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ERM

Symposium

Risk Aggregation: An analysis of inter-risk correlation between market and credit risk

Klaus Böcker, UniCredit Group

2008 Enterprise Risk Management Symposium
April 15, 2008, Chicago, IL, USA

References:

- Böcker, K. and Hillebrand, M.,
  Interaction of Market and Credit Risk: An Analysis of Inter-Risk Correlation and Risk Aggregation.
  Submitted for Publication. (2008)
- Böcker, K. and Klüppelberg, C.,
  Economic Capital Modelling and Basel II Compliance in the Banking Industry,
  In: Mathematics — Key Technology for the Future,
  Ed. Jäger, W. and Krebs, H.-J., Springer, Berlin. (2008)
- Böcker, K., *Aggregation by Risk Type and Inter Risk Correlation*,
  In: Pillar II in the New Basel Accord: the Challenge of the Internal Capital Adequacy Assessment Process, Ed. Resti, A., Risk Books,
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Disclaimer:

Presented risk control and measurement concepts are not necessarily used by UniCredit Group or any affiliates.

Agenda

- Motivation (from a practitioners point of view)

- Inter-risk correlation of credit and market risk

- One-factor model approximations

- Risk aggregation

The need for risk aggregation

What is the total risk of the bank? → Calculation of aggregated EC:

- Metric: VAR, ES, volatility,…
- Usage: ICAAP, rating agencies, shareholders,…

Two different approaches:

- Base-level approach: a common set of risk factors describe risk at "atomic" level (products) of the entire bank portfolio.
  → Requires a single bank-wide stochastic scenario generator.

- Top-level approach: separately pre-aggregated loss distributions for each risk type are combined "on-top" with an appropriate inter-risk dependence structure.
  → Notion of inter-risk correlation is born.

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Top-level approach: covariance approach

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- For multivariate normally distributed risk types, EC is simply a multiple of the standard deviation. Hence, for two risk positions

$$\sigma_{total} = \sqrt{\sigma_1^2 + \sigma_2^2 + 2\,\rho\,\sigma_1\sigma_2}$$

$$\Rightarrow \quad EC_{total} = \sqrt{EC_1^2 + EC_2^2 + 2\rho\,\mathrm{EC}_1\,\mathrm{EC}_2} \qquad (1)$$

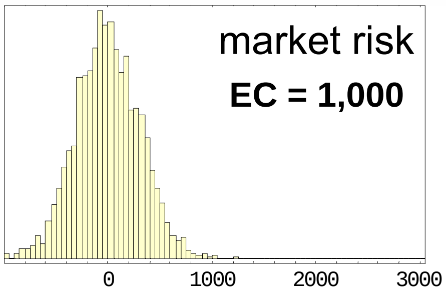
- Covariance approach: Use (1) as an approximation for total EC also in the case of non-normally distributed risk types:

$$EC_{total} \approx \sqrt{EC_1^2 + EC_2^2 + 2\boxed{\rho}\,EC_1\,EC_2}$$

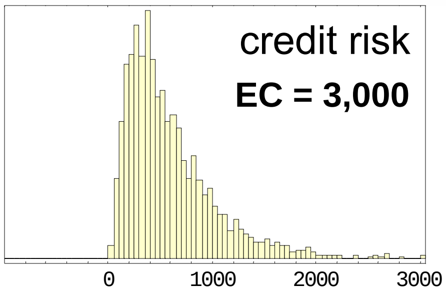
Inter-risk correlation

Top-level approach: copula approach

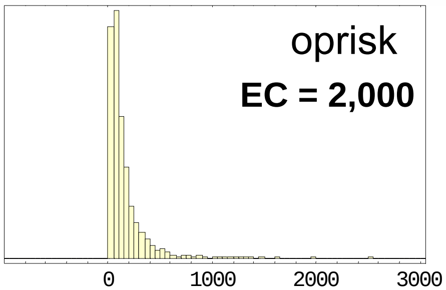
Model MR



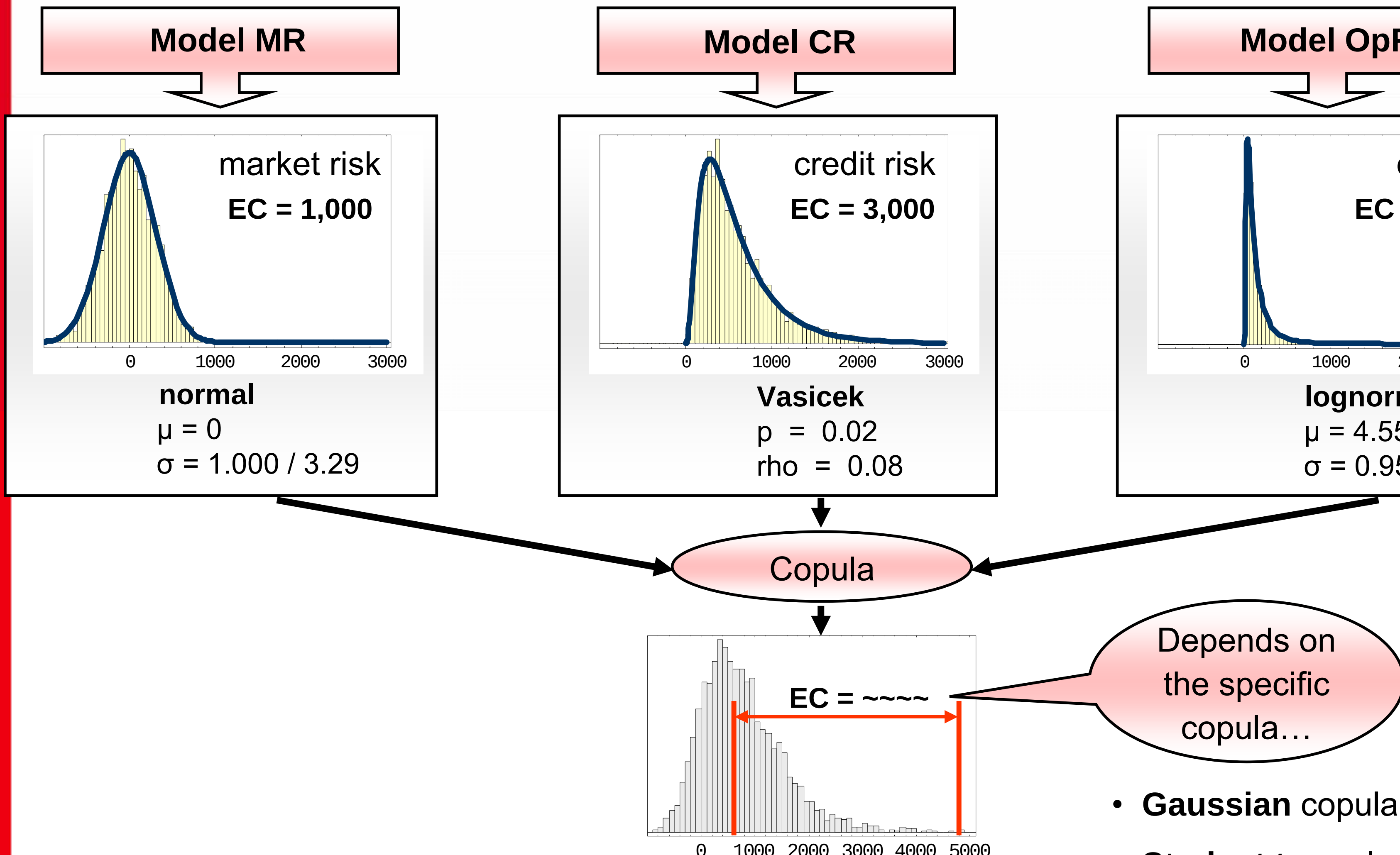
Model CR



Model OpR



Top-level approach: copula approach



- **Gaussian** copula
- **Student t** copula

Current practices for risk aggregation[1]

- 70 % of banks use a top-level approach for risk aggregation.
- The treatment of diversification (in % of banks):

Diversification ?

- 20 % → **No!** Summation approach
- 80 % → **Yes!**
  - 75 % → Covariance approach
  - 25 % → Simulation, copulas,…

$$EC_{tot} \approx \sqrt{EC_1^2 + EC_2^2 + 2\,\rho\; EC_1\, EC_2}$$

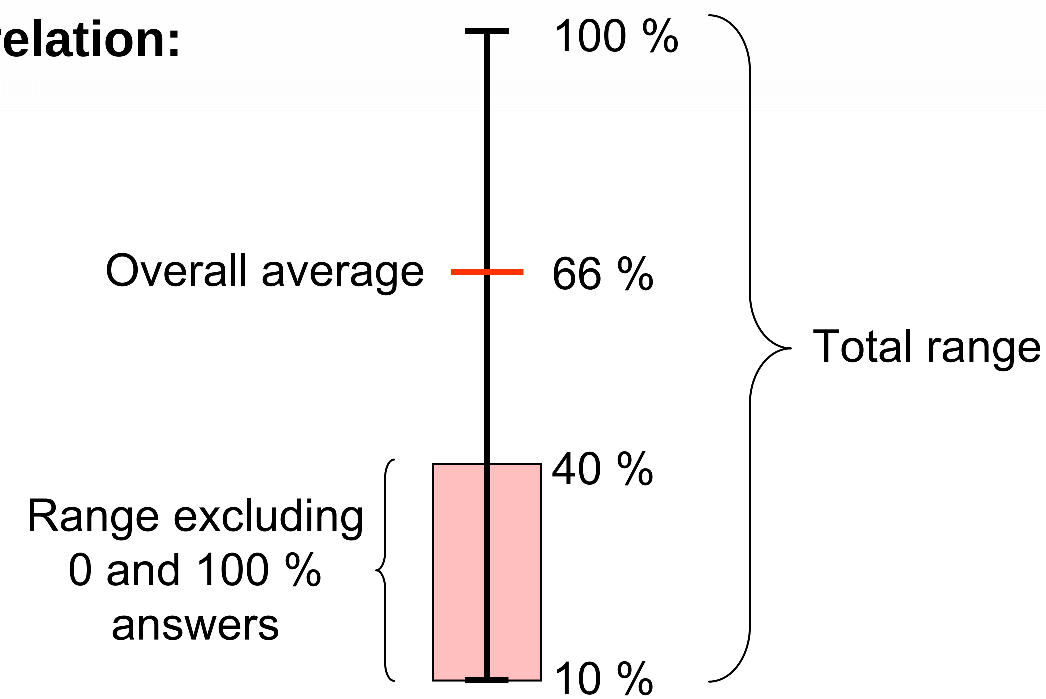
[1]See: *Insights from the joint IFRI/CRO Forum survey on Economic Capital practice and applications.* Available at `http://ifri.ch/publications/Final_position_paper.pdf`, henceforth cited as [IFRI/CFO]

Current practices for risk aggregation[1]

[IFRI/CFO]:

"Correlation estimates used vary widely, to an extend that is unlikely to be solely attributable to differences in business mix."

CR-MR correlation:



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Definition 1: Normal factor model for credit risk

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- Consider a credit portfolio of n loans with exposures e_i and default probabilities $p_i, i=1,\ldots n$.
- Portfolio loss is given by $L^{(n)} = \sum_{i=1}^{n} e_i L_i$ with $L_i = 1_{\{A_i < D_i\}}$.
- The asset-value logreturns A_i linearly depend on the factors $Y = (Y_1, \ldots, Y_K)$ and some idiosyncratic factor ε_i, all independent and standard normally distributed,

$$A_i = \sum_{k=1}^{K} \beta_{ik} Y_k + \sqrt{1 - \sum_{k=1}^{K} \beta_{ik}^2}\ \varepsilon_i, \quad i = 1, \ldots, n.$$

- Default dependence between different creditors is modelled by their joint dependence on Y.
- According to [IFRI/CFO], over 70% of banks use such a Merton-style approach.

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Definition 2: Normal factor model for market risk

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- Pre-aggregated market risk is described by a random variable Z.
- Z linearly depends on the factors Y and some specific risk η:

$$Z = -\sigma \left(\sum_{k=1}^{K} \gamma_k Y_k + \sqrt{1 - \sum_{k=1}^{K} \gamma_k^2}\ \eta \right)$$

where σ is the volatility of market risk Z.

- Z is normally distributed with variance σ^2.
- The factor weights γ_k and β_{ik} are allowed to be zero so that market or credit risk may only depend on a subset of Y.
- Z may also represent business risk, financial investment risk or real estate risk.

Definition 3: Shock model for market risk

- Similar to **Definition 2** (normal factor model MR); however, Z is disturbed by a positive mixing variable W, independent from Y and η,

$$\hat{Z} = -\sigma \left(W \cdot \sum_{k=1}^{K} \gamma_k Y_k + W \cdot \sqrt{1 - \sum_{k=1}^{K} \gamma_k^2}\ \eta \right)$$

where σ is the volatility of market risk Z.

- Henceforth we focus on $W = \sqrt{\nu / S_\nu}$ where S_ν is a χ_ν^2 distributed random variable with ν_Z degrees of freedom,

➡ $\hat{Z}$ follows a scaled t distribution with distribution function

$$F(x) = F_\nu(x/\sigma)$$

where F_ν is a t distribution with ν degrees of freedom.

Inter-risk correlation in the normal model

- Suppose MR and CR are described by the joint **normal** model $\left(L^{(n)}, Z\right)$.

- What can we say about the linear correlation between credit portfolio loss $L^{(n)}$ and market risk Z ?

$$\operatorname{corr}\left(L^{(n)}, Z\right) = \frac{\sum_i^n e_i r_i \exp\left(-\frac{1}{2} D_i^2(p_i)\right)}{\sqrt{2\pi \sum_{ij}^n e_i e_j \left(p_{ij} - p_i p_j\right)}}$$

- with: joint default probability p_{ij} ,

  default point $D_i(\cdot) = \Phi^{-1}(\cdot)$,

  correlation $r_i := \operatorname{corr}(A_i, Z) = \sum_{k=1}^K \beta_{ik} \gamma_k$, $\quad i = 1, \ldots, n$

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Exposure weights, decreasing function of the p_i.

- with: joint default probability p_{ij} ,

  default point $D_i(\cdot) = \Phi^{-1}(\cdot)$,

  correlation $r_i := \operatorname{corr}\left(A_i, Z\right) = \sum_{k=1}^K \beta_{ik} \gamma_k$, $i = 1, \ldots, n$

Inter-risk correlation bound in the normal model

- Assume we have
  - a parameterized credit risk model but
  - no information about market risk.
- What can we say about inter-risk correlation $\operatorname{corr}\left(L^{(n)}, Z\right)$?

Inter-risk correlation bound in the normal model

- Assume we have
  - a parameterized credit risk model but
  - no information about market risk.
- What can we say about inter-risk correlation $\operatorname{corr}\left(L^{(n)}, Z\right)$?
- Using the Cauchy-Schwarz inequality we obtain with $\sum_k^K \gamma_k^2 \leq 1$

$$|r_i| = \left|\sum_k^K \beta_{ik}\gamma_k\right| \leq \left(\sum_k^K \beta_{ik}^2\right)^{1/2} \left(\sum_k^K \gamma_k^2\right)^{1/2} \leq \left(\sum_k^K \beta_{ik}^2\right)^{1/2} \leq 1.$$

$$\left|\operatorname{corr}\left(L^{(n)}, Z\right)\right| \leq \frac{\sum_i^n e_i \sqrt{\sum_k^K \beta_{ik}^2} \exp\left(-\frac{1}{2} D_i^2(p_i)\right)}{\sqrt{2\pi \sum_{ij}^n e_i e_j \left(p_{ij} - p_i p_j\right)}}.$$

Depends only on credit model's parameters of the credit portfolio!

A hybrid model: heavy tails in MR

- Suppose MR and CR are described by the joint hybrid model $\left(L^{(n)}, \hat{Z}\right)$
  - MR follows a scaled *t*-distribution with ν degrees of freedom (**Definition 3**).
  - CR follows the normal factor model (**Definition 1**).

A hybrid model: heavy tails in MR

- Suppose MR and CR are described by the joint hybrid model $\left(L^{(n)}, \hat{Z}\right)$
  - MR follows a scaled *t*-distribution with ν degrees of freedom (**Definition 3**).
  - CR follows the normal factor model (**Definition 1**).

- Since $\mathrm{E}(Z) = 0$ we obtain with $\operatorname{cov}(L^{(n)}, \hat{Z}) = \mathrm{E}(W)\operatorname{cov}(L^{(n)}, Z)$ and $\operatorname{var}(\hat{Z}) = \mathrm{E}(W^2)\operatorname{var}(Z)$ for the inter-risk correlation

$$\operatorname{corr}\left(L^{(n)}, \hat{Z}\right) = \frac{\mathrm{E}(W)}{\sqrt{\mathrm{E}(W^2)}} \operatorname{corr}\left(L^{(n)}, Z\right).$$

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A hybrid model: heavy tails in MR

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$$\mathrm{corr}\left(L^{(n)}, \hat{Z}\right) = \frac{\mathrm{E}(W)}{\sqrt{\mathrm{E}(W^2)}}\,\mathrm{corr}\left(L^{(n)}, Z\right).$$

- From Cauchy-Schwarz it follows that $0 \le \mathrm{E}(W) \big/ \sqrt{\mathrm{E}(W^2)} \le 1$.

Given a positive inter-risk correlation $\mathrm{corr}(L^{(n)}, Z)$, the market risk shock W diminishes the shocked inter-risk correlation $\mathrm{corr}(L^{(n)}, \hat{Z})$.

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An application to one-factor models

- **Large homogenous portfolio (LHP) approximation for credit risk**
  - Well-known and popular for credit (e.g. IRB formula of Basel II).
  - We set $e_i = e,\; p_i = p,\; \beta_{ik} = \beta_k$ for $i = 1,\ldots,n$, and $k = 1,\ldots,K$.

  Normal one-factor model: $A_i = \sqrt{\rho}\,\tilde{Y} + \sqrt{1-\rho}\,\varepsilon_i,\; i = 1,\ldots,n$

  with uniform asset correlation $\rho = \sum_{k=1}^{K} \beta_k^2$ and $\tilde{Y} = \sum_{k=1}^{K} \beta_k Y_k \big/ \sqrt{\rho}$.

  - Asymptotic result credit portfolio loss (→Vasicek distribution):

$$\frac{L^{(n)}}{n\,e} \xrightarrow{\text{a.s.}} \Phi\left(\frac{D-\sqrt{\rho}\,\tilde{Y}}{\sqrt{1-\rho}}\right) =: L, \qquad n \to \infty$$

- **One-factor setup for market risk**
  - Rewrite the market risk models of **Definition 3** and **4** in terms of the single factor $\tilde{Y}$, e.g. for the shock model

$$\hat{Z} = -\,\sigma\, W\left(\tilde{\gamma}\,\tilde{Y} + \sqrt{1-\tilde{\gamma}^2}\;\tilde{\eta}\right)$$

An application to one-factor models

- **One-factor inter-risk correlation: normal model**
  - $$\operatorname{corr}(L,Z) = \frac{r\,\exp(-D^2/2)}{\sqrt{2\pi(p_{12}-p^2)}} \quad \text{with} \quad \begin{cases} r = \operatorname{corr}(Z,A_i) = \sqrt{\rho}\,\tilde{\gamma} \\ D = \Phi^{-1}(p) \\ p_{12} = \Phi_\rho(D,D) \end{cases}$$
  - $$|\operatorname{corr}(L,Z)| \le \frac{\sqrt{\rho}\,\exp(-D^2/2)}{\sqrt{2\pi(p_{12}-p^2)}} =: \psi(p,\rho)$$
  - for later usage observe that $\tilde{\gamma} = \operatorname{corr}(L,Z)/\psi$.

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  - for later usage observe that $\tilde{\gamma} = \operatorname{corr}(L,Z)/\psi$.

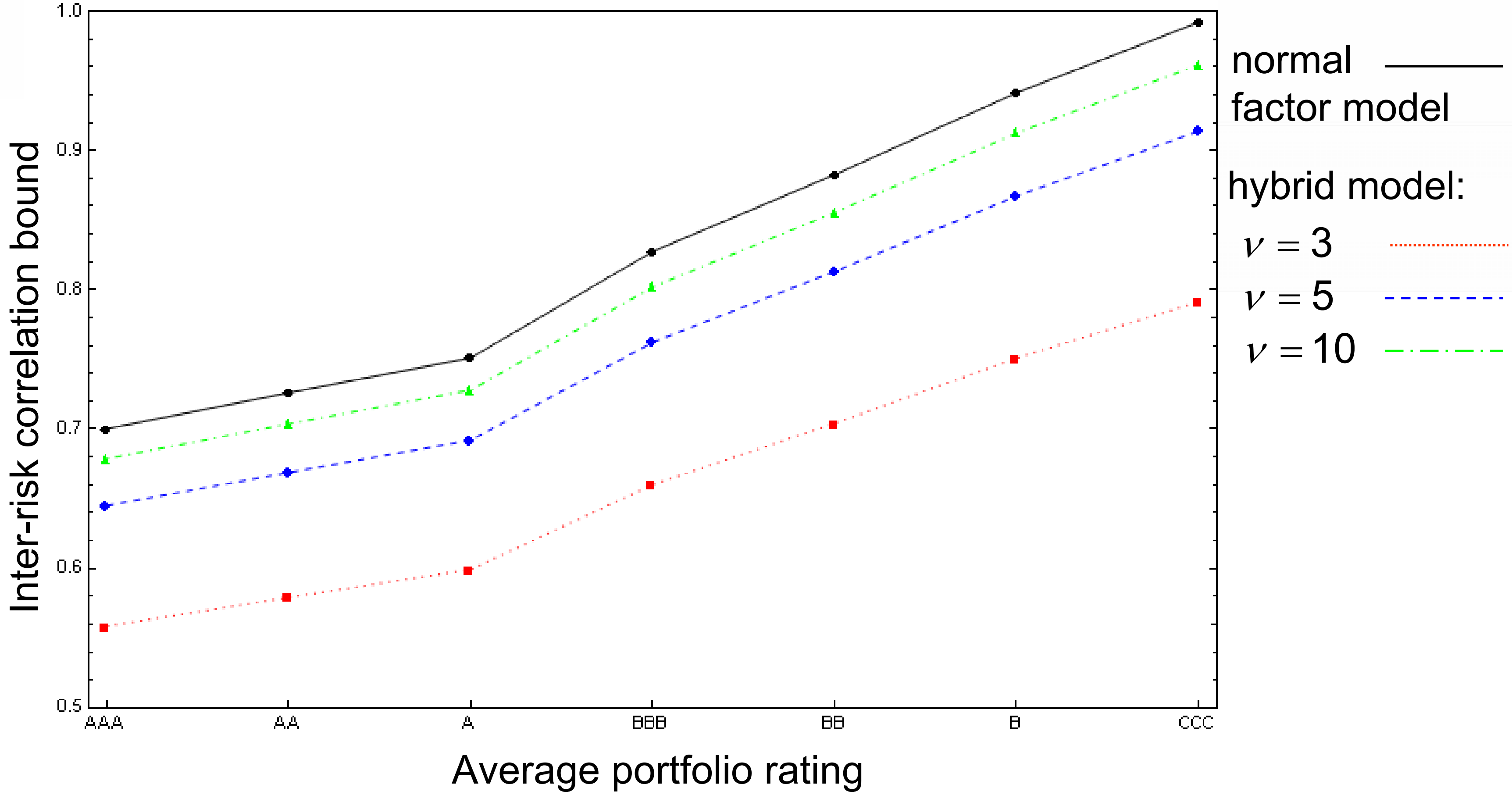
- **One-factor inter-risk correlation: hybrid model**

$$\operatorname{corr}(L,\hat{Z}) = \sqrt{\frac{\nu-2}{2}}\,\frac{\Gamma(\frac{\nu-1}{2})}{\Gamma(\frac{\nu}{2})}\operatorname{corr}(L,Z), \quad \nu > 2$$

An application to one-factor models

Inter-risk correlation bound as a function of the portfolio rating

(Average asset correlation $\rho = 10\ \%$.)



"IRB approach" for inter-risk correlation

- Recall the one-factor inter-correlation bound for the normal model:

$$\psi(p,\rho)=\frac{\sqrt{\rho}\,\exp\left(-\left(\Phi^{-1}(p)\right)^2/2\right)}{\sqrt{2\pi\left(p_{12}-p^2\right)}}$$

- Moment estimator for the inter-correlation bound for non-LHP portfolios:
  - Consider a credit portfolio with total exposure e_{tot}, expected loss μ, and variance $\operatorname{var}(L)$.
  - Then match: $\mu = e_{\text{tot}}\, p$

    $$\operatorname{var}(L) = e_{\text{tot}}^2\left(p_{12}-p^2\right)=e_{\text{tot}}^2\left(\Phi_\rho(\Phi^{-1}(p),\Phi^{-1}(p))-p^2\right)$$

  - Calculate p, ρ, and finally the inter-correlation bound $\psi(p,\rho)$.
- Sample portfolio: Estimates $p=0.54\,\%$, $\rho=23\,\%$ yield $\psi(p,\rho)=0.69$

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LHP approximation and copula aggregation

Consider the normal one-factor model:

- Portfolio loss $L = \Phi\left(\frac{\Phi^{-1}(p) - \sqrt{\rho}\,\tilde{Y}}{\sqrt{1-\rho}}\right)$ is monotonously increasing in $-\tilde{Y}$.
- Copulas are invariant under monotonously increasing transformations.

L and Z have the same copula as $-\tilde{Y}$ and Z.

- $-\tilde{Y}$ and Z are bivariate normally distributed. Hence, they are associated by a Gaussian copula with parameter

$$\mathrm{corr}\left(-\tilde{Y}, Z\right) = \tilde{\gamma}.$$

- Credit and market losses are coupled by a Gaussian copula with parameter $\tilde{\gamma}$.
- Problem: Estimation of $\tilde{\gamma}$ for non-LHP portfolios…

An "IRB approach" for copula aggregation

- Problem: For non-LHP portfolios, the copula parameter $\tilde{\gamma}$ cannot be calculated directly because the portfolio is not homogenous.
- Remedy: Recall that for the normal one-factor model we have

$$\tilde{\gamma} = \frac{\operatorname{corr}(L,Z)}{\psi}.$$

- Calculate the ratio on the r.h.s of this equation for a non-LHP portfolio by using
  1) the general formulas both for $\operatorname{corr}(L,Z)$ and $|\operatorname{corr}(L,Z)|$,or,
  2) the general formula for $\operatorname{corr}(L,Z)$ but the moment estimator for ψ.
- In the case of 1) we obtain the following estimator $\tilde{\gamma}^*$ for the copula parameter $\tilde{\gamma}$:

$$\tilde{\gamma}^* = \frac{\sum_i^n e_i r_i \exp\left(-\frac{1}{2} D_i^2(p_i)\right)}{\sum_i^n e_i \sqrt{\sum_k^K \beta_{ik}^2} \exp\left(-\frac{1}{2} D_i^2(p_i)\right)}, \quad \text{with} \begin{cases} D_i(\cdot) = \Phi^{-1}(\cdot), \\ r_i = \sum_{k=1}^K \beta_{ik}\gamma_k \,. \end{cases}$$

Thank very much!

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A Practical Concept of Tail Correlation

B John Manistre FSA, FCIA, MAAA
VP Risk Research, AEGON NV

April 15, 2008

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Agenda

- Context – Economic Capital
- Simple Aggregation Models
- More Complex Models
  – Presentation Tools
  – Closed Form Example
- Main Conclusion
- Other Contents of Paper

Context

- AEGON NV – a large multi-national multi-line life insurer
- Operates in US, UK, NL, CA, …
  - Several business units within each country
- Developing an internal capital model based on a market value balance sheet
- Need to meet needs of IFRS Phase II Solvency II, rating agencies etc

Context : Economic Capital Basics

- Hold sufficient capital to withstand a 99.5% event over the course of 1 year
- Major Risk Types
  - Underwriting Risk
  - Credit Risk
  - A/L Mismatch Risk
  - Operational Risk
- Capital determined at the individual risk and business unit level and then aggregated up

Aggregation Models

- Simple Models – assume all risks have an elliptical distribution (e.g. multivariate Gaussian or Student's- t)
  - Pro: can aggregate capital using a correlation matrix

$$C(c_1,\ldots,c_n) = \sqrt{\sum_{i,j} \rho_{ij} c_i c_j}$$

  - Con: elliptical models have an underlying spherical symmetry, may be too "special"
  - Con: For a large company the correlation matrix may have tens of thousands of entries
- Some brutal pragmatism required no matter what
-

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Aggregation Models

- More Complex Models
  - make more detailed assumptions about copula (dependency structure), marginal distributions etc.
  - Two potentially offsetting issues

1. Complex model can capture tail dependence
2. Another diversification benefit emerges when component risks have finite variance and the model does not have too much symmetry

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Complex Models – Presentation Tools

- For a complex model the aggregation process cannot be written as a simple formula $C = C(c_1, \ldots, c_n)$ BUT
- Under the reasonable assumption that the true capital aggregation process satisfies the scaling property $C(\lambda c_1, \ldots, \lambda c_n) = \lambda C(c_1, \ldots, c_n)$ paper shows that there is always a family of local formula approximations of the form

$$C \approx \sum_i D_i c_i \qquad D_i = \frac{\partial C}{\partial c_i} \qquad (1)$$

$$C \approx \sqrt{\sum_{i,j} D_{ij} c_i c_j} \qquad D_{ij} = \frac{1}{2} \frac{\partial^2 C^2}{\partial c_i \partial c_j} \qquad (2)$$

Presentation Tools

- diversification factors

$$D_i = \frac{\partial C}{\partial c_i}$$

- tail correlation matrix

$$D_{ij} = \frac{1}{2} \frac{\partial^2 C^2}{\partial c_i \partial c_j}$$

Non Elliptical Closed Form Example

- For $\xi > 0$ consider the formula $C = [\sum_i c_i^{1/\xi}]^{\xi}$
- Exact for aggregating independent stable risks
- Approximate formula for aggregating independent compound risks whose severity distributions have regularly varying tails
- If $\xi < 1/2$ then risks have finite variance
- If $\xi = 1/2$ then this is the standard aggregation formula

Non Elliptical Closed Form Example

$$C = [\sum_i c_i^{1/\xi}]^{\xi}$$

Table 1.1 $c_1 = c_2$

	$\xi =$ 0.35		$\xi =$ 0.50		$\xi =$ 0.65	
$D=C/(c_1+c_2)$	64%		71%		78%	
D_1, D_2	64%	64%	71%	71%	78%	78%
D_{ij}	116%	-35%	100%	0%	95%	28%
	-35%	116%	0%	100%	28%	95%

Non Elliptical Closed Form Example

- Example Suggests:
  - *The standard formula may be conservative when aggregating risks whose tail indices are less than ½.*

Main Conclusions

- In the absence of elliptical symmetry:
  - Tail dependence makes aggregation results more conservative (intuitive)
  - Lighter tails $\xi < 1/2$ make results more liberal
- In the presence of elliptical symmetry
  - Neither effect matters to the aggregation process
- Can always locally approximate a complex model with a simple one

Other Contents in the Paper

- Methods for estimating
  - diversification factors $$D_i = \frac{\partial C}{\partial c_i}$$
  - tail correlation matrices $$D_{ij} = \frac{1}{2}\frac{\partial^2 C^2}{\partial c_i \partial c_j}$$

  from real or simulated data
- Some insights into what is, and is not, important when choosing marginal distributions
- A number of more "realistic" examples

Multivariate Dependence Modeling using Pair–Copulas

Ernesto Schirmacher

Liberty Mutual Group

Agenda

1. Sklar's Theorem, Dependence, and Increasing Transformations

2. χ–Plots to help us visualize dependence

3. The Pair–Copula Construction

4. Example on currency rate changes

Sklar's Theorem

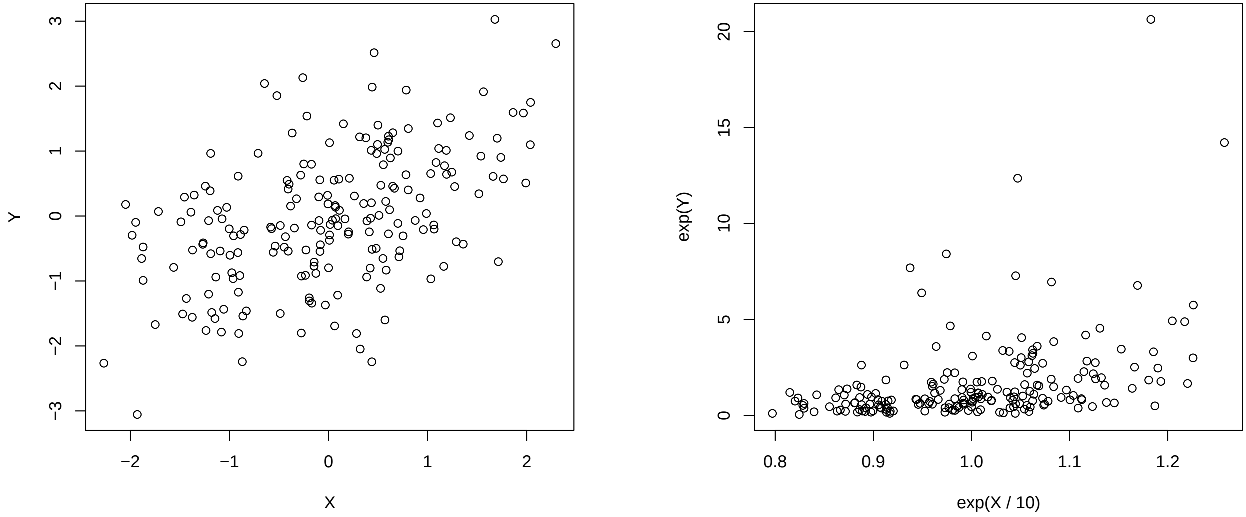
Let $F(x_1, \ldots, x_n)$ be an n–dimensional distribution function with continuous marginals $F_1, F_2, \ldots, F_n$. Then there exists a unique copula function $C\colon [0,1]^n \to [0,1]$ such that

$$F(x_1, x_2, \ldots, x_n) = C\left(F_1(x_1), F_2(x_2), \ldots, F_n(x_n)\right).$$

One can also move in the other direction:

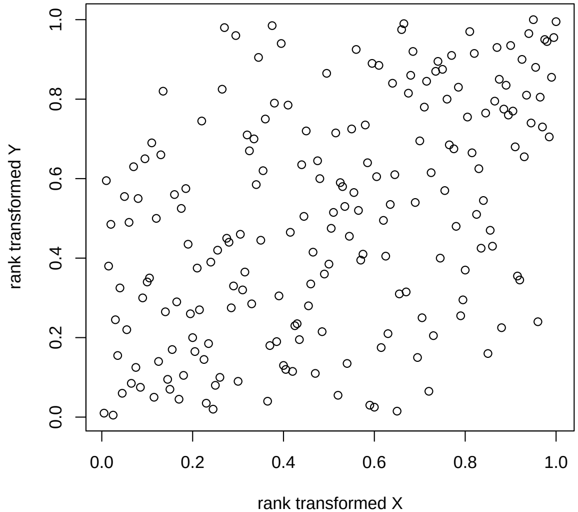
Copula + Marginals $\to$ Joint Distribution.

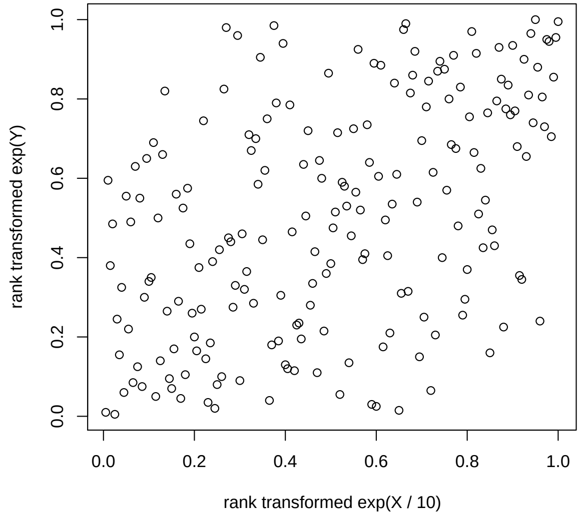
Copulas and increasing transformations



Marginals influence our perception!

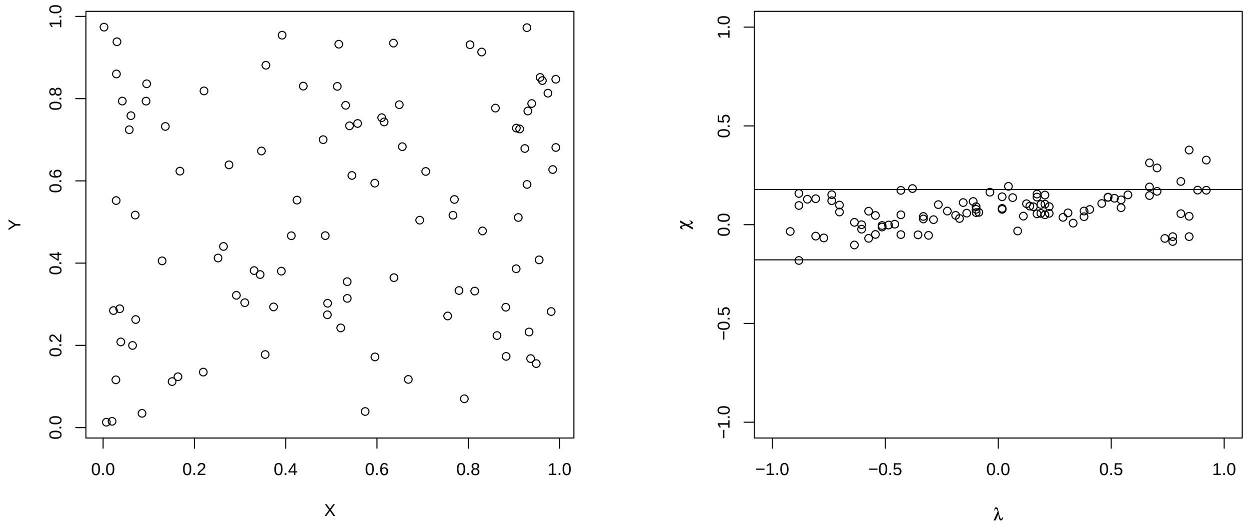
Remove marginals to study dependence



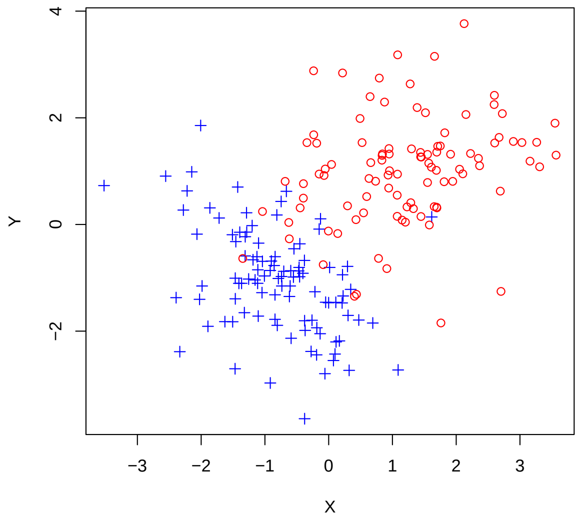


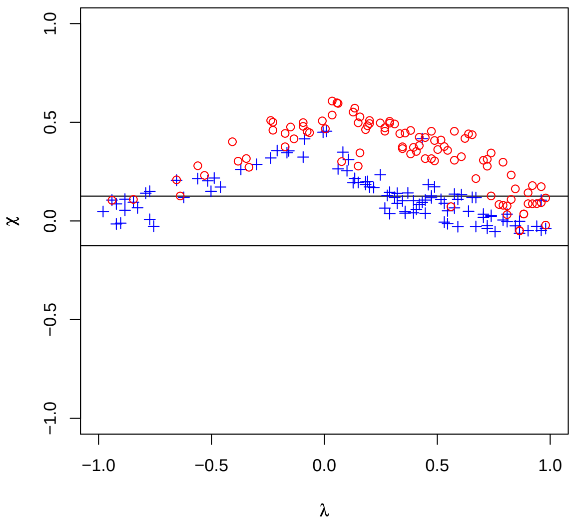
To understand dependence, rank transform your data to eliminate the marginals as **copulas are invariant under strictly increasing transformations**.

The χ-Plot helps visualize dependence

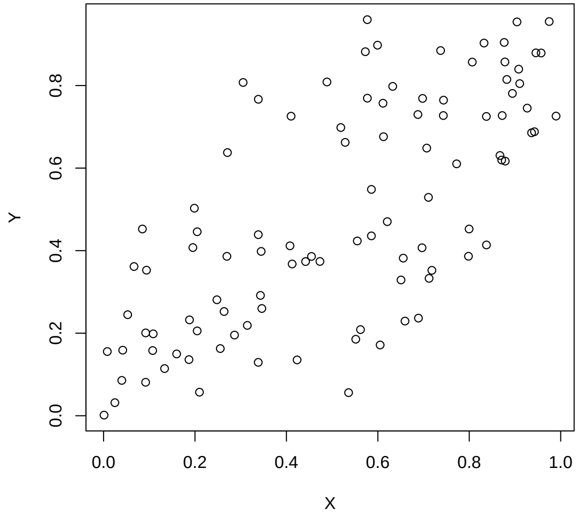


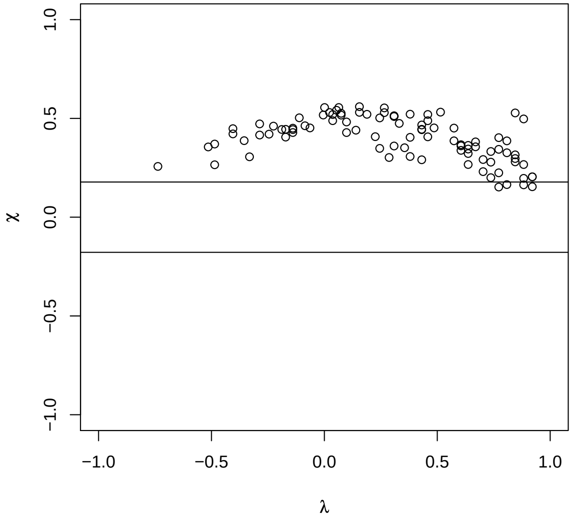
χ-Plot Examples





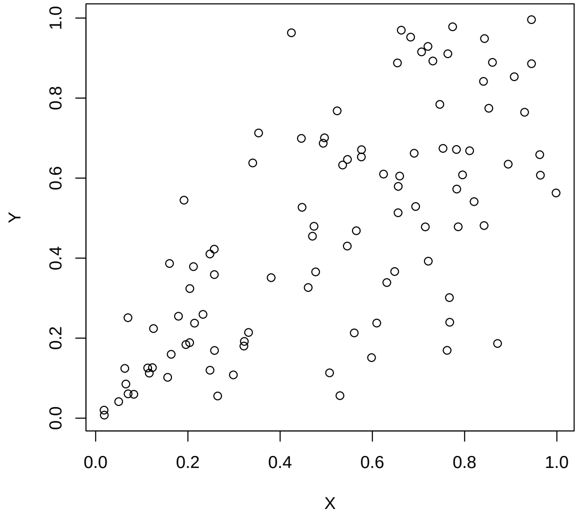
χ-Plot Examples (Normal copula)

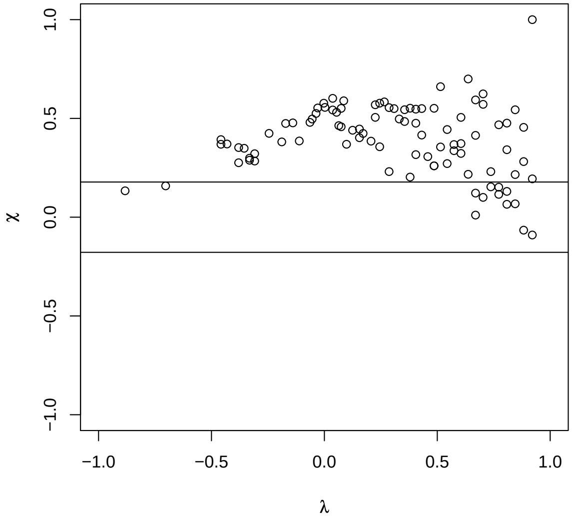




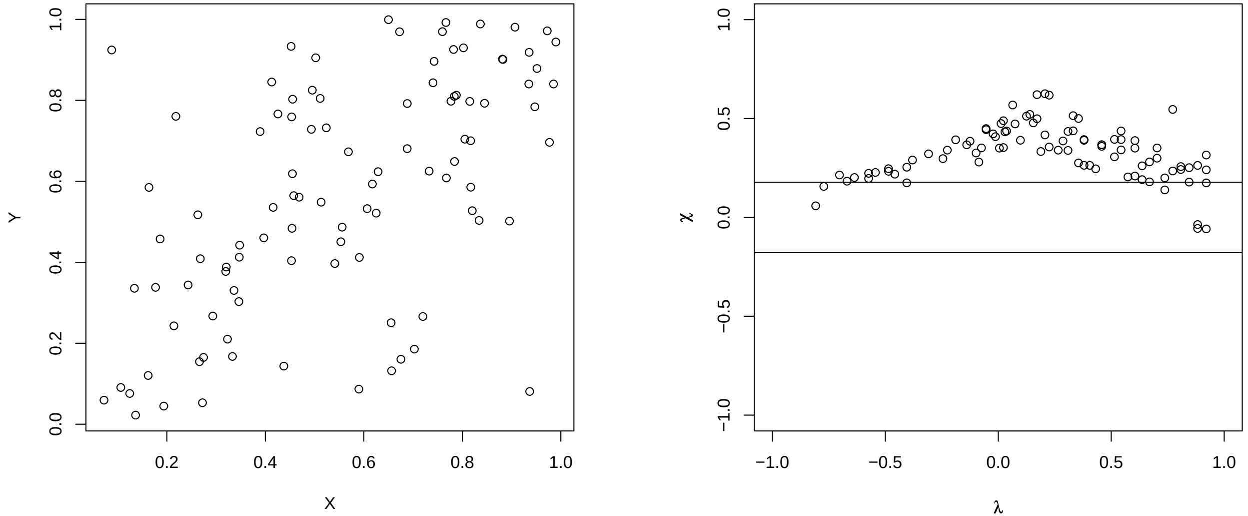
Liberty Mutual.

χ-Plot Examples (Clayton copula)





χ-Plot Examples (Frank copula)



The Pair–Copula Construction

Given an n–dimensional joint density function $f(x_1, \ldots, x_n)$ do the following:

1. 'Factorize' it into a product of conditional densities

2. Rewrite each conditional density from the previous step into a product of bivariate copulas and marginal densities

3. Model each bivariate copula via one of the many choices: normal, t, Frank, Gumbel, Galambos, Clayton, etc…

Three dimensional example

Given $f(x_1, x_2, x_3)$ we can apply steps (1) and (2) to get:

$$
\begin{aligned}
f(x_1, x_2, x_3) &= f_1(x_1) \cdot f_{2|1}(x_2|x_1) \cdot f_{3|12}(x_3|x_1, x_2) \\
&= f_1(x_1) \cdot \\
&\quad c_{12}\Big(F_1(x_1), F_2(x_2)\Big) \cdot f_2(x_2) \cdot \\
&\quad c_{13|2}\Big(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2)\Big) \cdot \\
&\quad c_{23}\Big(F_2(x_2), F_3(x_3)\Big) \cdot f_3(x_3).
\end{aligned}
$$

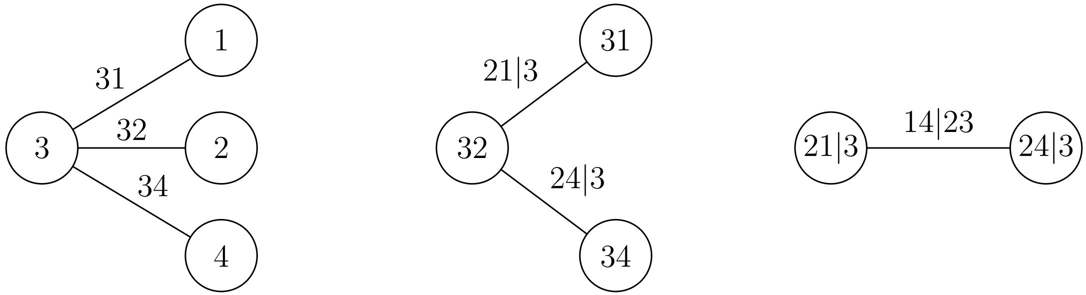
Vines to organize decompositions

The decomposition of $f(x_1, \ldots, x_n)$ in the previous slide into pair–copulas and marginal densities is not unique.

D-vines and canonical vines are two graphical models that help us organize a subset of all possible decompositions.

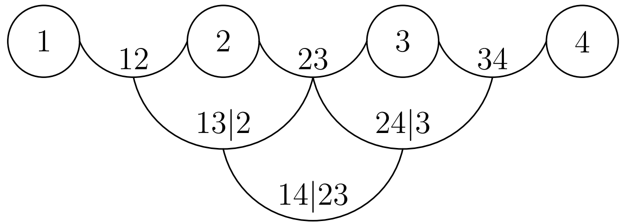
Both consists of sequences of trees that show us how to write a joint density function into pair–copulas and marginal densities.

Four dimensional canonical vine



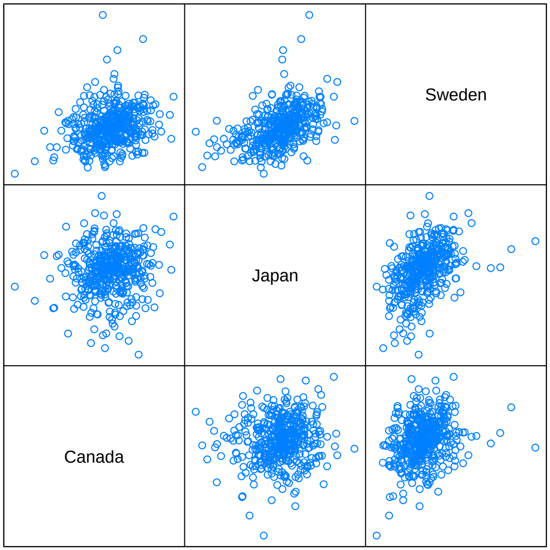
$$
\begin{aligned}
f(x_1, x_2, x_3, x_4) = {} & f_1(x_1) f_2(x_2) f_3(x_3) f_4(x_4) \\
& c_{31}\Big(F_3(x_3), F_1(x_1)\Big) c_{32}\Big(F_3(x_3), F_2(x_2)\Big) c_{34}\Big(F_3(x_3), F_4(x_4)\Big) \\
& c_{21|3}\Big(F_{2|3}(x_2|x_3), F_{1|3}(x_1|x_3)\Big) c_{24|3}\Big(F_{2|3}(x_2|x_3), F_{4|3}(x_4|x_3)\Big) \\
& c_{14|23}\Big(F_{1|23}(x_1|x_2, x_3), F_{4|23}(x_4|x_2, x_3)\Big)
\end{aligned}
$$

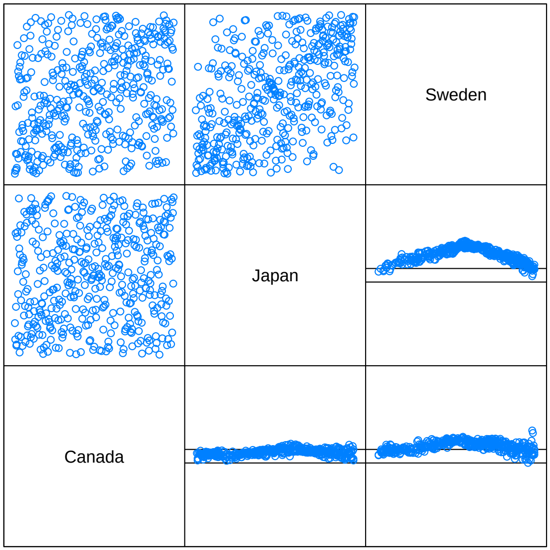
Four dimensional D-vine



$$
\begin{aligned}
f(x_1, x_2, x_3, x_4) = {} & f_1(x_1) f_2(x_2) f_3(x_3) f_4(x_4) \\
& c_{12}\Big(F_1(x_1), F_2(x_2)\Big) c_{23}\Big(F_2(x_2), F_3(x_3)\Big) c_{34}\Big(F_3(x_3), F_4(x_4)\Big) \\
& c_{13|2}\Big(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2)\Big) c_{24|3}\Big(F_{2|3}(x_2|x_3), F_{4|3}(x_4|x_3)\Big) \\
& c_{14|23}\Big(F_{1|23}(x_1|x_2, x_3), F_{4|23}(x_4|x_2, x_3)\Big)
\end{aligned}
$$

Example: Currency Rate Changes

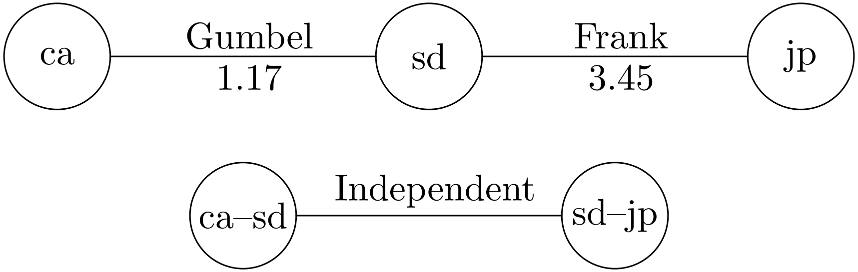




Monthly changes in foreign currency rates to US dollar.

Data Source: FRED database from the Federal Reserve Bank of St. Louis.

Initial ML–estimates for canonical vine



1. Bivariate ML–estimates are easy to calculate

2. These are just initial estimates used to start a global ML–estimation

Maximum likelihood parameter estimates

Pair-copula	Family	ML estimate
Canada–Sweden	Gumbel	1.11
Japan–Sweden	Frank	1.62
Canada–Japan *given* Sweden	Independent	