	0.562	0.4095	0.2301	0.0262	0.9688	15,213
1.785	0.560	0.2416	0.1353	0.0154	0.9842	14,991
1.790	0.559	0.1313	0.0734	0.0084	0.9926	14,772
1.795	0.557	0.0658	0.0367	0.0042	0.9968	14,557
1.800	0.556	0.0304	0.0169	0.0019	0.9987	14,344
1.805	0.554	0.0129	0.0072	0.0008	0.9995	14,135
1.810	0.552	0.0051	0.0028	0.0003	0.9998	13,929
1.815	0.551	0.0018	0.0010	0.0001	1.0000	13,726
1.820	0.549	0.0006	0.0003	0.0000	1.0000	13,526

SUMMARY OF RESULTS

	Classical Estimate	Posterior Mean	Posterior Std Dev	Approximate 95% Confidence Interval Low	High
For q_i	1.750	1.750	0.017	1.710	1.780
For $E[X\mid q_i]$	16,619	16,642	859	15,213	18,703

EXHIBIT 4

Prior Distribution			Given			Observed	
q_0	q_{30}	r_i	k	Retention	Limit	$\hat{q}$	n
1.000	2.500	q_i	100,000	1,000,000	5,000,000	1.75	100

q_i	r_i	$v(\hat{q}\vert q_i)$	$w(q_i\vert\hat{q})$	$b(q_i\vert\hat{q})$	$B(q_i\vert\hat{q})$	$E[X\vert q_i]$
1.000	1.000	0.0001	0.0001	0.0000	0.0000	160,944
1.050	1.050	0.0004	0.0004	0.0000	0.0000	137,822
1.100	1.100	0.0011	0.0012	0.0000	0.0001	118,085
1.150	1.150	0.0032	0.0037	0.0001	0.0002	101,229
1.200	1.200	0.0086	0.0103	0.0004	0.0006	86,826
1.250	1.250	0.0211	0.0264	0.0010	0.0016	74,512
1.300	1.300	0.0477	0.0620	0.0023	0.0038	63,979
1.350	1.350	0.0990	0.1337	0.0049	0.0088	54,964
1.400	1.400	0.1895	0.2653	0.0098	0.0185	47,245
1.450	1.450	0.3336	0.4837	0.0178	0.0364	40,631
1.500	1.500	0.5407	0.8110	0.0299	0.0662	34,961
1.550	1.550	0.8067	1.2504	0.0461	0.1123	30,099
1.600	1.600	1.1081	1.7730	0.0653	0.1777	25,926
1.650	1.650	1.4015	2.3124	0.0852	0.2629	22,343
1.700	1.700	1.6320	2.7744	0.1022	0.3651	19,265
1.750	1.750	1.7500	3.0625	0.1129	0.4780	16,619
1.800	1.800	1.7280	3.1104	0.1146	0.5926	14,344
1.850	1.850	1.5713	2.9070	0.1071	0.6997	12,387
1.900	1.900	1.3159	2.5002	0.0921	0.7919	10,702
1.950	1.950	1.0149	1.9790	0.0729	0.8648	9,251
2.000	2.000	0.7209	1.4418	0.0531	0.9179	8,000
2.050	2.050	0.4716	0.9669	0.0356	0.9535	6,922
2.100	2.100	0.2842	0.5968	0.0220	0.9755	5,992
2.150	2.150	0.1577	0.3392	0.0125	0.9880	5,189
2.200	2.200	0.0806	0.1774	0.0065	0.9946	4,496
2.250	2.250	0.0380	0.0855	0.0031	0.9977	3,897
2.300	2.300	0.0165	0.0379	0.0014	0.9991	3,380
2.350	2.350	0.0066	0.0155	0.0006	0.9997	2,932
2.400	2.400	0.0024	0.0058	0.0002	0.9999	2,545
2.450	2.450	0.0008	0.0020	0.0001	1.0000	2,210
2.500	2.500	0.0003	0.0006	0.0000	1.0000	1,920

SUMMARY OF RESULTS

	Classical Estimate	Posterior Mean	Posterior Std Dev	Approximate 95% Confidence Interval Low	High
For q_i	1.750	1.785	0.173	1.400	2.050
For $E[X\vert q_i]$	16,619	17,154	9,432	6,922	47,245

EXHIBIT 5

Prior Distribution			Given			Observed	
q_0	q_{30}	r_i	k	Retention	Limit	$\hat{q}$	n
1.612	1.732	$1/q_i$	100,000	1,000,000	5,000,000	1.75	100

q_i	r_i	$v(\hat{q}\mid q_i)$	$w(q_i\mid\hat{q})$	$b(q_i\mid\hat{q})$	$B(q_i\mid\hat{q})$	$E[X\mid q_i]$
1.612	0.620	1.1812	0.7328	0.0265	0.0265	25,015
1.616	0.619	1.2054	0.7459	0.0270	0.0536	24,719
1.620	0.617	1.2294	0.7589	0.0275	0.0811	24,427
1.624	0.616	1.2532	0.7717	0.0280	0.1090	24,138
1.628	0.614	1.2768	0.7843	0.0284	0.1374	23,852
1.632	0.613	1.3001	0.7967	0.0289	0.1663	23,570
1.636	0.611	1.3232	0.8088	0.0293	0.1956	23,292
1.640	0.610	1.3460	0.8207	0.0297	0.2253	23,016
1.644	0.608	1.3685	0.8324	0.0302	0.2555	22,744
1.648	0.607	1.3906	0.8438	0.0306	0.2860	22,476
1.652	0.605	1.4123	0.8549	0.0310	0.3170	22,210
1.656	0.604	1.4335	0.8657	0.0314	0.3484	21,948
1.660	0.602	1.4544	0.8761	0.0317	0.3801	21,689
1.664	0.601	1.4747	0.8863	0.0321	0.4122	21,433
1.668	0.600	1.4946	0.8960	0.0325	0.4447	21,180
1.672	0.598	1.5139	0.9054	0.0328	0.4775	20,931
1.676	0.597	1.5327	0.9145	0.0331	0.5106	20,684
1.680	0.595	1.5508	0.9231	0.0334	0.5440	20,440
1.684	0.594	1.5684	0.9314	0.0337	0.5778	20,199
1.688	0.592	1.5853	0.9392	0.0340	0.6118	19,961
1.692	0.591	1.6016	0.9466	0.0343	0.6461	19,726
1.696	0.590	1.6171	0.9535	0.0345	0.6806	19,494
1.700	0.588	1.6320	0.9600	0.0348	0.7154	19,265
1.704	0.587	1.6461	0.9660	0.0350	0.7504	19,038
1.708	0.585	1.6595	0.9716	0.0352	0.7856	18,814
1.712	0.584	1.6721	0.9767	0.0354	0.8210	18,593
1.716	0.583	1.6839	0.9813	0.0355	0.8565	18,374
1.720	0.581	1.6949	0.9854	0.0357	0.8922	18,158
1.724	0.580	1.7051	0.9890	0.0358	0.9280	17,945
1.728	0.579	1.7144	0.9921	0.0359	0.9640	17,734
1.732	0.577	1.7229	0.9947	0.0360	1.0000	17,526

SUMMARY OF RESULTS

	Classical Estimate	Posterior Mean	Posterior Std Dev	Approximate 95% Confidence Interval Low	High
For q_i	1.750	1.675	0.035	1.612	1.728
For $E[X\mid q_i]$	16,619	20,851	2,198	17,734	25,015