A GAME-THEORETIC APPROACH TO NON-LIFE INSURANCE MARKETS

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Outlines

TOPIC OF THE PRESENTATION

Insurance market :

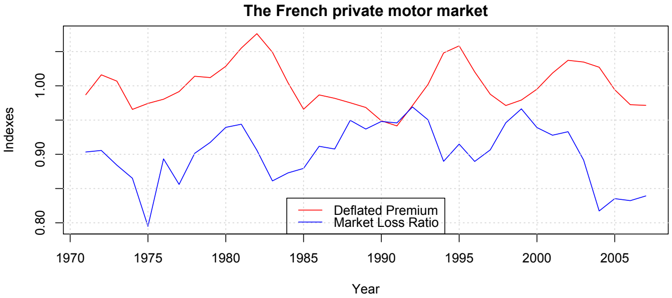
- In exchange of a premium paid to an insurer, an insured transfers part or all its risk to an insurer.
- In return, the insurer will pay an amount of money if a certain type of events occurs.

In this presentation, we focus on modelling premium by taking into account competition.

Indicators of insurance market cycles

Common Indicators :

- loss ratio (LR),
- market premium,
- underwriting profit,
- entry and exits,
- ...



"GENERALLY ACCEPTED" CAUSES

From [Fel01] or [Wei07],

- actuarial pricing : claim cost uncertainty, information lag (accounting, renewal), capacity constraint,
- underwriting philosophy : mass psychology, lack of coordination,
- interest rate movements : external capital cost, investment result,
- competitive strategy : fierce competition, almost no differentiation, entry-exits.

One thing remains true : it is widely admitted, one cause alone can't explain the presence of a cycle !

Time serie modelling

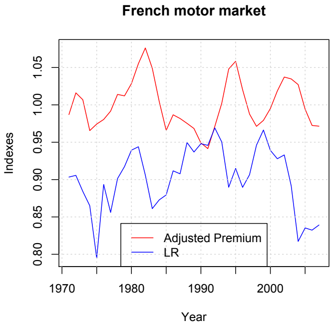
- basic AR(2) :

$$X_t - m = a_1(X_{t-1} - m) + a_2(X_{t-2} - m) + \epsilon_t.$$

  If $a_2 < 0$ and $a_1^2 + 4a_2 < 0$, then the period is

$$p = 2\pi \arccos\left(\frac{a_1}{2\sqrt{-a_2}}\right).$$

- example with deflated premium : $m = 1$, $a_1 = 1.175$, $a_2 = -0.613$. Thus $p = 8.707$.



Introductory example of game theory

Game theory – prisonmer dilemna :

- Two people are arrested for a crime, and police lack sufficient evidence to convict either suspect.
- Police need them to give testimony against each other. They put suspect in two different cells to prevent any communication.
- If only one suspect testify against the other, he will be released and the other will go to prison for 10 years.
- If both testity against each other, both will go to prison for 5 years.
- If neither testify, both will be released after 6 months.

The prison sentence can be represented by the following matrix

J1 | J2	silent (S)	confess (C)
silent (S)	(-1/2, -1/2)	(-10, 0)
confess (C)	(0, -10)	(-5, -5)

Each play will seek to minimize its potential stay in prison
$\Rightarrow$ (C,C) is a *Nash equilibrium*, i.e. a couple of strategies such that no player can unilateraly decrease its cost.

THE PROPOSED MODEL

Consider a repeated game, i.e. a one-shot game repeated over time.

The static game has four components :

1. a lapse model,
2. a loss model,
3. an objective function,
4. a solvency constraint.

A lapse model (1/4)

Consider $(n_1, \dots, n_I)$ as the initial portfolio size of each insurer s.t. $\sum_{j=1}^I n_j = n$).

- The n_j policyholders of Insurer j follows an I-dimensional multinomial distribution

$$\mathcal{M}_I(n_j, (p_{j\to 1}(x), \dots, p_{j\to I}(x)))$$

where $x \in [\underline{x}, \overline{x}]^I$ is the price vector and

$$p_{j\to j}(x) = \frac{1}{1 + \sum\limits_{l\neq j} e^{f_j(x_j, x_l)}}, \quad p_{j\to k}(x) = \frac{e^{f_j(x_j, x_k)}}{1 + \sum\limits_{l\neq j} e^{f_j(x_j, x_l)}}, k \neq j. \qquad (1)$$

- The function $f_j(x_j, x_l)$ represent the price-sensitivity of customers

$$\bar{f}_j(x_j, x_l) = \mu_j + \alpha_j \frac{x_j}{x_l} \text{ and } \tilde{f}_j(x_j, x_l) = \tilde{\mu}_j + \tilde{\alpha}_j (x_j - x_l).$$

- The portfolio size of Insurer j is

$$N_j(x) = B_{jj}(x) + \sum_{k=1, k\neq j}^{I} B_{kj}(x).$$

where $B_{kj} \sim \mathcal{B}(n_k, p_{k\to j}(x))$.

A LOSS MODEL (2/4)

Consider a collective model for claims.

- Total claim of insured i is

$$Y_i = \sum_{l=1}^{M_i} Z_{i,l},$$

where M_i is the claim number, $(Z_{i,l})_l$ the claim severities and $M_i \perp (Z_{i,l})_l$.

- Assumption : independence of claims $(Y_i)_i$ between insureds

- The aggregate claim of Insurer j is

$$S_j(x) = \sum_{i=1}^{N_j(x)} Y_i.$$

- Two instances tested : Poisson – lognormal (PLN) et binomial négative – lognormal (NBLN).

AN OBJECTIVE FUNCTION (3/4)

The objective function choice $x \mapsto O_j(x)$ is justified by

- economic criteria : given x_{-j}, the demand is a decreasing function of x_j and the insurer objective depends on a break-even premium π_j,
- mathematical criteria : $x_j \mapsto O_j(x)$ must be strictly concave.

We choose

$$O_j(x) = \frac{n_j}{n}\left(1 - \beta_j\left(\frac{x_j}{m_j(x)} - 1\right)\right)(x_j - \pi_j), \quad (2)$$

- where the break-even premium π_j and the market premium $m_j(x)$ are computed as

$$\pi_j = \omega_j \bar{a}_{j,0} + (1-\omega_j)\overline{m}_0 \text{ and } m_j(x) = \frac{1}{I-1}\sum_{k \neq j} x_k.$$

- $\bar{a}_{j,0}$, $\overline{m}_0$, ω_j represent the mean actuarial premium, the mean market premium and the credibility factor, respectively.

A model without competition would be $O_j(x) = O_j(x_j)$.

A SOLVENCY CONSTRAINT (4/4)

For the solvency constraint, we want an explicit concave function $g_j^1(.)$.

- We choose
$$K_j + n_j(x_j - \pi_j)(1 - e_j) \geq k_{99.5\%}\sigma(Y)\sqrt{n_j},$$
where e_j is the expense rate and $k_{99.5\%}$ tail coefficient verifying
$$E(Y)n_j + k_{99.5\%}\sigma(Y)\sqrt{n_j} \approx \text{VaR}_{99.5\%}\left(\sum_{i=1}^{n_j} Y_i\right).$$
In practice, we set $k_{99.5\%} = 3$.
- The overall constraint function g_j is defined as
$$\begin{aligned} g_j^1(x_j) &= \frac{K_j + n_j(x_j - \pi_j)(1 - e_j)}{k_{99.5\%}\sigma(Y)\sqrt{n_j}} - 1 \\ g_j^2(x_j) &= x_j - \underline{x} \\ g_j^3(x_j) &= \overline{x} - x_j \end{aligned} \quad (3)$$

Game sequence

Over one period, the game proceeds as follows

1. Insurers set their premium according to a Nash $x^\star$. solving for all $j \in \{1, \dots, I\}$

$$x_j^\star \in \arg\max_{x_j, g_j(x_j)\geq 0} O_j(x_j, x_{-j}^\star).$$

2. Policyholders randomly choose their new insurer according to probabilities $p_{k\to j}(x^\star)$: we get $N_j(x^\star)$.

3. For the one-year coverage, claims are random according to the frequency – average severity model relative to the portfolio size $N_j(x^\star)$.

4. Finally the underwriting result is determined by $UW_j(x^\star) = N_j(x^\star)x_j^\star(1 - e_j) - S_j(x^\star)$ and new capital is $K_j + UW_j(x^\star)$, where e_j denotes the expense rate and K_j the initial capital value.

One-shot model – properties

Proposition ([DAL13a])

The insurance game with I insurers whose objective functions and solvency constraint functions are defined in Equations (2) and (3), respectively, admits a unique Nash premium equilibrium.

Sketch of the proof.

O_j continue + $x_j \mapsto O_j(x)$ quasiconcave $\Rightarrow$ existence,
$x_j \mapsto O_j(x)$ strictly concave $\Rightarrow$ uniqueness. □

Proposition ([DAL13a])

Let $x^\star$ be the Nash premium equilibrium of the insurance game with I insurers. For each Insurer j, the insurer equilibrium $x_j^\star$ with $x_j^\star \in]\underline{x}, \overline{x}[$:

- *increases with break-even premium π_j, solvency coefficient k_{995}, loss standard deviation $\sigma(Y)$, expense rate e_j and*
- *decreases with sensitivity parameter β_j and capital K_j.*

Otherwise, $x_j^\star = \underline{x}$ or $\overline{x}$.

Sketch of the proof.

KKT conditions + implicit function theorem □

NUMERICAL ILLUSTRATION (SIMPLE)

Consider a three-player game with $n = 10000$ customers, $I = 3$. Assume $(n_1, n_2, n_3) = (4500, 3200, 2300)$; K_i and solvency ratio is 133% ; $E(X) = 1$, $\sigma_{PLN}(Y) = 4.472$ and $\sigma_{NBLN}(Y) = 10.488$.

	P1	P2	P3	market	P1	P2	P3
PLN/NBLN	1.10	1.15	1.05	1.10	1.10	1.1166	1.0833
		$\bar{a}_{j,0}$		$\bar{m}_0$		π_j	

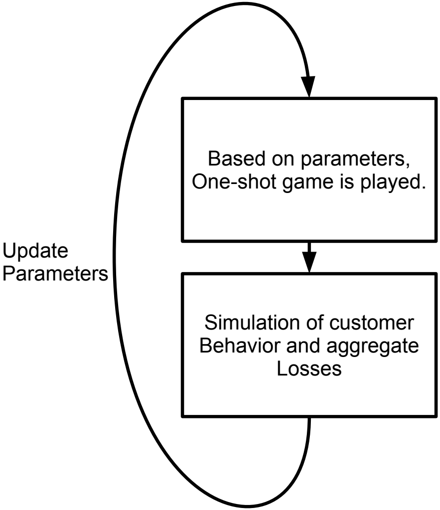
TABLE: Parameters $\bar{a}_{j,0}$, $\bar{m}_0$

Let $\Delta_i = E(N_i(x^\star)) - n_i$, $\rho_i = \frac{K_j + E(N_j(x^\star))(x_j^\star - \pi_j)(1 - e_j)}{k_{99.5}\sigma(Y)\sqrt{E(N_j(x^\star))}}$. Premium equilibrium are listed below.

	$x_1^\star$	$x_2^\star$	$x_3^\star$	Δ_1	Δ_2	Δ_3	ρ_1	ρ_2	ρ_3
PLN-$\tilde{f}_j$	1.544	1.511	1.471	-307.1	-10.58	317.7	3.524	2.993	2.729
PLN-$\bar{f}_j$	1.544	1.511	1.471	-256	-12.79	268.7	3.529	2.993	2.727
NBLN-$\tilde{f}_j$	1.544	1.511	1.471	-307.1	-10.58	317.7	2.295	2.042	1.881
NBLN-$\bar{f}_j$	1.544	1.511	1.471	-256	-12.79	268.7	2.292	2.043	1.887

TABLE: Equilibrium premium

GOING DYNAMIC



DYNAMIC PARAMETER

Let $\text{GWP}_{j,t}$ be the gross written premium, $n_{j,t}$ the portfolio size, $K_{j,t}$ the capital of Insurer j in t.

- At the beginning of each period, we determine

$$\bar{m}_{t-1} = \frac{1}{d}\sum_{u=1}^{d} \frac{\sum_{j=1}^{N} \text{GWP}_{j,t-u} \times x^{\star}_{j,t-u}}{\text{GWP}_{.,t-u}} \text{ et } \bar{a}_{j,t} = \frac{1}{1-e_{j,t}} \frac{1}{d}\sum_{u=1}^{d} \frac{s_{j,t-u}}{n_{j,t-u}}.$$

  Therefore, $\pi_{j,t} = \omega_j \bar{a}_{j,t} + (1-\omega_j)\bar{m}_{t-1}$.

- The objective and constraint functions are

$$O_{j,t}(x) = \frac{n_{j,t}}{n}\left(1 - \beta_{j,t}\left(\frac{x_j}{m_j(x)} - 1\right)\right)(x_j - \pi_{j,t}),$$

$$g^1_{j,t}(x_j) = \frac{K_{j,t} + n_{j,t}(x_j - \pi_{j,t})(1-e_{j,t})}{k_{995}\sigma(Y)\sqrt{n_{j,t}}} - 1.$$

- Some parameters are updated according to leader in turn's principle (based on $\text{GWP}_{j,t}$) : expense $e_{j,t}$, sensitivty $\beta_{j,t}$, lapse $\mu_{j,t}$, $\alpha_{j,t}$.

Repeated model – game sequence

For period t, the game proceeds as follows

1. Insurers set their premium according to a Nash $x^\star$. solving for all $j \in \{1, \dots, I\}$
$$x^\star_{j,t} \in \arg\max_{x_{j,t}, g_{j,t}(x_{j,t}) \geq 0} O_{j,t}(x_{j,t}, x^\star_{-j,t}).$$
2. Policyholders randomly choose their new insurer according to probabilities $p_{k\to j}(x^\star_t)$: we get realization $n^\star_{j,t}$ of $N_{j,t}(x^\star)$.
3. For the one-year coverage, claims $s_{j,t}$ are random according to the frequency – average severity model relative to the portfolio size $n^\star_{j,t}$.
4. The underwriting result is determined by
$$UW_{j,t} = n^\star_{j,t} \times x^\star_{j,t} \times (1 - e_j) - s_{j,t}.$$
5. The capital is updated
$$K_{j,t} = K_{j,t-1} + UW_{j,t}.$$
Insurer j gets bankrupt if $K_{j,t} < 0$ or $n^\star_{j,t} = 0$

Repeated game – properties

Proposition ([DAL13b])

Over one period, if for all $k \neq j$, $x_{j,t} \leq x_{k,t}$ et $x_{j,t}(1 - e_{j,t}) \leq x_{k,t}(1 - e_{k,t})$, then by-police underwriting result $uw_{j,t}$ are stochastically ordered $uw_{j,t} \leq_{icx} uw_{k,t}$.

Sketch of the proof.

Majorization order and convex order properties. □

Proposition ([DAL13b])

Infinitely repeated, the probability there exist at least two non-ruined insurers tends geometrically to zero with t.

Sketch of the proof.

Bounding of $P(\mathrm{Card}(I_t) > 1)$ where I_t is the set of survivors in t. □

REPEATED GAME – SAMPLE PATH

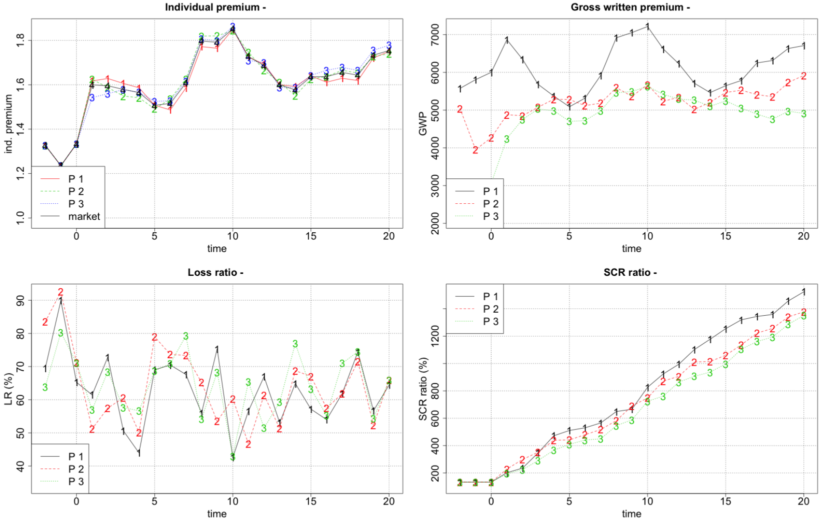


FIGURE: Loss model : NBLN and prince-sensitivity $\tilde{f}_j$

Leader and ruin probabilities

Computation over $2^{14} \approx 16000$ simulations and $T = 20$ period.

	Ruin before $t = 10$	Ruin before $t = 20$	Leader at $t = 5$	Leader at $t = 10$	Leader at $t = 20$
Insurer 1	6.1e-05	6.1e-05	0.593	0.381	0.331
Insurer 2	0	0	0.197	0.308	0.329
Insurer 3	0.000244	0.000244	0.21	0.312	0.34

Table: Ruin/leader probabilities

	Min.	1st Qu.	Median	Moy.	3rd Qu.	Max.
Insurer 1	-0.7905	0.2309	0.3617	0.3563	0.4869	1.2140
Insurer 2	-0.4340	0.2279	0.3600	0.3555	0.4869	1.1490
Insurer 3	-0.4730	0.2308	0.3627	0.3563	0.4871	1.0950

Table: By-policy underwriting result at $t = 20$

CYCLE PERIOD

- Fit AR(2) : $X_t = a_1 X_{t-1} + a_2 X_{t-2} + \mathcal{E}_t$.
- If $a_2 < 0$ et $a_1^2 + 4a_2 < 0$, then (X_t) is p-periodic with $p = 2\pi \cos^{-1}\left(\frac{a_1}{2\sqrt{-a_2}}\right)$.
- non-periodic random path : 240 out of 2^{14}

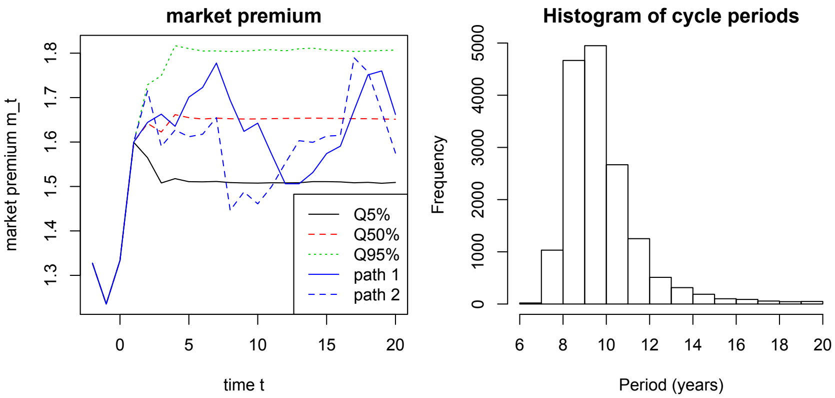


FIGURE: Market premium

CASE STUDY : THE FRENCH MOTOR MARKET

	Portfolio Size	Loss Ratio	Coverage Ratio	Top (GWP)
MAAF - MMA - GMF	7103	0.81	4.499	1
AXA	4799	0.77	3.291	2
Groupama - GAN	4066	0.9	7.693	3
Macif (SFEREN)	5721	0.78	4.416	4
AGF Allianz	3103	0.77	7.193	5
Maif (SFEREN)	3370	0.86	5.887	6
Generali	1341	0.85	3.971	7
Matmut (SFEREN)	2703	0.91	7.332	8
Assurance Credit Mutuel	1596	0.83	7.207	9
Credit Agricole	1220	0.91	4.61	10

TABLE: Parameters based on fact figures in 2002

STATISTICS OF CYCLE PERIODS

	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.	NA's	Std. Dev.
$\tilde{f}_j$-PLN	4.618	6.193	6.738	7.354	7.735	22.74	27%	2.433
$\tilde{f}_j$-NBLN	5.178	6.558	7.543	9.28	9.341	53.54	29%	7.277
$\bar{f}_j$-PLN	5.42	6.639	7.234	7.742	8.114	18.1	4%	1.912
$\bar{f}_j$-NBLN	5.852	7.367	8.405	9.621	10.26	33.02	7%	3.987

TABLE: Cycle period

Reminder :

- PLN - Poisson Lognormal ; NBLN - Negative Binomial Lognormal
- $\bar{f}_j(x_j, x_l) = \mu_j + \alpha_j \frac{x_j}{x_l}$ et $\tilde{f}_j(x_j, x_l) = \tilde{\mu}_j + \tilde{\alpha}_j(x_j - x_l)$.

- How to model competition in non-life insurance markets ?
  - repeated game to mimic insurer behaviors.
  - new point of view on cycles
  - possible extensions : different class of insurers, reinsurance, . . .

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