Session 4 Call Paper Program Topics

A Comparative Study of the Performance of Loss Reserving Methods through Simulation

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Agenda

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Loss Reserving Methods

Loss Development

$L_{i,j}$ = losses paid by period j for accident year i

$i, j = 1, \ldots n.$

$f_{i,j} = L_{i,j+1} / L_{i,j}$

Development factor $f_j = \sum_{i=1}^{n-j} f_{i,j} / (n - j)$

Age k to ultimate factor $U_k = \prod_{j=k}^{n-1} f_j$

$U_n = 1$

Ultimate for AYi $= L_{i,n-i+1} * U_{n-i+1}$

Reserve for AYi $= \text{Ultimate}(i) - L_{i,n-i+1}$

Loss Reserving Methods

Buhlman Complementary Loss Ratio

Incremental loss $s_{i,j} = L_{i,j} - L_{i,j-1}$

R = inflation rate

Inflated losses $S_{i,j} = s_{i,j} * (1 + R)^{n-i}$

Average $M_j = \sum_{i=1}^{n+j-1} S_{i,j} / (n - j + 1)$

$\hat{s}_{i,j} = M_j (1 + R)^{(i-n)}$

Reserve for AYi = $\sum_{j=n+2-1}^{n} \hat{s}_{i,j}$

Loss Reserving Methods

Regression Models

$$Z_{i,j} = \log s_{i,j}$$

Model 1:

$$Z_{i,j} = \mu + \alpha_i + \beta_j + e_{i,j}$$
$$\alpha_1 = \beta_1 = 0$$

Model 2:

$$Z_{i,j} = \mu + (i-1)\alpha + \beta_j + e_{i,j}$$
$$\beta_1 = 0$$

Model 3:

$$Z_{i,j} = \mu + (i-1)\alpha + (j-1)\beta + \gamma \log j + e_{i,j}$$

$e_{i,j}$ are random noise assumed $N(0, \sigma^2)$

Loss Reserving Methods

Regression Models

$$Z = X\theta + e$$

$$Z = [Z_{11}, Z_{12} \ldots Z_{1,n}, Z_{2,1} \ldots Z_{n,1}]^T$$

Model 1: $\theta = [\mu, \alpha_2 \ldots \alpha_n, \beta_2 \ldots \beta_n]$

$$X = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0, & 0, & \ldots & \ldots & 0 \\ 1 & 0 & \ldots & \ldots & 0 & 1 & \ldots & \ldots & 0 \\ \vdots \\ 1 & 0 & \ldots & \ldots & 0 & 0 & 0 & \ldots & 1 \\ 1 & 1 & 0 & 0 & \ldots & 0 & \ldots & \ldots & 0 \\ 1 & 1 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ \vdots \\ 1 & 0 & \ldots & \ldots & 1 & 0 & \ldots & \ldots & 0 \end{bmatrix}$$

e is a random vector.

$E(e) = 0, V(e) = \sigma^2 I$

Use method of least squares to estimate θ, σ^2

$$\hat{\theta} = (X^T X)^{-1} X^T Z$$

$$\hat{\sigma}^2 = (Z^T Z - \hat{\theta}^T X^T Z) / (r - p)$$

$$r = n(n+1)/2, \quad p = 2n - 1$$

Loss Reserving Methods

Regression Models

Model 2: $\theta = [\mu, \alpha, \beta_2, \ldots \beta_n]^T$

$$X = \begin{bmatrix} 1 & 0 & 0 & \ldots & \ldots & 0 \\ 1 & 0 & 1 & \ldots & \ldots & 0 \\ \vdots & & & & & \\ 1 & 0 & 0 & \ldots & \ldots & 1 \\ 1 & 1 & 0 & \ldots & \ldots & 0 \\ 1 & 1 & 1 & 0 & \ldots & 0 \\ \vdots & & & & & \\ 1 & n-1 & 0 & \ldots & \ldots & 0 \end{bmatrix}$$

Model 3: $\theta = [\mu, \alpha, \beta, \gamma]$

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & \log 2 \\ \vdots & & & \\ 1 & 0 & n-1 & \log n \\ 1 & 1 & 0 & 0 \\ \vdots & & & \\ 1 & (n-1) & 0 & 0 \end{bmatrix}$$

Loss Reserving Methods

Regression Models

Define:

$$g(m,t) = \sum_{k=0}^{\infty} \frac{m^k (m+2k)}{m(m+2)\cdots(m+2k)} \frac{t^k}{k!}$$

Unbiassed estimate of $s_{i,j}$

$$\hat{s}_{i,j} = \exp\left(A_{i,j}\,\hat{\theta}\right) \quad g\left(t-p, \tfrac{1}{2}\left(1 - A_{i,j}\left(X^T X\right)^{-1} A^T{}_{i,j}\right)\hat{\sigma}^2\right)$$

Where $E\,Z_{i,j} = \left[A_{i,j}\right] \quad \theta$.

Look for details in Verral (1994).

Once $s_{i,j}$ are estimated, accident year reserves can be computed.

Simulation of Random Loss Triangles

Random Reporting Factors

Step 1 Generate random number of claims.

Step 2 Random claim size for each claim.

Step 3 Add all the claims together to get S_1.

Step 4 Uniform random numbers $X_2, X_3 \ldots X_{10}$

Step 5 $T_j = 0.1 + 0.5 \; X_j + 0.5 \log j$

Step 6 $U_j = T_1 + T_2 + \ldots T_j$

Step 7 $L_{1,j} = S_1 * \left(1 - \exp(Uj)\right), \; j = 1, 2, \ldots 10. \quad L_1{,}11 = S_1$

Step 8 Repeat steps 1 through 7 and multiply accident year i losses by $(1.06)^{(i-1)}$.

Note: Claim frequency Poisson with mean 100.

Claim severity is log normal with $\mu = 7.36$, $\sigma^2 = 1.510$.

Simulation of Random Loss Triangles

Backward Development Factors

Steps 1 - 3 As in Method 1

Step 4 Simulate $X_j \sim N(\mu_j, \sigma^2{}_j),\ j = 1, \ldots 10$

$\mu_j = (11 - j + (10 - j)^2) / 100$

$\sigma_j{}^2 = (11 - j + (10 - j)^2) / 500$

Step 5 $f_j = \exp(X_j)$

$$F_j = \prod_{k=j}^{10} f_k$$

$L_{i,j} = S_1 / F_j$

Step 6 Repeat Steps 1 through 5 for each accident year and inflate the losses.

Simulation of Random Loss Triangles

Individual Losses with Changing Severity

Step 1 Generate frequency of claims N, Poisson with mean 100.

Step 2 For each claim, simulate:

A. Occurrence delay x_1; uniform (0,1).

B. Reporting delay x_2; exponential with mean 2.

C. Claim closing delay x_3; exponential with mean 5.

D. Cap x_2 at $11 - x_1$, and x_3 at $11 - x_1 - x_2$.

Step 3 Simulate percentile level x for each claim; uniform (0,1).

Step 4 Claim size ℓ_k for a claim is

$$\begin{aligned} \ell_k &= 0 \qquad k < x_1 + x_2 \\ &= \lambda(k)((1-x)^{1/\theta(k)} - 1), \quad x_1 + x_2 \le k \le x_1 + x_2 + x_3 \\ &= \lambda(r)((1-x)^{1/\theta(r)} - 1), \quad k > x_1 + x_2 + x_3 \end{aligned}$$

Where r is the smallest integer $\ge x_1 + x_2 + x_3$

$$\lambda(k) = (1000 + (k-1)\,50)\,1.06^{k-1}$$

$$\theta(k) = 2.5 - 0.05(k-1)$$

Step 5 Add the individual claims ℓ_k.

Step 6 Inflate losses 6% for accident years 2,…11.

Simulation of Random Loss Triangles

Pentikainen Rantala Method

Step 1 Simulate $y \sim N(0.4,\ 0.05^2)$ for each i, j

Step 2 Compute $q(i, j) = 0.6 * q(1, j-1) + y$

Where $q(i,0) = 1$

Step 3 Simulate $y \sim N(0, 0.015^2)$ and

Compute $x(i) = \inf(1) + 0.6(\inf(i) - \inf(1)) + y$

$\inf(i+i) = \max[x(i), 0.03)]$

Where $\inf(1) = 0.06$ and repeat for $i = 1, \ldots 20$

Step 4 Compute $\text{infe}(i) = 1.01^{(i-1)}$, $i = 1, \ldots 11$

Step 5 Compute the incremental losses

$s_{i,j} = 500{,}000 * q(i, j) * \text{infe}(i) * \inf(i + j - 1) * xp(j)$

Where $x\,p(j)$ is the jth element of the vector

[.22, .18, .15, .12, .1, .08, .06, .04, .027, .016, .007]

Step 6 Loss triangle can be computed from $s_{i,j}$

Criteria for Comparing Results

1. Bias: Average of (actual - estimated reserves)
2. R.M.S.E.: $\sqrt{\text{Average}(\text{actual} - \text{estimated reserves})^2}$
3. Avg. Abs. Dev.: Average $\left|\text{actual} - \text{estimated reserves}\right|$
4. Average % Error: Average $\left[\dfrac{\text{actual} - \text{estimated reserves}}{\text{actual}}\right]$
5. Correlation of Actual and Estimated Reserves

Note: Criterion 5 used for total reserve only.

Simulation Results

Loss Triangle Simulation_Random Reporting Factors

TABLE 1

Actual Total Reserve:	Average =	1,108,298		SD=	244,287
		Buhlmann		**Regression**	
Method of Forecast:	**Dev Factor**	**Loss Ratio**	**Model 1**	**Model 2**	**Model 3**
		Five Thousand Iterations			
Bias	151,681	5,222	36,486	31,240	51,367
RMSE	466,055	266,874	395,819	328,870	341,537
Avg Abs Dev	364,628	204,674	314,829	254,069	263,444
Avg %	16.84%	4.84%	6.22%	6.75%	8.69%
Corr Actual vs Est	0.25	0.09	0.25	0.15	0.14

Simulation Results

Loss Triangle Simulation_Random Backward Development Factors

TABLE 2

Actual Total Reserve: Average = 3,665,734 SD= 485,206

		Buhlmann	**Regression**		
Method of Forecast:	**Dev Factor**	**Loss Ratio**	**Model 1**	**Model 2**	**Model 3**
		Five Thousand Iterations			
Bias	157,684	(8,088)	55,356	15,393	3,125
RMSE	512,092	639,187	481,727	542,257	519,705
Avg Abs Dev	391,022	485,769	373,282	420,438	403,056
Avg %	4.38%	1.23%	1.58%	0.79%	0.47%
Corr Actual vs Est	0.70	0.11	0.70	0.57	0.58

Simulation Results

Loss Triangle Simulation_Individual Losses with Changing Severity

TABLE 3

Actual Total Reserve:	Average =	1,634,559	SD=	252,631

		Buhlmann	Regression		
Method of Forecast:	**Dev Factor**	**Loss Ratio**	**Model 1**	**Model 2**	**Model 3**
		Five Thousand Iterations			
Bias	30,566	(83,039)	(144,192)	(52,327)	(176,089)
RMSE	413,137	441,109	375,367	299,099	340,506
Avg Abs Dev	356,932	347,340	314,629	259,057	280,243
Avg %	1.39%	-4.36%	-9.49%	-3.31%	-9.52%
Corr Actual vs Est	0.62	0.39	0.68	0.66	0.32

Simulation Results

Loss Triangle Simulation_Pentikainen & Rantala Method

TABLE 4

Actual Total Reserve: Average = 3,183,654 SD= 330,776

Method of Forecast:	Dev Factor	Buhlmann Loss Ratio	Regression Model 1	Regression Model 2	Regression Model 3
Five Thousand Iterations					
Bias	10,106	(21,441)	5,326	4,789	34,136
RMSE	186,688	186,916	183,351	195,148	201,012
Avg Abs Dev	147,536	147,830	145,029	153,675	157,283
Avg %	0.23%	-0.24%	0.07%	0.06%	0.98%
Corr Actual vs Est	0.89	0.84	0.89	0.88	0.88

Conclusion

1. Regression models compete well with traditional methods.
2. Parsimony is important for regression models.
3. LDF method provides reasonable answers.
4. Variance is not considered, but RMSE is an indicator of variance if the method is unbiased.

Note: i) Regression model results should be better in practice because of user interaction in selecting an appropriate model.

ii) For individual accident year results, see CAS 1997 Forum.