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Reserving Using Paid and Incurred Data

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Why Use Paid and Incurred Data?

- Paid data is “real”
- BUT
- Paid claims data may be unstable
 - Incurred data may include some useful information

“Munich chain-ladder”

Use paid claims and reported numbers



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Papers

Huijuan Liu and Richard Verrall: Bootstrapping MCL, to appear in Variance

Richard Verrall, Jens Perch Nielsen, Anders Jessen, Maria Martinez-Miranda: 2 papers in ASTIN on a new model using reported numbers and paid amounts



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Notation

We use the notation of Quarg and Mack, so that cumulative claims for accident year i and development year j are denoted by C_{ij}^P and C_{ij}^I .

$$F_{ij}^P = \frac{C_{i,j+1}^P}{C_{ij}^P}$$

$$F_{ij}^I = \frac{C_{i,j+1}^I}{C_{ij}^I}$$

$$Q_{ij} = \frac{C_{ij}^P}{C_{ij}^I}$$



Assumptions

$$E[F_{ij}^P | P_i(j)] = f_j^P \quad \text{Var}[F_{ij}^P | P_i(j)] = \frac{(\sigma_j^P)^2}{C_{ij}^P}$$

$$E[F_{ij}^I | I_i(j)] = f_j^I \quad \text{Var}[F_{ij}^I | I_i(j)] = \frac{(\sigma_j^I)^2}{C_{ij}^I}$$

$$E[Q_{ij}^{-1} | P_i(j)] = q_j^{-1} \quad \text{Var}[Q_{ij}^{-1} | P_i(j)] = \frac{(\tau_j^P)^2}{C_{ij}^P}$$

$$E[Q_{ij} | I_i(j)] = q_j \quad \text{Var}[Q_{ij} | I_i(j)] = \frac{(\tau_j^I)^2}{C_{ij}^I}$$



Residuals

$$r_{ij}^P = \frac{F_{ij}^P - E[F_{ij}^P | P_i(j)]}{\sqrt{\text{Var}[F_{ij}^P | P_i(j)]}} \quad r_{ij}^I = \frac{F_{ij}^I - E[F_{ij}^I | I_i(j)]}{\sqrt{\text{Var}[F_{ij}^I | I_i(j)]}}$$

$$r_{ij}^{Q^{-1}} = \frac{Q_{ij}^{-1} - E[Q_{ij}^{-1} | P_i(j)]}{\sqrt{\text{Var}[Q_{ij}^{-1} | P_i(j)]}} \quad r_{ij}^Q = \frac{Q_{ij} - E[Q_{ij} | I_i(j)]}{\sqrt{\text{Var}[Q_{ij} | I_i(j)]}}$$



Correlations

$$E[r_{ij}^P | B_i(j)] = \rho^P r_{ij}^{Q^{-1}}$$

$$\text{Cov}[r_{ij}^P, r_{ij}^{Q^{-1}} | P_i(j)] = \text{Corr}[r_{ij}^P, r_{ij}^{Q^{-1}} | P_i(j)] = \text{Corr}[F_{ij}^P, Q_{ij}^{-1} | P_i(j)] = \rho^P$$

$$E[r_{ij}^I | B_i(j)] = \rho^I r_{ij}^Q$$

$$\text{Cov}[r_{ij}^I, r_{ij}^Q | B_i(j)] = \text{Corr}[r_{ij}^I, r_{ij}^Q | B_i(j)] = \text{Corr}[F_{ij}^I, Q_{ij} | B_i(j)] = \rho^I$$



MCL Adjusted development factors

$$E[F_{ij}^P | B_i(j)] = E[F_{ij}^P | P_i(j)] + \sqrt{\frac{\text{Var}[F_{ij}^P | P_i(j)]}{\text{Var}[Q_{ij}^{-1} | P_i(j)]}} \text{Corr}[F_{ij}^P, Q_{ij}^{-1} | B_i(j)] (Q_{ij}^{-1} - E[Q_{ij}^{-1} | P_i(j)])$$

$$\hat{\lambda}_{ij}^P = \hat{f}_j^P + \hat{\rho}^P \frac{\hat{\sigma}_j^P}{\hat{\tau}_j^P} (Q_{ij}^{-1} - \hat{q}_j^{-1})$$

$$E[F_{ij}^I | B_i(j)] = E[F_{ij}^I | I_i(j)] + \sqrt{\frac{\text{Var}[F_{ij}^I | I_i(j)]}{\text{Var}[Q_{ij} | I_i(j)]}} \text{Cov}[F_{ij}^I, Q_{ij} | B_i(j)] (Q_{ij} - E[Q_{ij} | I_i(j)])$$

$$\hat{\lambda}_{ij}^I = \hat{f}_j^I + \hat{\rho}^I \frac{\hat{\sigma}_j^I}{\hat{\tau}_j^I} (Q_{ij} - \hat{q}_j)$$



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Reserving and Bootstrapping

Define and fit statistical model

Obtain residuals and pseudo data
Re-fit statistical model to pseudo data

Obtain forecast, including process error

Any model that can be clearly defined can be bootstrapped

See England and Verrall: "Predictive Distributions of Outstanding Liabilities in General Insurance," *Annals of Actuarial Science* 1, 2007



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Residuals

Residuals are $U_{ij} = \left\{ (r_{ij}^P), (r_{ij}^{Q^{-1}}), (r_{ij}^I), (r_{ij}^Q) \right\}$

$$r_{ij}^P = \frac{F_{ij}^P - \hat{f}_j^P}{\hat{\sigma}_j^P} \sqrt{C_{ij}^P}$$

$$r_{ij}^{Q^{-1}} = \frac{Q_{ij}^{-1} - \hat{q}_j^{-1}}{\hat{\tau}_j^P} \sqrt{C_{ij}^P}$$

$$r_{ij}^I = \frac{F_{ij}^I - \hat{f}_j^I}{\hat{\sigma}_j^I} \sqrt{C_{ij}^I}$$

$$r_{ij}^Q = \frac{Q_{ij} - \hat{q}_j}{\hat{\tau}_j^I} \sqrt{C_{ij}^I}$$

Bootstrap bias correction $\sqrt{\frac{(n-j)}{(n-j-1)}}$



Bootstrap procedure

Resample grouped residuals, with replacement

$$U_{ij}^B = \left\{ \left(r_{ij}^P \right)^B, \left(r_{ij}^{Q^{-1}} \right)^B, \left(r_{ij}^I \right)^B, \left(r_{ij}^Q \right)^B \right\}$$

Construct bootstrap samples

$$\left(F_{ij}^P \right)^B = \frac{\left(r_{ij}^P \right)^B \hat{\sigma}_j^P}{\sqrt{C_{i,j}^P}} + \hat{f}_j^P \quad \left(Q_{ij}^{-1} \right)^B = \frac{\left(r_{ij}^{Q^{-1}} \right)^B \hat{\tau}_j^P}{\sqrt{C_{i,j}^P}} + \hat{q}_j^{-1}$$

$$\left(F_{ij}^I \right)^B = \frac{\left(r_{ij}^I \right)^B \hat{\sigma}_j^I}{\sqrt{C_{i,j}^I}} + \hat{f}_j^I \quad \left(Q_{ij} \right)^B = \frac{\left(r_{ij}^Q \right)^B \hat{\tau}_j^I}{\sqrt{C_{i,j}^I}} + \hat{q}_j$$



Bootstrap estimates

Calculate bootstrap parameter estimates (using MCL), and

$$\left(\hat{\lambda}_{ij}^P \right)^B = \left(\hat{f}_j^P \right)^B + \left(\hat{\rho}^P \right)^B \frac{\left(\hat{\sigma}_j^P \right)^B}{\left(\hat{\tau}_j^P \right)^B} \left(\left(Q_{ij}^{-1} \right)^B - \left(\hat{q}_j^{-1} \right)^B \right)$$

$$\left(\hat{\lambda}_{ij}^I \right)^B = \left(\hat{f}_j^I \right)^B + \left(\hat{\rho}^I \right)^B \frac{\left(\hat{\sigma}_j^I \right)^B}{\left(\hat{\tau}_j^I \right)^B} \left(\left(Q_{ij} \right)^B - \left(\hat{q}_j \right)^B \right)$$



Predictive distributions

Bootstrapping recursive models, England and Verrall (2007)

$$X_{kl}^P \sim \text{Normal} \left(\left(\hat{\lambda}_{i,l-1}^P \right)^B \hat{X}_{k,l-1}^P, \left(\left(\hat{\sigma}_{l-1}^P \right)^2 \right)^B \hat{X}_{k,l-1}^P \right)$$

$$X_{kl}^I \sim \text{Normal} \left(\left(\hat{\lambda}_{i,l-1}^I \right)^B \hat{C}_{k,l-1}^I, \left(\left(\hat{\sigma}_{l-1}^I \right)^2 \right)^B \hat{X}_{k,l-1}^I \right)$$



Example: Quarg and Mack data

	<i>j</i> = 1	<i>j</i> = 2	<i>j</i> = 3	<i>j</i> = 4	<i>j</i> = 5	<i>j</i> = 6	<i>j</i> = 7		<i>j</i> = 1	<i>j</i> = 2	<i>j</i> = 3	<i>j</i> = 4	<i>j</i> = 5	<i>j</i> = 6	<i>j</i> = 7
<i>i</i> = 1	576	1804	1970	2024	2074	2102	2131	<i>i</i> = 1	978	2104	2134	2144	2174	2182	2174
<i>i</i> = 2	866	1948	2162	2232	2284	2348		<i>i</i> = 2	1844	2552	2466	2480	2508	2454	
<i>i</i> = 3	1412	3758	4252	4416	4494			<i>i</i> = 3	2904	4354	4698	4600	4644		
<i>i</i> = 4	2286	5292	5724	5850				<i>i</i> = 4	3502	5958	6070	6142			
<i>i</i> = 5	1868	3778	4648					<i>i</i> = 5	2812	4882	4852				
<i>i</i> = 6	1442	4010						<i>i</i> = 6	2642	4406					
<i>i</i> = 7	2044							<i>i</i> = 7	5022						

	Bootstrap		MCL		Mack	
	Paid	Incurred	Paid	Incurred	Paid	Incurred
<i>i</i> = 1	0	43	0	43	0	43
<i>i</i> = 2	37	94	35	96	32	97
<i>i</i> = 3	109	131	103	135	158	88
<i>i</i> = 4	277	321	269	326	331	277
<i>i</i> = 5	299	296	289	302	407	191
<i>i</i> = 6	657	651	646	655	919	465
<i>i</i> = 7	5492	5646	5505	5606	4063	6380
Overall Total	6871	7182	6846	7163	5911	7540



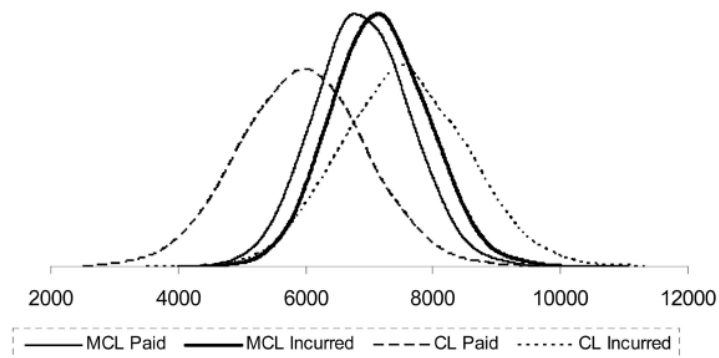
Example: Prediction errors

	MCL		Mack	
	Paid	Incurred	Paid	Incurred
$i = 1$	0	0	0	0
$i = 2$	5	5	15	9
$i = 3$	48	70	53	82
$i = 4$	61	86	68	105
$i = 5$	72	104	72	117
$i = 6$	215	208	289	216
$i = 7$	735	716	897	869
Overall Total	776	782	991	980

	MCL		Mack	
	Paid	Incurred	Paid	Incurred
$i = 1$	—	0%	—	0%
$i = 2$	14%	5%	45%	9%
$i = 3$	44%	53%	33%	93%
$i = 4$	22%	27%	21%	38%
$i = 5$	24%	35%	18%	61%
$i = 6$	33%	32%	31%	46%
$i = 7$	13%	13%	22%	14%
Total	11%	11%	17%	13%



Example: Predictive distributions





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Modelling the claims process

Model payment numbers and amounts separately (if the data are available).

Should we use paid or incurred data?

Verrall, Nielsen and Jessen (to appear in ASTIN) assume we have reported numbers of claims and paid aggregate claims. We build a model using the methods set out in the literature by (for example):

Bühlmann, H., Schnieper, R. and Straub, E. (1980): Claims reserves in casualty insurance based on a probability model. Mitteilungen der Vereinigung Schweizerischer Versicherungsmathematiker.

Norberg, R., (1986): A contribution to modelling of IBNR claims. Scandinavian Actuarial Journal, 155-203.

Norberg, R., (1993): Prediction of outstanding liabilities in non-life insurance. Astin Bulletin, vol. 23, no. 1, 95-115.

Norberg, R., (1999): Prediction of outstanding claims: Model variations and extensions. Astin Bulletin, vol. 29, no. 1, 5-25.



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Modelling the claims process

We have reported numbers of claims N_{ij} . We build a payment delay model for the (latent) paid numbers of claims, by first considering

$N_{ijk}^{paid} | N_{ij}$ which we assume has a multinomial distribution. This is the payment delay.

The number of paid claims can be found by summing these:

$$N_{ij}^{paid} = N_{ij0}^{paid} + N_{i,j-1,1}^{paid} + \dots + N_{i,0,j}^{paid}$$

The IBNR delay is considered by modelling the incurred number of claims.

The RBNS delay is considered through the model for $N_{ijk}^{paid} | N_{ij}$.



Paid claims

Denoting paid claims by X_{ij} ,

$$X_{ij} = \sum_{k=1}^{N_{ij}^{paid}} Y_{ij}^{(k)}$$

$$E[X_{ij}] = E[N_{ij}^{paid} | \mathbb{N}^I] E[Y_{ij}^{(k)}]$$

$$V[X_{ij}] = E[N_{ij}^{paid} | \mathbb{N}^I] V[Y_{ij}^{(k)}] + V[N_{ij}^{paid} | \mathbb{N}^I] (E[Y_{ij}^{(k)}])^2$$

Assuming claims are iid

$$E[X_{ij}] = E[N_{ij}^{paid} | \mathbb{N}^I] \mu$$

$$V[X_{ij}] = E[N_{ij}^{paid} | \mathbb{N}^I] \sigma^2 + V[N_{ij}^{paid} | \mathbb{N}^I] \mu^2$$



Paid claims

$$E[N_{ij}^{paid} | \mathbb{N}^I] = E\left[\sum_{k=0}^{\min\{j,d\}} N_{i,j-k,k}^{paid} | \mathbb{N}^I\right] = \sum_{k=0}^{\min\{j,d\}} E[N_{i,j-k,k}^{paid} | \mathbb{N}^I] = \sum_{k=0}^{\min\{j,d\}} N_{i,j-k} p_k$$

$$V[N_{ij}^{paid} | \mathbb{N}^I] = \sum_{k=0}^{\min\{j,d\}} N_{i,j-k} p_k (1 - p_k)$$

$$E[X_{ij}] = \sum_{k=0}^{\min\{j,d\}} N_{i,j-k} p_k \mu$$

$$V[X_{ij}] = \sum_{k=0}^{\min\{j,d\}} N_{i,j-k} p_k \sigma^2 + \sum_{k=0}^{\min\{j,d\}} N_{i,j-k} p_k (1 - p_k) \mu^2$$

$$= \sum_{k=0}^{\min\{j,d\}} N_{i,j-k} p_k (\sigma^2 + (1 - p_k) \mu^2) \approx \sum_{k=0}^{\min\{j,d\}} N_{i,j-k} p_k (\sigma^2 + \mu^2)$$



Model for Paid claims

Hence, with this approximation, we can use an ODP model for paid claims: similar to chain-ladder, except there is more going on in the parameters.

$$V[X_{ij}] \approx \sum_{k=0}^{\min\{j,d\}} N_{i,j-k} p_k (\sigma^2 + \mu^2) = \varphi E[X_{ij}]$$

$$\varphi = \frac{\sigma^2 + \mu^2}{\mu^2}$$



Applying the model

Apply a chain-ladder (Poisson or ODP) to the reported numbers of claims. The projected numbers of reported claims will allow us to analyse IBNR claims (separately).

Apply an ODP model to the triangle of paid aggregate claims. The mean is

$$E[X_{ij}] = \sum_{k=0}^{\min\{j,d\}} N_{i,j-k} \psi_k$$

$$p_k = \frac{\psi_k}{\sum_{k=0}^d \psi_k} \quad \mu = \sum_{k=0}^d \psi_k \quad \sigma^2 = \varphi \sum_{k=0}^d \psi_k - \left(\sum_{k=0}^d \psi_k \right)^2$$



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Delay functions

Development IBNR	
j Factor	Delay
0	0.8752
1 1.1353	0.1184
2 1.0038	0.0038
3 1.0009	0.0009
4 1.0003	0.0003
5 1.0003	0.0003
6 1.0002	0.0002
7 1.0001	0.0001
8 1.0003	0.0003
9 1.0004	0.0004

The average IBNR
delay is 0.14 years.
The average RBNS
delay is 1.52 years.

j	0	1	2	3	4	5	6	7
p	0.3637	0.2881	0.1134	0.0852	0.0661	0.0358	0.0255	0.0222



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Estimated outstanding claims

i	IBNR	RBNS	TOTAL	CHAIN LADDER
2	628	605	1,233	1,685
3	1,350	4,514	5,863	29,379
4	1,510	43,623	45,133	60,638
5	1,967	94,526	96,493	101,158
6	2,579	171,633	174,212	173,802
7	3,168	299,136	302,304	249,349
8	5,349	509,334	514,684	475,992
9	14,280	852,144	866,423	763,919
10	254,499	1,135,678	1,390,177	1,459,860
Total	285,329	3,111,192	3,396,521	3,315,779



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Conclusions

Many methods, including chain-ladder, use just one triangle of aggregated data. Using a little more information, you can get more information out. This is what Munich chain-ladder tries to do, but it starts from the same point as the chain-ladder technique (aggregate triangles) rather than considering the way the data is generated.

When considering scenarios for capital modelling, it is better to be able to look at quantities that have real meaning.