# Optimal Reinsurance under VaR and CVaR Risk Measures: A Simplified Approach

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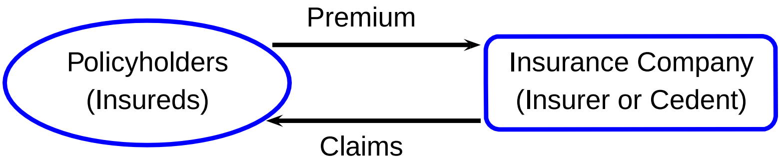
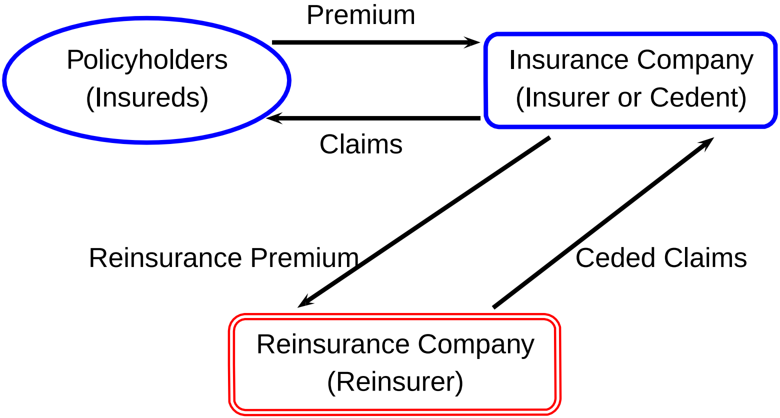
# Outline

- Introduction and motivation
- Risk measure based optimal reinsurance models
  - Cai and Tan (2007)
  - Chi and Tan (2011)
- Conclusion

# Without Reinsurance



# With Reinsurance



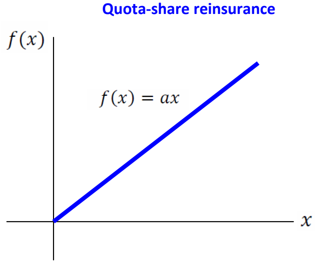
# Reinsurance as a Risk Management Tool

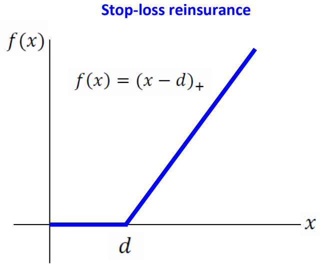
- Reinsurance can be an effective risk management tool for insurers
- Some reasons for Reinsurance:
    - limitation of exposure to risk
    - avoidance of large single losses
    - increasing capacity to accept risk
    - availability of expertise
- The primary goal of reinsurance is to maintain, at an acceptable level, the random fluctuations of the business operation of the insurers:
    - earning volatilities
    - variance of the underlying risk

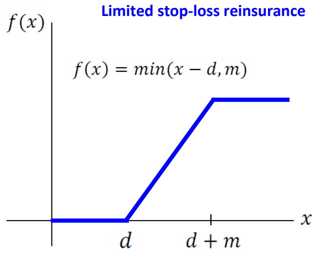
# Reinsurance Contracts

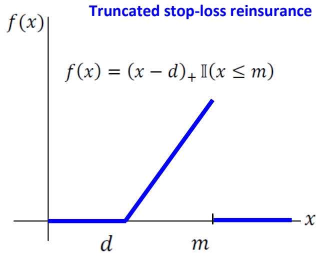
- Let $X$ denote the loss initially assumed by an insurer
- $X$ is a non-negative r.v. with
  - c.d.f. $F_X(x) = \Pr(X \leq x)$,
  - survival function $S_X(x) = \Pr(X > x)$, and
  - $\mathrm{E}[X] < \infty$
- In the presence of reinsurance, the insurer cedes part of its loss, say $f(X)$, to a reinsurer
  - $f(x)$ is known as a ceded loss function
  - $R_f(x) = x - f(x)$ is the retained loss function
  - $0 \leq f(x) \leq x$ and $0 \leq R_f(x) \leq x$

# Samples Ceded Loss Functions



# Reinsurance Premium

- By ceding part of its risk to a reinsurer, the insurer incurs an additional cost in the form of reinsurance premium which is payable to a reinsurer

- Let $\Pi_f(X)$ denote the reinsurance premium which corresponds to a ceded loss function $f(x)$
- Under expected premium principle:

$$\Pi_f(X) = (1 + \rho)\mathrm{E}[f(X)]$$

where $\rho > 0$ is the relative safety loading

# Optimal Reinsurance

- There's a tradeoff between the amount of loss retained and the reinsurance premium payable to reinsurer
  - **optimal reinsurance design?**
- Some plausible optimal reinsurance models:
  - Minimizes insurer's ruin probability
  - Classical result (Borch 1960):

$$\min_{f} \quad \text{Variance}(R_f(X))$$

$$\text{subject to} \qquad \text{Premium} = (1+\rho)\mathrm{E}[f(X)]$$

$\Rightarrow$ Stop-loss reinsurance is optimal

  - By maximizing expected utility of insurer's terminal wealth, Arrow (1963) shows that stop-loss reinsurance is optimal

⋮

  - Our approach exploits the risk measure based optimal reinsurance model of Cai and Tan (2007)

# Risk Measure based Optimal Reinsurance Model

- Define total “risk exposure" of the insurer in the presence of stop-loss reinsurance as

$$T_f(X) = R_f(X) + \Pi_f(X) = X - f(X) + \Pi_f(X)$$

$\Rightarrow$ **implications?**

- Risk measure based optimal reinsurance model:

$$\min_{f \in \mathcal{C}} \psi(T_f(X))$$

  - $\psi(\cdot)$ is a risk measure
  - $\mathcal{C}$ is the set of admissible ceded loss functions

  - Complexity of this model?

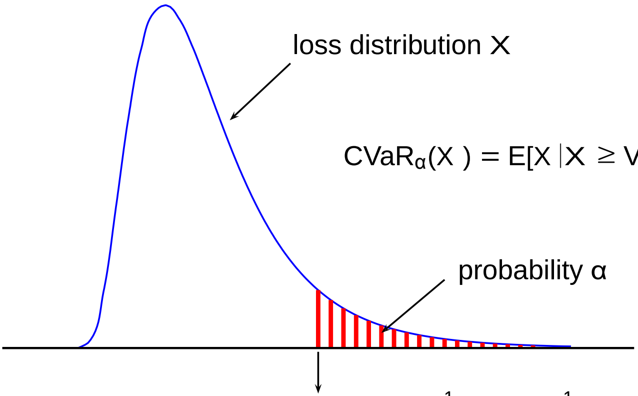
# Cai and Tan (2007): Assumptions

- $\mathcal{C}$ is the stop-loss reinsurance with retention $d > 0$;

$$f(x) = (x - d)_+$$

- $\Pi$ is the expected premium principle
- $\psi$ is either
  - Value at Risk (VaR) or
  - Conditional VaR (CVaR)/Conditional Tail Expectation (CTE)

# VaR vs CVaR at Confidence Level $1 - \alpha$




$$\mathrm{CVaR}_\alpha(X) = E[X \mid X \geq \mathrm{VaR}_\alpha(X)]$$

$$\mathrm{VaR}_{\alpha}(X) \equiv S_X^{-1}(\alpha) = F_X^{-1}(1-\alpha)$$

$$\Pr\{X > \mathrm{VaR}_\alpha(X)\} = \alpha \quad \Leftrightarrow \quad \Pr\{X \leq \mathrm{VaR}_\alpha(X)\} = 1 - \alpha$$

# Cai and Tan (2007): Results

- **VaR-optimization:**

$$d^* \to \min_{d>0}\{\mathrm{VaR}_\alpha(T_f(X); d)$$

- **CVaR-optimization:**

$$\tilde{d} \to \min_{d>0}\{\mathrm{CVaR}_\alpha(T_f(X); d)\}.$$

# An Alternate Justification

- Let $p_X$ be the premium payable by the insured to the insurer.
- Let $r_X$ be the minimum capital set aside by the insurer so that the insurer's probability of insolvency is at most $\alpha$; i.e.

$$\Pr\{T_f > r_X + p_X\} \leq \alpha.$$

- From the definition of VaR:

$$r_X = \mathrm{VaR}_\alpha(T_f) - p_X.$$

$$\Rightarrow \quad \boxed{\min_f \mathrm{VaR}_\alpha(T_f(X)) \Leftrightarrow \min_f r_X}$$

# Cai and Tan (2007): VaR-Optimization

- The optimal retention $d^* > 0$ that minimizes $\text{VaR}_\alpha(T_f(X))$ exists **if and only if** both

$$\alpha < \rho^* < S_X(0)$$

and

$$S_X^{-1}(\alpha) \;\geq\; S_X^{-1}(\rho^*) + \Pi\left(S_X^{-1}(\rho^*)\right)$$

hold, where $\rho^* = \dfrac{1}{1+\rho}$.

- When the optimal retention $d^*$ exists, then $d^*$ is given by

$$d^* = S_X^{-1}(\rho^*)$$

and the minimum VaR of $T$ is given by

$$\text{VaR}_\alpha(T_f(X), d^*) = d^* + \Pi(d^*).$$

# Examples

## $X \sim$ Exponential Distribution

- $S_X(x) = e^{-0.001x}$
- E[$X$] = 1,000, $\alpha = 0.1$, $\rho = 0.2$
- optimal retention $d^*$ exists and equals to

$$d^* = S_X^{-1}(\rho^*) = 1,000 \log(1 + \rho) = 182.32.$$

## $X \sim$ Pareto Distribution

- $S_X(x) = \left(\frac{2,000}{x + 2,000}\right)^3, \; x \geq 0.$
- E[$X$] = 1,000, $\alpha = 0.1$, $\rho = 0.2$
- optimal retention $d^*$ exists and equals to

$$d^* = S_X^{-1}(\rho^*) = 125.32$$

# Cai and Tan (2007): CVaR-Optimization

- The optimal retention $\tilde{d} > 0$ that minimizes $\mathrm{CVaR}_\alpha(T_f(X); d)$ exists **if and only if**

$$0 < \alpha \leq \rho^* < S_X(0).$$

- When the optimal retention $\tilde{d} > 0$ exists, $\tilde{d}$ is given by

$$\tilde{d} = S_X^{-1}(\rho^*) \quad \text{if} \quad \alpha < \rho^*,$$

and

$$\tilde{d} \geq S_X^{-1}(\rho^*) \quad \text{if} \quad \alpha = \rho^*,$$

# Cai and Tan (2007): Summary

- The optimal reinsurance model is simple and intuitive
- It exploits two prevalent risk measures
- The optimal retention has a very simple analytic form
- If optimal solutions exist, then both VaR- and CVaR-based optimization criteria yield the same optimal retentions, except when $\alpha = \rho^*$
  - $d^* = \tilde{d} = S_X^{-1}\left(\dfrac{1}{1+\rho}\right)$
  - The optimal retention depends only on the assumed loss distribution and the reinsurer's safety loading factor
- Limitations?

# Chi and Tan (2011)

- Generalize Cai and Tan (2007) by considering more general admission sets of ceded loss functions:

$$\mathcal{C}^1 \triangleq \{0 \le f(x) \le x : f(x) \text{ is an increasing convex function}\}$$

$$\mathcal{C}^2 \triangleq \{0 \le f(x) \le x : \text{both } R_f(x) \text{ and } f(x) \text{ are increasing functions}\}$$

$$\mathcal{C}^3 \triangleq \{0 \le f(x) \le x : R_f(x) \text{ is an increasing and l.c. function}\}.$$

## Properties:

- $\mathcal{C}^1 \subsetneqq \mathcal{C}^2 \subsetneqq \mathcal{C}^3$
- What is the significance of imposing increasing condition on both retained and ceded loss functions?

# VaR-Optimization under $\mathcal{C}^1$

## Optimal Reinsurance Model:

$$\min_{f \in \mathcal{C}^1} VaR_\alpha(T_f(X))$$

## Optimal Solution:

- $$f^{*1}(x) = \begin{cases} (x - d^*)_+ \,, & VaR_\alpha(X) > \beta; \\ c\,(x - d^*)_+ \,, \; \forall c \in [0,1], & VaR_\alpha(X) = \beta; \\ 0, & \text{otherwise}, \end{cases}$$

  where

  $$\begin{aligned} d^* &= VaR_{\rho^*}(X), \\ \beta &= d^* + (1+\rho)\mathrm{E}\left[(X - d^*)_+\right]. \end{aligned}$$

- $VaR_\alpha(T_{f^{*1}}(X)) = \min_{f \in \mathcal{C}^1} VaR_\alpha(T_f(X)) = \min(\beta,\ VaR_\alpha(X))$

  $\Rightarrow$ **Stop-loss reinsurance** is optimal under $\mathcal{C}^1$

# A Special Case

- If $\rho^* \geq S_X(0)$, then $d^* = 0$.
- The optimal ceded loss function $f^{*1}$ simplifies to

$$f^{*1}(x) \triangleq \begin{cases} x, & VaR_\alpha(X) > (1+\rho)\mathbb{E}[X]; \\ cx, \ \forall c \in [0,1], & VaR_\alpha(X) = (1+\rho)\mathbb{E}[X]; \\ 0, & \text{otherwise.} \end{cases}$$

$\Rightarrow$ **Quota-share ceded loss function**

# VaR-Optimization under $\mathcal{C}^2$

## Optimal Reinsurance Model:

$$\min_{f \in \mathcal{C}^2} VaR_\alpha(T_f(X))$$

## Optimal Solution:

- $f^{*2}(x) \begin{cases} \min\left\{(x-d^*)_+, VaR_\alpha(X)-d^*\right\}, & d^* < VaR_\alpha(X); \\ 0, & \text{otherwise}, \end{cases}$
- $VaR_\alpha(T_{f^{*2}}(X)) = \min[d^*, VaR_\alpha(X)]$

$$+(1+\rho)\mathrm{E}\left[\min\left\{(X-d^*)_+, (VaR_\alpha(X)-d^*)_+\right\}\right].$$

$\Rightarrow$ **Limited stop-loss reinsurance** is optimal under $\mathcal{C}^2$

# VaR-Optimization under $\mathcal{C}^3$

## Optimal Reinsurance Model:

$$\min_{f\in\mathcal{C}^3} VaR_\alpha(T_f(X))$$

## Optimal Solution:

- Let $\gamma = \alpha + \rho^*$, then

$$f^{*3}(x) = (x-\gamma)_+ \mathbb{I}(x \leq VaR_\alpha(X)),$$

- $VaR_\alpha(T_{f*3}(X)) = \gamma + (1+\rho)\mathrm{E}\left[(X-\gamma)_+ \mathbb{I}(X \leq VaR_\alpha(X))\right].$

$\Rightarrow$ **Truncated stop-loss reinsurance** is optimal under $\mathcal{C}^3$

# CVaR-Optimization under $\mathcal{C}^j, j = 1, 2, 3$

## Optimal Reinsurance Model:

$$\min_{f \in \mathcal{C}^j} VaR_\alpha(T_f(X))$$

## Optimal Solution:

- $f^*(x) = \begin{cases} (x - d^*)_+, & \alpha < \rho^*; \\ 0, & \text{otherwise}, \end{cases}$
- $CVaR_\alpha(T_{f^*}(X)) = \min_{f \in \mathcal{C}^j} CVaR_\alpha(T_f(X))$

$$= \begin{cases} \beta, & \alpha < \rho^*; \\ CVaR_\alpha(X), & \text{otherwise} \end{cases}$$

$\Rightarrow$ **Stop-loss reinsurance** is optimal under $\mathcal{C}^j, j = 1, 2, 3$

# Summary/Conclusion

- We extended the reinsurance model of Cai and Tan (2007) by analyzing the solutions to the VaR- and CVaR-based optimal reinsurance models over different classes of ceded loss functions with increasing generality.
- The impact of the optimal reinsurance design on the assumed feasible set of ceded loss functions is highlighted in the case of VaR criterion.
  - This suggests a difference in risk management strategy depending on the adopted optimal reinsurance model.
  - The different optimal reinsurance policies also suggest the differences in insurer's style toward risk management and its attitude towards risk.
- The CVaR-based optimal reinsurance model is quite robust in the sense that the stop-loss reinsurance is always the optimal solution.

**Thank You For Your Attention**