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REINSURER RISK LOADS FROM MARGINAL SURPLUS REQUIREMENTS

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1. INTRODUCTION

Writing an insurance risk increases the variability of an insurer's results. This has direct economic costs to the insurer, such as not being able to write other attractive risks or to comfortably maintain the desired degree of risk in its asset portfolio. Extra risk also reduces the value of its future profits in the capital market, that is, its stock price or similar valuation. Insurers require premiums that allow enough expected profit to overcome these costs.

In "Reinsurer Risk Loads from Marginal Surplus Requirements" and "Investment-Equivalent Reinsurance Pricing," published in this volume, Rodney Kreps has increased our understanding of how a reinsurer (actually, any insurer) commits its capital to risks. Based on simple microeconomic assumptions and consequential expressions, Kreps has developed powerful models relating an insurer's capitalization and a prospective contract's risk profile to develop a minimum acceptable premium for the contract. Premiums below this minimum cause an insurer to dilute its earnings and should be rejected. Even though reinsurers do not appear to manage their capital on an individual contract basis as Kreps' calculations assume, the model has received the highest actuarial compliment: it is actively used to price business. This is notable at several of the newly established catastrophe reinsurance markets.

2. THE KREPS MODEL

Fill your bowl to the brim and it will spill.
Keep sharpening your knife and it will blunt.

— The *Tao*[1]

Kreps begins by assuming that every insurance contract can be uniquely associated with a marginal amount of an insurer's surplus. This amount is computed assuming that each insurer selects and maintains a certain probability of ruin and then finds the amount by which its surplus must increase to maintain that probability if a proposed contract is written. For proposed contracts that are small relative to the insurer's existing business, this is equivalent to requiring that the ratio of the insurer's surplus to the standard deviation of its results does not change after the contract is written. A proposed contract must have an expected profit that adequately rewards the required marginal amount of surplus, or else it would dilute the insurer's return and is thus declined.

To see how Kreps' results are used, recall his Equation 2.4,

$$\text{Minimum premium for a contract} = \mu + \sigma\mathscr{R} + E - yB/(1+y),$$

where μ and σ are the mean and standard deviation of the losses on the proposed contract and y is the insurer's target return on equity. E is the insurer's marginal expense. B is the "bank," if any, that the insured has "built up." Kreps defines $\mathscr{R}$ as the insurer's "reluctance" to assume additional degrees of risk.

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[1]See Mitchell [11, Chapter 9]. The *Tao*, literally the "Way," is a short collection of Chinese philosophical writing that was probably first gathered in the sixth century B.C., but which is still influential for its simple, natural, and organic way of describing human perception and behavior.

This excerpt and others from Mitchell's readable translation help illustrate the forces that insurers must seek to balance. Readers may want to consider that critical thinking about focus, competition, success, and control is much older and deeper than our current microeconomic analysis of insurers.

For high-level catastrophe coverage on national ceding companies, each proposed contract has a correlation close to 1, since these treaties are only exposed by a very few physical hazards that already expose the reinsurers' other business. Kreps argues that for contracts that are relatively small additions to the reinsurers' portfolio, the average ratio of marginal surplus to marginal standard deviation is equal to the average ratio. With a correlation of 1, and using z to represent the ratio of the insurer's current surplus to the standard deviation of the losses on its existing portfolio, Kreps' reluctance becomes:

$$\mathcal{R} = yz/(1+y). \tag{2.1}$$

At this writing, few reinsureds claim large positive "banks" (to be kind) and few reinsurers see "banks" as economic rather than rhetorical obligations; so the B term is ignored as well, and Kreps' conclusion (with a simple substitution) shows that the minimum premium must be:

$$\mu + (yz/(1+y))\sigma + E. \tag{2.2}$$

This minimum premium has two contract-specific terms: the expected losses plus a charge for the marginal contribution to the insurer's standard deviation. The latter term can be thought of as an interest rate, $y/(1+y)$, applied to a marginal amount of surplus, $z\sigma$. The two terms are generally independent. This conclusion is in sharp contrast to earlier actuarial theory and practice, which based risk charges either directly on the expected losses (incurred or unpaid) or, indirectly, through the premiums on notional allocations of surplus.

The values of $\mathcal{R} = yz/(1+y)$ are similar at many catastrophe reinsurance markets and share derivations based on similar views of acceptable ruin scenarios and required returns to capital. While it is inappropriate to detail specific market pricing in an industry forum, I can also note that prices often show similar variations in the relative contribution of the μ and $\mathcal{R}\sigma$ terms for different contracts. The following four examples are several brokers' consensus estimates of prices at a recent date for different property catastrophe reinsurance layers for a hypothetical U.S. nationally-exposed cedant, expressed as annual rates-on-line (ROL):[2]

Annual Layer and Retention ($ Millions)	Layer Penetration Recurrence Time (Years)	Estimated Price (ROL)	Pure Premium (ROL)	Standard Deviation (ROL)	Implied Loss Ratio
10 xs 10	10	18.00%	10.0%	30.0%	55.5%
10 xs 20	40	9.50%	2.5%	15.6%	26.3%
30 xs 30	100	5.00%	1.0%	9.9%	20.0%
40 xs 60	1,000	2.25%	0.1%	3.1%	4.4%

For many cedants, the expected loss ratios vary across the different layers of their programs by this factor of ten or more, decreasing in the higher layers as the risk charge contribution takes on more importance compared to the expected losses. Kreps' formula easily explains this and other surprising[3] variations in risk loads visible in the current reinsurance market.

[2]The annual losses to each layer are approximated as a binomial process. Pure Premium = 1/Recurrence Time. Standard deviation is the square root of the product of pure premium and the complement of pure premium.

[3]Another excellent reinsurance example where Kreps' approach improves our understanding of current pricing is "second event" covers. Reinsurance actuaries frequently treat prices expressed as rate-on-line as if they were probabilities. This common shorthand cannot explain second-event cover prices.

For the hypothetical cedant reviewed earlier, a $10 million excess $10 million second-event layer (i.e., pays up to $10 million for a second loss during the year in excess of $10 million) has a pure premium of approximately 1/2% (as a rate-on-line). Using Poisson assumptions, the pure premium for the original layer, when expressed as a rate-on-line, is actually the probability of one or more losses; so the complement of the pure premium is the probability of no losses, here 90%. This produces a Poisson frequency of $-\ln(90\%) = 10.54\%$. It follows that the probability of exactly one loss is $90\% \times 10.54\%$ or 9.48%. (The Poisson probability of exactly one loss is the probability of no losses times the frequency.) The probability of no more than one loss is 99.48%. The probability of two or more losses is 0.52%.

Many actuaries are tempted to perform a similar calculation on the price for the original layer, which is an 18% rate on line. If 18% is a fair compensation for assuming the risk of one or more losses to the layer, including the value of assuming the variability in the layer, then 82.0% is the consistent price for a "no losses" cover. Continuing with this common logic, and treating the price as a risk-adjusted probability, produces a Poisson "risk-adjusted" frequency of 19.8%; and the risk-adjusted price for coverage of exactly one loss is 16.3%. Thus the price for a second-event coverage, under this logic, would be

The Kreps minimum premium formulation is clear, understandable and powerful. It also avoids the problematic assumptions[4] needed to allocate an insurer's total surplus to product. However, the model ignores important considerations:

- Other things equal, an insurer prefers to reduce its probability of ruin below the current level.
- If marginal results are very attractive, an insurer may choose to grow and increase its probability of ruin beyond the current target.
- The market capitalization rate applied to an insurer's future profits must depend on the kind and amount of business that the insurer assumes.
- Insurers identify and separately manage distinct risk categories, such as lines of business and exposure zones. They do not directly examine the covariance between a proposed contract and their entire existing portfolios. This is particularly true for catastrophe reinsurers that analyze contracts using modern event-modeling software.
- Insurers do not always calculate unlimited means of the losses for their contracts. They generally evaluate expectations only over scenarios with realistic probabilities. For example, current event-modeling software includes only foreseeable events with

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1.7% (100%, less 82.0% for the value of "covering" the no loss case, less 16.3% for the value of exactly one loss.) In financial economics, this probability-like measure is called a *martingale*, and some recent research has developed utility functions that produce risk loaded prices that are martingales.

Unfortunately, brokers agree that in today's market this second-event cover would actually cost something above a 5% rate-on-line. Real-world prices are not martingales and the common arithmetic of treating a reinsurance rate-on-line as a risk-adjusted frequency is empirically wrong. Kreps' approach correctly indicates the higher price by noting that the standard deviation of the second-event layer is above that of the third excess layer and is even more highly correlated with reinsurers' results. The risk load must be a significantly greater part of the limit than the 4% in that higher layer (5% price less 1% pure premium).

Second-event cover pricing, as well as the up-front discount for a mandatory 100% reinstatement premium, is strong empirical evidence for a formulation like Kreps'.

[4]See Kneuer [6], Miller and Rapp [10], Roth [12], and Bass and Khury [1].

estimated annual recurrence probabilities above 10^{-4} to 10^{-7}. Less frequent (or apparent) events are omitted, so the reported means are understated. There is no theoretical reason why the unlimited mean even needs to be finite.

- Kreps' process is circular.[5] Insurers evaluate proposed contracts based on their expected return on marginal surplus. But the marginal surplus requirements depend on the order in which proposed contracts are evaluated. In Kreps' calculation, an insurer compares each proposed contract's contribution to the variance of a portfolio consisting of every other current contract. This is equivalent to assuming, a priori, that each contract is equally desirable. That may not be the case because some contracts may be selected before others. A different amount of imputed marginal surplus will be found when the comparison base is some contracts, rather than all. Different minimum premiums result.

These considerations matter and the users of Kreps' formula need to consider how the limitations in his assumptions may distort their analyses. Fortunately, the distortions are not fatal. We can avoid the first five considerations listed above. Starting with similar, but broader assumptions, a more realistic model can support results much like Kreps' contributions. However, the last concern, circularity, is not directly avoided (at least, not yet).

Let's explore a model that allows an insurer the flexibility to pick a portfolio of risks so as to adjust its level of risk compared to its capital base. The alternative model, like Kreps', will omit tractable real world considerations including taxes and reserves and their associated investment income. A more complete model would reflect multiple risk factors, but is also deferred here for simplicity.

[5] See Gogol [5] and Mango [9] for illustrations of these differences and a suggested cure to the problem.

3. AN ALTERNATIVE MODEL

Nothing in the world is as soft and yielding as water.
As for dissolving the hard and inflexible nothing can surpass it.[6]

An insurer's job, like any firm's, is to maximize its worth, the expected present value of its future free cash flows. This present value reflects the riskiness of its business, among other things. We simplistically assume that the insurer's management's only decision variables are the portion of each proposed contract that the insurer will assume. That is, it can choose to assume between 0% and 100% of each contract that has been offered to it. Further, with Kreps, we assume that the insurer is a price-taker. Its individual decision does not change the price at which a contract is offered.

Like Kreps, let's also assume that our insurer is looking forward one period and examining how a proposed contract changes its probability of ruin. However, we do not assume an inflexible maximum probability of ruin. Instead we consider a fluid distribution of the insurer's future value to its owners (stockholders, or policyholders if a non-stock insurer). We use a nearly linear relationship to (GAAP) surplus that seems close to current market valuations:

$$V_1 = \text{Value of Insurer (at } t = 1)$$
$$= \begin{cases} 0, & \text{if} \quad \text{Surplus is less than some value, } G_1; \\ M \times \text{Surplus}, & \text{if} \quad \text{Surplus} \geq G_1. \end{cases} \tag{3.1}$$

Like a shark, when an insurer stops moving, it drowns. G_1, which is significantly greater than zero, represents the $t = 1$ surplus level below which the insurer ceases to be a going concern. The discontinuity point is likely much less than S_0, the current sur-

[6]The *Tao* [11, Chapter 78].

plus level. M, the insurer's book-value multiple,[7] is a number that is rarely below 1.0 nor often as high as 3.0.

If there were an absolutely efficient market in insurer capital, then $G_1 = 0$ and $M = 1$, because V_1 would always equal S_1 (or zero, when S_1 is negative). However, regulation and clients' security concerns limit the flexibility to move capital through insurers[8] and this allows market valuations higher than book values ($M > 1$). Let us assume that the *franchise value*, this capitalized value above the break-up value, $(M - 1)S_0$, is positive and much larger than the expected value of the losses that might be avoided by bankruptcy. Our insurer is solid now and underwrites believing it will stay that way.

We can analyze our insurer's microeconomic underwriting decision in light of this more general model. Our insurer has already selected a portfolio of contracts with premium P and random losses L, with known expectation, $\mathrm{E}(L)$. Our insurer is considering a new contract with premium p and random losses ℓ. The insurer will choose to insure some part of the risk, Q, between 0 and 1.

Thus its total premium will be $P + Qp$ and its total losses will be $L + Q\ell$. Ignoring investments, taxes and operating expenses for simplicity here, and assuming that the proposed contract expires in time for the measurement of the insurer's value that we assume occurs at $t = 1$, we find that the insurer's final surplus is:

$$S_1 = \max(0, S_0 + P - L + Q(p - \ell)), \tag{3.2}$$

where L and ℓ are random variables and S_0, P, and p are known to the insurer.

Consider how our insurer looks at the distribution and expectation of the net present value (NPV) of its total future value,

[7] Investment bankers often use book-value multiples for valuations of P/C insurers because these multiples are more stable than Price/Earnings ratios and also control for leverage differences.

[8] See Kneuer [7].

including the franchise value:

$$\text{NPV} = \begin{cases} 0, & \text{if} \quad S_1 < G_1; \\ \text{NPV}(M \times S_1), & \text{if} \quad S_1 \geq G_1. \end{cases} \tag{3.3}$$

$$\text{E(NPV)} = M \times \text{Prob}(S_1 \geq G_1) \times \text{E}(S_1 \mid S_1 \geq G_1)/(1+y). \tag{3.4}$$

Abbreviating, $\text{E(NPV)} = M \times \text{Z} \times \Psi/(1+y)$, where Z is the probability that $S_1 \geq G_1$, Ψ is the conditional expectation of S_1, given that $S_1 \geq G_1$, and y is Kreps' assumed management target yield rate, which approximates in concept the appropriately risk-adjusted market discount rate in effect between $t = 0$ and $t = 1$.

For any price on the proposed contract, our insurer seeks to maximize its current worth, its expected NPV, by choosing a value of Q. It will maximize this market value by differentiating E(NPV) over Q and examining the derivative at $Q = 0$. If the derivative is positive, the insurer will decide to assume at least some of the proposed contract. The insurer will decide to assume more as long as this derivative remains positive at higher values of Q:

$$d/dQ\,\text{E(NPV)} = d/dQ(\text{Z} \times M \times \Psi/(1+y)) \tag{3.5}$$

$$= M\left(\frac{\text{Z}'\Psi}{1+y} + \frac{\Psi'\text{Z}}{1+y} - \frac{\Psi \text{Z} y'}{(1+y)^2}\right) \tag{3.6}$$

where Z', Ψ', and y' are derivatives with respect to Q.

The interpretation of this formula is direct. The present value that our insurer expects to add (or subtract) by writing some of the proposed contract (in other words, the derivative with respect to Q) is:

- the increase (decrease) in the probability of remaining a going concern (Z') times the current present value of the firm as a going concern ($M\Psi/(1+y)$), plus

- the increase (decrease) in the current present value of the firm as a going concern ($M\Psi'/(1+y)$), times the current probability of remaining a going concern (Z), minus
- the current present value of the firm ($MZ\Psi/(1+y)$) times the relative increase (decrease) in the risk-adjustment in the market discount rate ($y'/(1+y)$).

When is d/dQ of E(NPV) positive? Since $1+y$ and M are both greater than zero for any conceivable insurer, this derivative has the same sign as:

$$Z'\Psi(1+y) + Z\Psi'(1+y) - Z\Psi y'. \tag{3.7}$$

$d/dQ\,$E(NPV) will be positive whenever

$$Z\Psi'(1+y) > \Psi Z y' - \Psi Z'(1+y). \tag{3.8}$$

Or since $Z > 0$,

$$\Psi' > \frac{\Psi Z y' - \Psi Z'(1+y)}{Z(1+y)} \tag{3.9}$$

$$= \Psi\left(\frac{y'}{1+y} - \frac{Z'}{Z}\right). \tag{3.10}$$

But Ψ' is just the increase in the expected value of the insurer (assuming it survives) caused by assuming some of the proposed contract.

Define $\hat{\mu} = \mathrm{E}(\ell \mid S_1 \geq G_1)$, the limited expectation of the losses on the proposed contract, given that our insurer is not impaired. Many insurers implicitly calculate something like $\hat{\mu}$ by modeling contract losses only under certain not-too-extreme scenarios.[9]

[9]For example, event-modeling software analyzes the probabilities of the 1906 San Francisco earthquake and the 1938 New England Hurricane, but not of the 1906 earthquake recurring in Boston! For some familiar loss distributions, such as the Pareto, μ may not be finite, so a limited mean is essential for any pricing analysis.

This definition allows us to find Ψ':

$$\Psi(Q) = \mathrm{E}(S_1 \mid S_1 \geq G_1) = \mathrm{E}(S_0 + P - L + Qp - Q\ell \mid S_1 \geq G_1) \tag{3.11}$$

$$= \mathrm{E}(S_0 + P - L \mid S_1 \geq G_1) + Q\mathrm{E}(p - \ell \mid S_1 \geq G_1), \qquad \text{or} \tag{3.12}$$

$$= \Psi(0) + Q(p - \hat{\mu}), \qquad \text{and thus} \tag{3.13}$$

$\Psi' = p - \hat{\mu}$. Substituting, we find that d/dQ of E(NPV) is positive when

$$\Psi' = p - \hat{\mu} > \Psi(y'/(1+y) - \mathrm{Z}'/\mathrm{Z}), \qquad \text{or,} \tag{3.14}$$

$$p > \hat{\mu} + \Psi(y'/(1+y) - \mathrm{Z}'/\mathrm{Z}). \tag{3.15}$$

If p is greater than the right-hand side then our insurer would accept more of the proposed contract, expecting to increase its own present value. This offers a minimum premium for the proposed contract without the concept of marginal surplus. The calculation also allows the probability of ruin to vary. The minimum premium is equal to the sum of:

- the losses that our insurer expects from the proposed contract, ignoring here any loss scenario that would impair it, plus,
- its expected amount of surplus (at $t = 1$) multiplied by
  - the relative increase in the discount on that future surplus caused by adding the risk of the additional contract, and
  - the relative decrease in the probability of surviving as a going concern, reflecting here those extreme loss scenarios not considered above in $\hat{\mu}$.

Like Kreps' result, this minimum premium consists of the sum of expected losses and a "reluctance" term that is positively related to the insurer's surplus and the variability of the proposed contract. $\Psi y'$ is analogous to Kreps' $z\sigma$; however, there is an additional component here ($\Psi Z'/Z$) reflecting the reduction in the probability of survival. While $\Psi Z'/Z$ is denominated in terms of the insurer's surplus, it is not Kreps' marginal surplus. Kreps allocates an amount of surplus to a contract, and while it is not expected to be lost, a marginal return is required because the surplus cannot be allocated to other uses. $\Psi Z'/Z$ is the expected surplus that will be lost by taking on the risk of the proposed contract. The minimum premium includes a charge for this expected loss of capital, not a return on it. The difference here is that the charge reflects principal lost to default ($\Psi Z'/Z$), versus only interest on principal outstanding (Kreps' $(y/(1+y))z\sigma$).

Under the alternative model, the reluctance term and the expected losses term are distinct and not in general dependent upon each other, as Kreps has also found. We will next examine how (re)insurers consider the risk of a proposed contract under this more general model of incentives. Then we will see separately how the marginal risk changes the probability of survival and the market discount rate. Combining these results produces a usable minimum premium under the more general model. The result has a strong symmetry with Kreps' simpler formula.

4. WHAT IS THE MARGINAL RISK OF A PROPOSED CONTRACT?

Think of the small as large and the few as many.
Confront the difficult while it is still easy;
accomplish the great task by a series of small acts.[10]

For simplicity, we have assumed that insurers are only concerned with one risk factor. (For national U.S. catastrophe reinsurance accounts that is a fair approximation.) Let's denote this one risk factor by R, and assume that it is a real-valued random variable that fully describes all of the common elements of risk in the insurer's portfolio. For simplicity here, let's also re-scale R to be positively correlated with L and have a standard devia-

[10] The *Tao*, [11, Chapter 63].

tion of one.[11] Further assume that the cumulative density function (c.d.f.) of R is continuous over its range, except perhaps at a finite number of points.

For the loss processes of the existing portfolio (L) and the proposed contract (ℓ), define L_0 and ℓ_0, the unsystematic parts of the loss processes, the parts that can't be explained by R.

$$L_0 = L - \frac{C}{\mathrm{Var}(R)}R, \qquad \text{and} \tag{4.1}$$

$$\ell_0 = \ell - \frac{c}{\mathrm{Var}(R)}R, \tag{4.2}$$

where $C = \mathrm{Cov}(L,R)$ and $c = \mathrm{Cov}(\ell,R)$.

Since we have assumed a single risk factor, $\mathrm{Cov}(L_0,\ell_0) = 0$. (If not, there is another external factor that affects at least two contracts and that must be quantified. We've restricted ourselves to a one-factor model for now.) It's also easy to show that $\mathrm{Cov}(L_0,R) = \mathrm{Cov}(\ell_0,R) = 0$. Finally, observe that L_0 is the sum of the many unsystematic risk elements of the contracts in the insurer's current portfolio: L_0 and $L_0 + Q\ell_0$ are normally distributed. While $L + Q\ell$ will not necessarily be normal, its c.d.f. will be continuous and differentiable for changes in either the mean or standard deviation of the loss process.

Our insurer has defined R to decompose the loss processes into a quantified external risk factor and the unsystematic part of its risks. It understands $\mathrm{Var}(L)$ in terms of the variances and covariances of L_0 and R.

$$\mathrm{Var}(L) = \mathrm{Var}(L_0) + [C/\mathrm{Var}(R)]^2\mathrm{Var}(R). \tag{4.3}$$

[11]For property catastrophe reinsurance coverage, this R might mean something like "aggregate insured property losses in the U.S. during the next year, in excess of \$2 billion per event, divided by \$5 billion." If the standard deviation of the annual excess losses is \$5 billion, as roughly true in the last decade, this variable has a standard deviation of one as required.

And similarly for $\mathrm{Var}(\ell)$:

$$\mathrm{Var}(\ell) = \mathrm{Var}(\ell_0) + [c/\mathrm{Var}(R)]^2 \mathrm{Var}(R). \tag{4.4}$$

We can use these expressions to find the new portfolio variance, if our insurer assumes Q of the proposed contract.

$$\mathrm{Var}(L + Q\ell) = \mathrm{Var}(L) + Q^2 \mathrm{Var}(\ell) + 2Q\,\mathrm{Cov}(L,\ell). \tag{4.5}$$

Since L_0 and ℓ_0 are uncorrelated with each other and R, we find

$$\mathrm{Cov}(L,\ell) = \mathrm{Cov}\left(L_0 + \frac{CR}{\mathrm{Var}(R)}, l_0 + \frac{cR}{\mathrm{Var}(R)}\right) \tag{4.6}$$

$$= \mathrm{Cov}\left(\frac{CR}{\mathrm{Var}(R)}, \frac{cR}{\mathrm{Var}(R)}\right) \tag{4.7}$$

$$= \frac{Cc}{\mathrm{Var}(R)^2}\mathrm{Cov}(R,R) = \frac{Cc}{\mathrm{Var}(R)^2}\mathrm{Var}(R) \tag{4.8}$$

and can substitute to show that

$$\begin{aligned}\mathrm{Var}(L + Q\ell) = {} & \mathrm{Var}(L_0) + [C/\mathrm{Var}(R)]^2 \mathrm{Var}(R) + Q^2 \mathrm{Var}(\ell_0) \\ & + Q^2 [c^2/\mathrm{Var}(R)]^2 \mathrm{Var}(R) \\ & + [2QCc/\mathrm{Var}(R)^2]\mathrm{Var}(R).\end{aligned} \tag{4.9}$$

Recall that we re-scaled R to make $\mathrm{Var}(R) = \mathrm{SD}(R)^2 = 1^2 = 1$, so

$$\mathrm{Var}(L + Q\ell) = \mathrm{Var}(L_0) + C^2 + Q^2 \mathrm{Var}(\ell_0) + Q^2 c^2 + 2QCc. \tag{4.10}$$

We can differentiate with respect to Q and find the marginal risk, which is the rate of increase in our insurer's portfolio variance with respect to changes in Q,

$$d/dQ\,\mathrm{Var}(L + Q\ell) = 2Q\,\mathrm{Var}(\ell_0) + 2Qc^2 + 2Cc. \tag{4.11}$$

5. MARGINAL RISK AND THE PROBABILITY OF SURVIVAL

If you realize that all things change,
There is nothing you will try to hold onto.
If you aren't afraid of dying,
There is nothing you can't achieve.[12]

One of the terms of the minimum premium is $\Psi Z'/Z$. To calculate this term, we need to see how Z, the probability of survival, changes with the marginal risk from assuming more of the proposed contract. Since G_1, P, and S_0 don't vary with Q, we can also view Z as Z^*, a function of only the mean (Ψ) and standard deviation (Λ) of the surplus amount. Within our assumptions both parameters depend on a single variable, Q,

$$Z(P,L,Q,p,\ell,G_1,S_0) = Z^*(\Psi(Q\ell,Qp),\Lambda(Q\ell)). \tag{5.1}$$

We find

$$\frac{dZ^*}{dQ} = \left(\frac{\partial Z^*}{\partial \Psi}(p-\hat{\mu}) + \frac{\partial Z^*}{\partial \Lambda}\frac{d\Lambda}{dQ}\right). \tag{5.2}$$

Clearly, greater resources always improve the probability of survival, so $\partial Z^*/\partial\Psi$ is positive; and increasing levels of variability can increase the chance of ruin, so $\partial Z^*/\partial\Lambda$ is negative, at least for the range of Z values that concern us, fairly solid companies.[13]

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[12] The *Tao*, [11, Chapter 74].

[13] An unstable company may actually increase its survival probability by adding variance. The non-linear valuation caused by a floor of zero is equivalent to the shareholders owning an out-of-the-money put option. Any option increases in value as the variability increases. See Brealey and Myers [3, p. 498]. A more familiar illustration may be the "Hail Mary" passes by losing football teams during the last minutes of a game. While the marginal expected value of these plays (in yards) is very small, the increased variability significantly increases the teams' small probabilities of victory. It is equally important to note that winning teams don't throw "Hail Marys." They often "run out the clock," sacrificing all marginal gain to eliminate variance. As with our sound insurers, they believe that their $\partial Z/\partial\Lambda$ vastly outweighs $\partial Z/\partial\Psi$.

The standard deviation component depends on Q more strongly than the mean term does for two reasons: (1) $p - \hat{\mu}$ is small, while $d\Lambda/dQ$ is at least $2Cc$; and (2) for these solid companies, there's just more room for Z to go down than up. So by analyzing the Λ term the insurer sees whether Z'/Z (and thus the minimum premium) rises or falls as Q is increased above zero.

Evaluating the derivative of the variance (in Equation 4.11) at $Q = 0$,

$$d/dQ\,\mathrm{Var}(L + Q\ell) = 2Cc, \qquad (5.3)$$

which is positive except in the not-often-found-in-nature case where the signs of C and c differ, a contract that could serve as a hedge against the existing book. Barring this curiosity, this contribution of Q to the portfolio variance is always positive. The sign of the derivative of the standard deviation is the same as the sign of the derivative of the variance.[14] So this result is also true for the marginal standard deviation of the combined loss process with respect to changes in Q: $d\Lambda/dQ$ is positive at $Q = 0$.

The non-systematic part of the risk of a proposed contract does not initially contribute to the marginal variance at all. But when Q is greater than zero, two additional positive terms ($Q\,\mathrm{Var}(\ell_0) + Qc^2$) add to the marginal variance. With a little more arithmetic, it is easy to show that the second derivative of the standard deviation with respect to Q is always positive. The marginal standard deviation is at its minimum at $Q = 0$ and increases monotonically and rapidly thereafter. As Q grows, the marginal standard deviation grows, and Z', the change in the probability of survival with respect to Q becomes more negative quickly. The ratio Z'/Z is monotonically decreasing,[15] at least for these solid companies.

[14] $\Lambda = \mathrm{Var}^{1/2}$. Differentiating with respect to Q, $\Lambda' = \frac{1}{2}(1/\Lambda)\mathrm{Var}'$. $\frac{1}{2}(1/\Lambda)$ is positive. The signs of Λ' and Var' are the same.

[15] If the loss processes are normal there is a direct proof of this conclusion. We have assumed that the probability of survival is quite high, so that the standard deviation has much more influence on changes in the survival probability than does the mean. We are

When we examine y', the change in the market discount rate caused by adding marginal risk, we will find that it cannot change the value of Q at which the minimum premium is lowest. The minimum premium at $Q = 0$ is its lowest value.

6. MARGINAL RISK AND THE MARKET DISCOUNT RATE

Money or happiness: which is more valuable?
Success or failure: which is more destructive?[16]

Under our frictionless, one-period, one-factor assumptions, the Capital Asset Pricing Model dictates[17] the market discount rate applied to the future earnings of our insurer,

$$y = r_f + \beta\Pi, \qquad (6.1)$$

where r_f is the risk-free interest rate in effect between $t = 0$ and $t = 1$, β is the systematic risk of our insurer, and Π is the market risk premium.

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interested in the derivative of the ratio Z'/Z with respect to changes in Q. $Z' = d/dQ\,Z$ is a function of the mean and Λ. However, the influence of the mean is approximately zero, so we can treat Z as a function only of Λ. Now we apply the chain rule

$$d/dQ(Z'(Q)/Z) = d/d\Lambda(Z'(\Lambda)/Z(\Lambda))\,d\Lambda/dQ,$$

with $Z'(\Lambda)$ evaluated at $\Lambda = \Lambda(Q)$. We have seen that Λ is an increasing function of Q, so $d\Lambda/dQ > 0$, and the sign of the derivative of Z'/Z with respect to Q is the same as the sign of the derivative with respect to Λ.

The derivative of a quotient has a positive denominator so $d/d\Lambda\,(Z'/Z)$ will be negative wherever $Z''Z - (Z')^2$ is. It suffices to show that $Z'' < (Z')^2$ because Z is no more than one. By differentiating the c.d.f. of the normal with respect to Λ twice, squaring the first derivative, and expanding both in powers of Λ we can compare Z'' and $(Z')^2$. When $S - G_1 + P$ is sufficiently large compared to $E(L)$ (greater than the mean = median more than suffices, i.e., a survival probability of at least 50%), we can compare and conclude that Z'' is always less than $(Z')^2$. So Z'/Z is a monotonically decreasing function of Λ, and thus also of Q.

Charles A. Thayer helped develop this proof and other derivations in this review.

[16]The *Tao*, [11, Chapter 44].

[17]These assumptions from Kreps, taken with familiar and reasonable assumptions about rationality, risk-free borrowing, and available information, meet the requirements of the CAPM. We conclude that it will apply here.

To understand the β of our insurer, we need to examine its market-based rate of return between $t = 0$ and $t = 1$

$$r(i) = \frac{V_1}{V_0} - 1 \tag{6.2}$$

$$= \begin{cases} \dfrac{S_0 + P - L}{S_0} - 1, & \text{if} \quad S_1 \geq G_1 \\ 0/S_0 - 1, & \text{if} \quad S_1 < G_1 \end{cases} \tag{6.3}$$

$$= \begin{cases} (P - L)/S_0, & \text{if} \quad S_1 \geq G_1 \\ -1 & \text{if} \quad S_1 < G_1. \end{cases} \tag{6.4}$$

The β of our insurer is:

$$\beta = \frac{\mathrm{Cov}(r(i), r(m))}{\mathrm{Var}(r(m))}, \qquad \text{where} \quad r(m) \text{ is the average return in the capital market.} \tag{6.5}$$

Our insurer ignores the small distortion caused by the possibility of its own impairment, so we can find

$$\beta = \frac{\mathrm{Cov}((P - L)/S_0, r(m))}{\mathrm{Var}(r(m))} \tag{6.6}$$

$$= -1/S_0 \frac{\mathrm{Cov}(L, r(m))}{\mathrm{Var}(r(m))}. \tag{6.7}$$

Now we can add the facts that

$$L = L_0 + CR \tag{6.8}$$

and that L_0 by assumption is independent of $r(m)$ (or else $r(m)$ is a risk factor, which we have assumed it isn't). So,

$$\beta = -\frac{C}{S_0} \times \frac{\mathrm{Cov}(R, r(m))}{\mathrm{Var}(r(m))} \tag{6.9}$$

$$= -\frac{C}{S_0}\beta_R. \tag{6.10}$$

β_R is the beta, the systematic risk measure, of R, our one external risk factor. Substituting in Equation 6.1,

$$y = r_f - \frac{C}{S_0}\beta_R \Pi, \tag{6.11}$$

$$\mathrm{Cov}(R, L + Q\ell) = \mathrm{Cov}(R, L) + Q\,\mathrm{Cov}(R, \ell) = C + Qc, \tag{6.12}$$

so, similarly, when we include Q,

$$y(Q) = r_f - \frac{C + Qc}{S_0}\beta_R \Pi, \tag{6.13}$$

and observe that

$$\frac{dy}{dQ} = -\frac{c}{S_0}\beta_R \Pi; \tag{6.14}$$

that is, the derivative is independent of the level of Q. This discount rate contribution will be insignificant if, as some investment bankers suggest, β_R is zero or small. Unfortunately, for very high-level catastrophe reinsurance, the available history suggests that catastrophe risk is not zero-beta.[18]

The change in the discount rate caused by assuming a marginal amount of risk does not depend on the amount already assumed. As promised, the discount rate term cannot affect the point at which the minimum premium is lowest.

[18] Kozik [8] notes practical and theoretical difficulties in computing and applying the betas applicable to the underwriting operations of insurers. This analysis is especially valid for diversifiable, low-level coverages. However, high-level reinsurance contracts address a small set of rare physical events, and the potential systemic correlations are both larger and clearer. If sizable, these systematic correlations are very relevent to the owners of insurance companies.

Interested readers may want to consider the notable falls both in the equity and bond markets and in affected currency values in the periods following the 1906 San Francisco earthquake in the United States and the 1995 Kobe earthquake in Japan. These two observations suggest that β_R can be significantly negative. Large physical catastrophes are correlated with losses in the capital markets.

7. THE MINIMUM PREMIUM IN THE ALTERNATIVE MODEL

All streams flow to the sea
Because it is lower than they are.
Humility gives it its power.[19]

Combining the contributions from the changes in the probability of survival and in the discount rate, and adding the assumption that the capital market valuation of the insurer is rational, although perhaps inefficient, we can solve for the insurer's minimum premium in an accessible way.

The minimum premium for a contract to be attractive to our insurer (any insurer) is (from Equation 3.15)

$$p = \hat{\mu} + \Psi(y'/(1+y) - (\mathrm{Z}'/\mathrm{Z})).$$

Substituting from Equation 5.2 and solving for p,

$$p = \hat{\mu} + \Psi\frac{y'/(1+y) - (\partial \mathrm{Z}^*/\partial\Lambda)(d\Lambda/dQ)/\mathrm{Z}}{1 + (\partial \mathrm{Z}^*/\partial\Psi)(\Psi/\mathrm{Z})}. \qquad (7.1)$$

This can be expressed differently when we further assume that the market valuation of our insurer is consistent with expectations, and that M is stable. Consistent expectations require that:

$$MS_0(1+y) = M\mathrm{E}(S_1) \qquad (7.2)$$

and further assuming that M is stable between $t = 0$ and 1 produces:

$$S_0(1+y) = \mathrm{E}(S_1 \mid S_1 \geq G_1)\mathrm{Prob}(S_1 \geq G_1), \qquad (7.3)$$

or abbreviating and regrouping,

$$\Psi = S_0(1+y)/\mathrm{Z}. \qquad (7.4)$$

This yields

$$p = \hat{\mu} + \frac{S_0 y' - \Psi(\partial \mathrm{Z}^*/\partial\Lambda)(d\Lambda/dQ)}{\mathrm{Z} + \Psi(\partial \mathrm{Z}^*/\partial\Psi)}; \qquad (7.5)$$

[19] The *Tao*, [11, Chapter 16].

and we know that (from Equation 6.14)

$$y' = (-c/S_0)\beta_R\Pi, \qquad \text{so}$$

$$p = \hat{\mu} + \frac{-c\beta_R\Pi - \Psi(\partial Z^*/\partial\Lambda)(d\Lambda/dQ)}{Z + \Psi(\partial Z^*/\partial\Psi)}. \tag{7.6}$$

This result applies at any level of Q, but we have seen that the lowest minimum premium occurs for a marginal participation. For our ideal price-taking insurer, with an offered premium near p, the marginal increase in its NPV quickly falls as the share of a proposed contract rises above zero. Our assumed insurer, like many real ones, maximizes its value by assuming and retaining very small parts of every possible risk.

The problem of insurers seeking geographic diversification can be restated from the insureds' perspective, as its dual problem of insurance risks seeking maximum spread among the world's insurers. If worldwide capacity meets the demand then our hypothetical contract would be fully placed with these ideal marginal participations. Q is approximately zero and the marginal variance of the contract (see Equation 5.3) for our insurer becomes

$$d/dQ\,\text{Var}(L + Ql) = 2Cc, \tag{7.7}$$

and the marginal standard deviation is

$$d\Lambda/dQ = (\tfrac{1}{2})(1/\text{SD}(L))(2Cc) \tag{7.8}$$

$$= Cc/\text{SD}(L). \tag{7.9}$$

So the insurer's minimum premium becomes

$$p = \hat{\mu} + \frac{-c\beta_R\Pi - \Psi(\partial Z^*/\partial\Lambda)Cc/\text{SD}(L)}{Z + \Psi(\partial Z^*/\partial\Psi)}. \tag{7.10}$$

We have seen that β_R and $\partial Z^*/\partial\Lambda$ are both less than zero, so the latter term is generally a positive number. The premium is the limited expected losses plus a risk load. The risk load depends on the covariance of the proposed contract with the risk factor of concern to the insurer, the capital market valuation for that risk, the insurer's capital structure, and the partial derivatives of its survival probability with respect to changes in its expected profits and variability. Using the normal distribution as a strong practical approximation,[20] there are closed-form expressions for these partial derivatives.

Since $\mathrm{SD}(R) = 1$, we see that

$$c = \mathrm{Corr}(\ell, R)\sigma \qquad \text{and} \qquad C = \mathrm{Corr}(L, R)\Lambda,$$

[20] The essential nature of insurance is the transfer and pooling of risks. In practice, catastrophe reinsurers track and control their risk accumulations in between six and more than thirty distinct zones. See the *1996 Annual Report* of CAT Limited for a clear example of the high end. Reinsurers' results are driven by the sum of these independent random processes. Their results will be close to normally distributed. (These underwriters can also rely on the exact derivation of the conclusion about decreasing values of Z'/Z in note 15.) If we define $T = S_0 + P - G_1$, then $\mathrm{Z}^* = \mathrm{Prob}(L < T)$, where L is normally distributed with mean $W = \mathrm{E}(L)$ and standard deviation Λ

$$\mathrm{Z}^* = \frac{1}{\sqrt{2\pi}\Lambda}\int_{-\infty}^{T} \exp\left(-\frac{1}{2}\left(\frac{x-W}{\Lambda}\right)^2\right) dx. \tag{20.1}$$

Since Ψ is independent of T and Λ, we can find the $\partial\mathrm{Z}^*/\partial\Psi$ by bringing the differentiation within the integration

$$\frac{\partial\mathrm{Z}^*}{\partial\Psi} = \frac{1}{\sqrt{2\pi}\Lambda}\int_{-\infty}^{T} \frac{d}{d\Psi}\exp\left(-\frac{1}{2}\left(\frac{x-W}{\Lambda}\right)^2\right) dx. \tag{20.2}$$

Over the range of integration, the conditional expectation of surplus, Ψ, is exactly and inversely related to the expectation of losses, W:

$$d/d\Psi = -d/dW \tag{20.3}$$

$$\frac{\partial\mathrm{Z}^*}{\partial\Psi} = \frac{1}{\sqrt{2\pi}\Lambda}\int_{-\infty}^{T} -\frac{d}{dW}\exp\left(-\frac{1}{2}\left(\frac{x-W}{\Lambda}\right)^2\right) dx \tag{20.4}$$

$$= \frac{1}{\sqrt{2\pi}\Lambda}\int_{-\infty}^{T} \exp\left(-\frac{1}{2}\left(\frac{x-W}{\Lambda}\right)^2\right)\left(\frac{x-W}{\Lambda}\right) dx$$

$$= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{T-W}{\Lambda}\right)^2\right). \tag{20.5}$$

$\partial\mathrm{Z}^*/\partial\Psi$ also gives us $\partial\mathrm{Z}^*/\partial\Lambda$. Since Z^* is a function of both W and Λ, we can express the two derivatives using the chain rule and find a simple relationship between them and

and the denominator is approximately one for our solid companies; so

$$p = \hat{\mu} + -(\beta_R \Pi + \Psi(\partial Z^*/\partial\Lambda)\mathrm{Corr}(L,R))\mathrm{Corr}(\ell,R)\sigma. \quad (7.11)$$

The alternative minimum premium formula roughly matches the dimensions in Kreps' analysis. $\mathscr{R}$, the reluctance, is directly related to the proposed contract's correlation with the relevant part of the existing portfolio, which is the risk factor of the insurer.[21]

$Z^{*\prime}((T-W)/\Lambda)$

$$\partial Z^*/\partial\Psi = -\partial Z^*/\partial \mathbf{W} \quad (20.6)$$

$$= -Z^{*\prime} d/d\mathbf{W}((T-W)/\Lambda) \quad (20.7)$$

$$= -Z^{*\prime}(-1/\Lambda) \quad (20.8)$$

$$= Z^{*\prime}/\Lambda, \quad \text{and} \quad (20.9)$$

$$\partial Z^*/\partial\Lambda = Z^{*\prime} d/d\Lambda((T-W)/\Lambda) \quad (20.10)$$

$$= Z^{*\prime}(T-W) d/d\Lambda(1/\Lambda) \quad (20.11)$$

$$= -Z^{*\prime}(T-W)(1/\Lambda)^2 \quad (20.12)$$

$$= -(T-W)/\Lambda Z^{*\prime}/\Lambda, \quad \text{or} \quad (20.13)$$

$$\partial Z^*/\partial\Lambda = -(T-W)/\Lambda\, \partial Z^*/\partial\Psi. \quad (20.14)$$

To illustrate, if an insurer believes that $(T-W)/\Lambda = 3.0$ (not unrealistically, $Z = .9975$, a one-in-four-hundred years probability of ruin) and that the standard deviation of the loss process on its existing contracts is, say, \$100,000,000, then

$$\frac{\partial Z^*}{\partial\Psi} = \frac{1}{\sqrt{2\pi}} \frac{1}{\$100{,}000{,}000 e^{4.5}} = +4.432 \times 10^{-11}/(\$ \text{ of mean}), \quad (20.15)$$

and

$$\frac{\partial Z^*}{\partial\Lambda} = \frac{1}{\sqrt{2\pi}} \frac{1}{\$100{,}000{,}000 e^{4.5}}(-3) = -1.130 \times 10^{-10}/(\$ \text{ of standard deviation}). \quad (20.16)$$

Results like these could be used in Equations 7.10 or 7.11 to solve for minimum premiums in a direct way.

[21] Bault [2] analyzes several common risk load approaches with different assumptions and concludes that each can be re-expressed as a covariance measure between the proposed

Of course, in the real world, regulation, occasional capacity shortages, frictional costs, information barriers, and some economies of scale will prevent perfect diversification. Higher prices result. However, this analysis provides a fair estimate of the market premium, establishes a lower bound, and shows a scale by which the costs of market inefficiencies can be seen.

8. NEXT STEPS

As it acts in the world, the Way is like the bending of a bow.
The top is bent downward; and the bottom is bent up.
It adjusts excess and deficiency so that there is perfect balance.
It takes from what is too much and gives to what isn't enough.[22]

This review has mimicked the development of Modern Portfolio Theory (MPT) and found a result like one of MPT's fundamental tenets. A diversified, rational, risk-averse insurer, like a similar investor, will accept a potential addition to its portfolio only after a comparison between the addition's systematic, non-diversifiable risk and its price in the market.

MPT goes on to show that since most investors price assets that way, the market pricing of assets must be based on only the value of their systematic risks. These investors get the best return for the least total risk (at market pricing) by distributing their portfolios in proportion to the asset distribution of the total market. Insurers find a similar optimal return for their risk: either writing a balanced worldwide spread or placing their riskier coverages with reinsurers who do.

Based on Kreps' and other recent results and the generalization added in this review, actuaries should be able to raise our knowledge of market risk pricing up to the level that MPT

contract and the insurer's surplus. This review attempts to show that this conclusion holds with realistic assumptions about insurers' incentives and the insurance and capital markets.

[22]The *Tao*, [11, Chapter 77]. For clarity, "Way" replaces "Tao" in Mitchell's translation.

has reached for asset pricing.[23] If reinsurers choose to diversify their exposures, as these results suggest they must and recent acquisitions[24] suggest they do, market pricing will be based only on each proposed contract's systematic risks. Any insurer whose capital structure or distribution of net risks varies significantly from the industry average will find that its minimum premiums will be higher than the market clearing prices in its areas of relative over-concentration. It will reduce its exposures (by reinsurance, securitization or direct volume reduction) until its minimum premiums fall to the market level. Since much of the industry follows this process, market prices will only be denominated by contracts' covariances with the one (actually more) risk factor(s) affecting the global insurance market.

In effect, every insurer trying to maximize its risk-adjusted NPV must act as if it desires a spread of net risks like the worldwide industry average. Again, regulation, returns to scale, and frictional and information costs prevent this in practice.

The minimum market premium for a proposed contract does not depend on its covariance with a particular group of an insurer's other contracts. It depends more on the covariances with the significant risks that influence the results of all possible contracts worldwide. No proposed contract is considered first. Or last. This final result eliminates the circularity in Kreps' analysis and mine.

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[23] See Feldblum [4] for a very rigorous attempt at this analysis. However, this result only developed relative operating profit provisions and cannot develop specific targets without assumptions about allocated capitalization and cannot be reconciled to MPT results for equities because they consider different universes. See also Turner's article in Cummins and Harrington (Note 8, above), for an equilibrium analysis, but without product distinctions. A more general model, like Kreps' concept of marginal surplus requirements or the approach suggested in this review, can support a risk-specific price that is in equilibrium with the capital market valuations of the insurer.

[24] Numerous recent transactions, but note two common themes: property companies acquiring books on other continents (Cologne Re, Sphere Drake, SAFR, M&G, American Re, SOREMA-UK) to "balance all the buckets," and liability companies acquiring property operations (Tempest, GCR, IRI, Mid-Ocean, CAT Limited). Both can be explained by the search for a broader mix of the world's exposures.

9. CONCLUSIONS

1. Kreps' analysis is a significant addition to both the practice and theoretical understanding of reinsurance and insurance pricing in that
   - the risk load required for a contract to be attractive (or indifferent) to an insurer must be based on the risk of the proposed contract and not on its expected losses, and
   - the risk load an insurer must require depends on the covariance of the proposed contract with the insurer's existing risks (that is, with the product of the correlation between the proposed contract and others, and the standard deviation of the contract).
2. These results still hold with a more realistic model of insurers' incentives.
3. The required risk load also reflects the relative correlations between insurance risk factors and movements in the overall capital markets. This is true even though there are frictional barriers to moving capital through insurers.
4. The minimum amount of this required risk load occurs for marginal participations. Insurers thus have strong incentives to diversify. Since most do, market prices are based on the covariances of the proposed contract with the general risk factors exposing all other possible contracts, that is, the entire insured market. Risk loads should not reflect any diversifiable risks.

Knowing others is intelligence;
knowing yourself is true wisdom.
Mastering others is strength;
mastering yourself is true power.

If you realize that you have enough,
you are truly rich.[25]

[25]The *Tao*, [11, Chapter 33].

SUMMARY OF NOTATION

In Kreps' Original

μ, expected losses for proposed contract

$\mathscr{R}$, insurer's reluctance to assume contract

σ, standard deviation of losses for proposed contract

E, insurer's marginal expense

y, risk-adjusted discount rate

B, reinsured's "bank," amount reinsurer is willing to concede (proposing to extract) in renewal price.

Z, insurer's ratio of surplus to standard deviation of existing loss portfolio

Added in Discussion

V_i, market value of the insurer at time $t = i$

S_i, surplus, GAAP book value, at time $t = i$

G_1, minimum going-concern surplus level at $t = 1$

M, insurer's book-value multiple

P, premium for existing portfolio

L, random loss process for existing portfolio

p, premium for proposed contract

ℓ, random loss process for proposed contract

Q, decision variable, portion of the proposed contract assumed

μ, expected losses for the proposed contract, $E(l)$

$\hat{\mu}$, expected losses for the proposed contract limited to the scenarios where $S_1 \geq G_1$

R, a measure of the one external risk factor of concern to insurer, re-scaled here to be positively correlated with L and with standard deviation = 1

r_f, risk-free interest rate between $t = 0$ and $t = 1$

β, systematic risk measure of insurer's return

Π, market risk premium expected between $t = 0$ and $t = 1$

$r(i)$, insurer's market-based return between $t = 0$ and $t = 1$

$r(m)$, return on overall capital market between $t = 0$ and $t = 1$

β_R, systematic risk measure of R, the external risk factor

Abbreviations

NPV, risk-adjusted net present value at $t = 0$ of insurer's market value at $t = 1$

Z, probability that $S_1 \geq G_1$, that the insurer survives

$$\mathrm{Z} = f(P, L, Q, p, \ell, G_1, S_0) \qquad \text{and}$$

$$\mathrm{Z}^* = f(\Psi(Q\ell, Qp), \Lambda(Q\ell))$$

$\Psi = \mathrm{E}(S_1 \mid S_1 \geq G_1)$

$C = \mathrm{Cov}(L, R)$

$c = \mathrm{Cov}(\ell, R)$

$L_0 = L - (C/\mathrm{Var}(R))R = L - CR$

$\ell_0 = \ell - (c/\mathrm{Var}(R))R = \ell - cR$

$\Lambda = \mathrm{SD}(S_1) = \mathrm{SD}(L + Q\ell)$

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